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FUZZY SET-VALUED MAPPINGS AND FIXED POINT THEOREMS

DUONG TRONG NHAN
Institute of Mathematics
Hanoi

INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh ([9]) and subsequently extensively studied by many authors. One of the most important concepts in fuzzy set theory which has applications to various branches of applied mathematics is the concept of fuzzy set-valued mapping. In [4] Chang gave a definition of continuity, called F-continuity, for point-valued mappings regarded as fuzzy point-valued mappings. This concept of continuity was extended by Butnariu to fuzzy set-valued mappings defined on a topological space equipped with the so-called local fuzzy topology ([2]).

In the present paper we shall deal with fuzzy set-valued mappings defined on a fuzzy topological space. In Section 2 we shall give a definition of F-upper semicontinuous fuzzy set-valued mappings and study their basic properties as well as the relationship of this concept of F-upper semicontinuity to that of F-continuity given in [4] and in [2]. Section 3 is devoted to the fixed point problem. The main results are Theorems 3.1 and 3.2 which represent fuzzy versions of Kakutani-Ky Fan's fixed point theorem. An application of these theorems to two persons fuzzy games is also discussed in this section.

For completeness, a preliminary section is also included.

I. PRELIMINARIES

We first recall some basic definitions and elementary properties. For more details, the readers are referred to [9], [3] and [8].

Let X be a set. A *fuzzy set* in X is a function $A: X \to [0, 1]$. By $L(X)$ we denote the collection of all fuzzy sets in X. For any $A \in L(X)$, the *complement* of A is a fuzzy set A' in X defined by $A'(x) = 1 - A(x)$ for any x in X. For $A, B \in L(X)$ we say that A is contained in B and write $A \subseteq B$ if $A(x) \leqslant B(x)$, $\forall\ x \in X$. The notation $A \subset B$ means that $A \subseteq B$ and there exists a point x_o in X such that $A(x_o) < B(x_o)$, $A = B$ *if* $A(x) = B(x)$, $\forall\ x \in X$. We denote by $Supp\ A$ the support of A, i. e. the set $Supp\ A = \{x \in X / A(x) > 0\}$. A *fuzzy point* is a fuzzy set the support of which contains only one element. We say that $\tilde{x}$ is a fuzzy point of a fuzzy set A and write $\tilde{x} \in A$ if $\tilde{x}(x) < A(x)$ where x is the support of $\tilde{x}$. For any family $\{A_i\}_{i \in I}$ of fuzzy sets in X the *intersection* $\bigcap_{i \in I} A_i$ and the *union* $\bigcup_{i \in I} A_i$ are the fuzzy sets defined respectively by

$$\Big(\bigcap_{i \in I} A_i\Big)(x) = \mathop{Inf}_{i \in I}\ A_i(x) \quad (\forall\ x \in X)$$

and

$$\Big(\bigcup_{i \in I} A_i\Big)(x) = \mathop{Sup}_{i \in I}\ A_i(x) \quad (\forall\ x \in X).$$

PROPOSITION 1.1. (See[8])

If $A = \bigcup_{i \in I} A_i$, *then* $x_o \in A$ *if and only if* $x_o \in A_{i_o}$ *for some* $i_o \in I$.

OBSERVATION 1.1.

A fuzzy set is composed of its fuzzy points, i. e. $A = \bigcup_{\tilde{x} \in A} \tilde{x} = \bigcup_{x \subseteq A} \tilde{x}$.

Indeed.

$$\Big(\bigcup_{\tilde{x} \in A} \tilde{x}\Big)(x) = \begin{cases} Sup\ \{t \in [0,1],\ t < A(x)\} = A(x), & if\ x \in Supp\ A \\ 0 & else \end{cases}$$

In [4] Chang introduced the notion of fuzzy topology as follows:

A *fuzzy topology* on X is a family $\mathcal{F}$ of fuzzy sets in X satisfying the following axioms:

1. $\chi_\phi \in \mathcal{F}$, $\quad \chi_X \in \mathcal{F}$,
2. $\bigcup_{i \in I} A_i \in \mathcal{F}$ $\quad$ if $A_i \in \mathcal{F}$ $\quad$ for any $i \in I$,
3. $A \cap B$ $\quad$ if $A, B \in \mathcal{F}$,

where ϕ denote the empty set and χ_A the characteristic function of a set A in X. The fuzzy set χ_ϕ is called *empty fuzzy set*.

The pair $(X, \mathcal{F})$ is called then a *fuzzy topological space* (briefly, f. t. s.). The elements of $\mathcal{F}$ are called *open* and their complements *closed*.

A fuzzy topology $\mathcal{G}$ on X is said to be weaker than $\mathcal{F}$ (or $\mathcal{F}$ is stronger than $\mathcal{G}$) if $\mathcal{G} \subseteq \mathcal{F}$.

A fuzzy set U in a fuzzy topological space X is called a *neighborhood* (briefly, n. b. hd.) of a fuzzy set A if there exists an open fuzzy set G in X such that $A \subseteq G \subseteq U$.

PROPOSITION 1. 2. ([4])

A fuzzy set A is open if and only if for each fuzzy set B contained in A, A is a n. b. hd. of B.

PROPOSITION 1.3. ([4])

The intersection of a finite number of neighborhoods of a fuzzy set A is also a n. b. hd of A and each fuzzy set which contains a n. b. hd of A is a n. b. hd of A, too.

Let A,B be in $L(X)$ with $B \subseteq A$. The set B is called an *interior fuzzy set* of A if A is a n. b. hd of B. The union of all interior fuzzy sets of A is called the *interior* of A and denote by A^o.

PROPOSITION 1.4. ([4])

The interior A^0 of a fuzzy set A is open and it is the largest open fuzzy set contained in A. The fuzzy set A is open if and only if $A = A^0$.

We denote by $\omega_r(A)$ and $\sigma_r(A)$ the weak and strong r-cut of A respectively, i. e.

$$\omega_r(A) = \{x \in X \mid A(x) \geqslant r\}, \quad \sigma_r(A) = \{x \in X \mid A(x) > r\}.$$

If X is a linear space, then a fuzzy set A in X is called convex if

$$A(tx_1 + (1-t)x_2) \geqslant Min\,\{A(x_1), A(x_2)\}.$$

whenever x_1, $x_2 \in X$, $t \in [0,1]$. In other words, A is *convex* if $\omega_r(A)$ is convex for each $r \geqslant 0$ (i. e. A is quasiconcave function) or equivalently, if $\sigma_r(A)$ is convex for each $r \geqslant 0$. So a convex fuzzy set has a convex support. The intersection $\bigcap_{i \in I} A_i$ of an arbitrary family of convex fuzzy sets $\{A_i\}_{i \in I}$ is a convex fuzzy set.

With any fuzzy topology $\mathcal{G}$ on X we associate an usual topology τ on X defined by $\tau = \{Supp\ G / G \in \mathcal{G}\}$ and we call it the *associated topology* of $\mathcal{G}$ (it is easy to verify that τ is a topology on X). Conversely with any topology τ on X we can associate a fuzzy topology $\mathcal{G}$ on X defined by $\mathcal{G} = \{G \in L(X)/Supp\, G \in \tau\}$ and call it the *associated fuzzy topology* of τ.

We recall that the *induced fuzzy topology* of a topological space X is the family of all fuzzy sets in X which are lower semicontinuous functions (see[7]). A fuzzy set A is open in the induced fuzzy topology if and only if for each $r \geqslant 0$ the set $\sigma_r\ (A)$ is open in the usual topology. So an open fuzzy set in the induced fuzzy topology has an open support. That means the induced fuzzy topology is weaker than the associated one.

The family of all characteristic functions of open sets in a topological space X is a fuzzy topology on X, namely, *local fuzzy topology* (see [2]). Since the characteristic function of an open set is lower semicontinuous, the induced fuzzy topology is stronger than the local one.

We observe that if we identify a subset of X with its characteristic function then an ordinary set can be understood as a fuzzy set and the usual topology on X can be regarded as a fuzzy topology on X, it is just the local fuzzy topology.

PROPOSITION 1.5.

A fuzzy set A *in a f.t.s* X *is open if and only if it is a neighborhood of all its fuzzy points.*

Proof. If A is open then A is a n.b.hd of all its fuzzy points. If for any $\tilde{x} \in A$, A is a n.b.hd of $\tilde{x}$ then since $\tilde{x}$ is a fuzzy interior set of A we have

$$A = \bigcup_{\tilde{x} \in A} \tilde{x} \subseteq A^{\circ}.$$

Hence $A = A^{\circ}$, i.e. A is open.

Remark 1.2.

Let (X, τ) be an usual topological space, let $\mathcal{G}$ be the associated fuzzy topology of τ. Then $\mathcal{G}$ is stronger than both the induced fuzzy topology and the local one. But all these fuzzy topologies have the same associated topology, namely the initial topology τ.

There are different definitions of **compactness.** One given by Chang can be stated as follows:

DEFINITION 1.1 ([4])

A fuzzy set A *in a f.t.s. is called compact if every open cover of A has a finite subcover.* Here an open cover of A is defined as a collection of open fuzzy sets the union of which contains A.

We can give a definition of compactness as follows:

DEFINITION 1. 2.

A fuzzy set A *in a fuzzy topological space* (X, $\mathcal{J}$) *is called Ass-compact if* Supp A *is compact in the associated topology* τ *of* $\mathcal{J}$.

PROPOSITION 1.6.

Let X *be a topological space equipped with the local fuzzy topology or any stronger one* (for example the induced or the associated).

Then a compact fuzzy set has a compact support.

Proof. Let A be a compact fuzzy set. Let $\{G_i\}_{i \in I}$ be an open cover of *Supp A*. Then the family $\{\chi_{G_i}\}_{i \in I}$ is an open cover of A in the local fuzzy topology (and in any stronger topology). Since A is compact there exists a finite subcover $\{\chi_{G_i}\}_{i=\overline{1,n}}$ of A. This implies $Supp\ A \subseteq \bigcup_{i=1}^{n} G_i$.

From this proposition we have that any compact fuzzy set in the local fuzzy topology (respectively in the induced or in the associated one) is Ass-compact.

DEFINITION 1.3.

A fuzzy set A in a fuzzy topological space (X, $\mathfrak{F}$) *is called Ass-closed if Supp A is closed in the associated topology* τ *of* $\mathfrak{F}$.

2. FUZZY SET-VALUED MAPPINGS AND CONTINUITY CONCEPTS

Let X, Y be two arbitrary sets. By a *fuzzy set-valued mapping* R from X into Y is meant a mapping from X into $L(Y)$. So a fuzzy set-valued mapping is a fuzzy set in the product set $X \times Y$ defined by

$$R(x,y) = R_x(y) \text{ for } x \in X,\ y \in Y$$

where $R_x(.)$ denotes the image of x by R.

For any $A \in L(X)$, we denote by $R\,[A]$ the *image* of A by R, i.e., the fuzzy set in Y defined by:

$$R[A](y) = \sup_{x \in X} \{R(x,y) \,.\, A(x)\},\ \forall y \in Y.$$

For any $B \in L(Y)$ we denote by $R^{-1}[B]$ the *inverse image* of B by R i.e., the fuzzy set in X defined by

$$R^{-1}[B](x) = \operatorname{Sup}_{y \in Y} \{R(x,y) \,.\, B(y)\}, \ \forall x \in X.$$

The following properties are obtained immediately from the definitions:

1. *For any* A_1, A_2 *in* $L(X)$ *with* $A_1 \subseteq A_2$ *we have*

$$R[A_1] \subseteq R[A_2].$$

2. *For any* B_1, B_2 *in* $L(Y)$ *with* $B_1 \subseteq B_2$ *we have*

$$R^{-1}[B_1] \subseteq R^{-1}[B_2].$$

3. *For any fuzzy point* $\tilde{x}$ *with support* **x** *we have*

$$(R[\tilde{x}])\,(y) = R(x,y) \,.\, \tilde{x}(x), \ \forall y \in Y.$$

4. *For any fuzzy point* $\tilde{y}$ *with support* y *in* Y *we have*

$$(R^{-1}[\tilde{y}])\,(x) = R(x,y) \,.\, \tilde{y}(y), \ \forall x \in X.$$

5. $$R[A] = \bigcup_{\tilde{x} \in A} R[\tilde{x}] \quad \textit{for any fuzzy set } A.$$

Proof. *The properties 1/, 2/, 3/, 4/, are obvious. Let us check 5/. Let* $y \in Y$*. It follows from Observation 1.1 that:*

$$(R[A])(y) = \operatorname{Sup}_{z \in X} \left\{ R(z,y) \,.\, A(z) \right\} = \operatorname{Sup}_{z \in X} \left\{ R(z,y) . \operatorname{Sup}_{x \in A} \tilde{x}(z) \right\}.$$

On the other hand we have

$$\left(\bigcup_{\tilde{x} \in A} R[\tilde{x}] \right)(y) = \operatorname{Sup}_{\tilde{x} \in A} \left\{ R[\tilde{x}](y) \right\} = \operatorname{Sup}_{\tilde{x} \in A} \left\{ \operatorname{Sup}_{z \in X} \left\{ R(z,y) \,.\, \tilde{x}(z) \right\} \right\} =$$

$$= \operatorname{Sup}_{z \in X} \operatorname{Sup}_{\tilde{x} \in A} \left\{ R(z,y) \,.\, \tilde{x}(z) \right\} = \operatorname{Sup}_{z \in X} \left\{ R(z,y) \,.\, \operatorname{Sup}_{\tilde{x} \in A} \tilde{x}(z) \right\}.$$

So

$$(R[A])(y) = \left(\bigcup_{\tilde{x} \in A} R[\tilde{x}] \right)(y), \ \forall y \in Y.$$

DEFINITION 2.1.

Let $(X, \mathcal{F})$ *and* $(Y, \mathcal{G}_Y)$ *be two fuzzy topological spaces,* R *be a fuzzy set-valued mapping from* X *into* Y. *We say that* R *is* F*-upper semicontinuous at a fuzzy point* $\tilde{x}$ *if for any open fuzzy set* G *in* Y *with* $R[\tilde{x}] \subseteq G$ *there exists a fuzzy neighborhood* V *of* $\tilde{x}$ *such that* $R[V] \subseteq G$.

R *is called* F-*upper semicontinuous if it is* F-*upper semicontinuous at every fuzzy point.*

DEFINITION 2.2.

A fuzzy set-valued mapping R *from* X *into* Y *is said to be a fuzzy point-valued mapping if for any* $x \in X$, *there exists an unique element* $y \in Y$ *such that* $R(x,y) > 0$ or, equivalently, R_x *is a fuzzy point for every* $x \in X$. It follows from the property 3/ that $Supp\, R_x = Supp\, R[\tilde{x}]$ for any fuzzy point $\tilde{x}$ with support x. So R is a fuzzy point-valued mapping if and only if the image of any fuzzy point by R is a fuzzy point.

In what follows, $R^+[B]$ denotes the fuzzy set

$R^+[B] = \cup \{\tilde{x} / \tilde{x} \in L(X),\ R[\tilde{x}] \subseteq B\} = \cup \{A \in L(X),\ R[A] \subseteq B\}$ for any $B \in L(Y)$, where $\tilde{x}$ is a fuzzy point.

THEOREM 2.1.

R is F-upper semicontinuous if and only if $R^+[B]$ is an open fuzzy set in X *for any open fuzzy set B in Y.*

Proof. Suppose that R is F-upper semicontinuous. Let B be an open fuzzy set in Y. Let $\tilde{x} \in R^+[B] = \bigcup \{\tilde{x} / \tilde{x} \in L(X), R[\tilde{x}] \subseteq B\}$. Since R is F-upper semicontinuous, there exists a fuzzy n. b. hd V of $\tilde{x}$ such that $R[V] \subseteq B$. This means $V \subseteq R^+[B]$, so $R^+[B]$ is a n. b. hd of $\tilde{x}$. Since $\tilde{x}$ is chosen arbitrarily, $R^+[B]$ is an open fuzzy set by Proposition 1. 5.

Conversely, let $\tilde{x}$ be an arbitrary fuzzy point and B an open fuzzy set in Y such that $R[\tilde{x}] \subseteq B$, that is $\tilde{x} \in R^+[B]$. Since $R^+[B]$ is open, $R^+[B]$ is a fuzzy n. b. hd of $\tilde{x}$. Moreover it is obvious that $R[R^+[B]] \subseteq B$.

So R is F-upper semicontinuous.

THEOREM 2.2.

If R is F-*upper semicontinuous with compact values, then the image $R[K]$ of any compact fuzzy set K in X is a compact fuzzy set of Y.*

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of $R[K]$. Since $R[\tilde{x}]$ is compact for any fuzzy point $x \in K$ and $\{G_i\}_{i \in I}$ covers $R[\tilde{x}]$, there exists a finite subcover of $R[\tilde{x}]$, say $\{G_i\}_{i = \overline{1,n}}$

$$(2.1) \quad R[\tilde{x}] \subseteq \bigcup_{i=1}^{n} G_i \underset{def}{=} G_{\tilde{x}} .$$

By virtue of Theorem 2.1 $R^+[G_{\tilde{x}}]$ is an open fuzzy set. Moreover (2.1) means $\tilde{x} \subseteq R^+[G_{\tilde{x}}]$, so we get

$$K = \bigcup_{\tilde{x} \in K} \tilde{x} \subseteq \bigcup_{\tilde{x} \in K} R^+[G_{\tilde{x}}],$$

i. e. $\{R^+[G_{\tilde{x}}]\}_{\tilde{x} \in K}$ constitutes an open cover of the compact fuzzy set K. Hence

$$K = \bigcup_{j=1}^{m} R^+[G_{\tilde{x}_j}].$$

Consequently, K is covered by a finite number of the sets G_i ($i \in I$). This completes the proof.

Now let X and Y be two topological spaces. Any set-valued mapping $T : X \to 2^Y$ can be regarded as a fuzzy set-valued mapping denoted also T_F, which is given by $T_F = \chi_{GraphT}$. The image and the inverse image of a fuzzy set A by T_F is written, respectively, as follows:

$$(T_F[A])(y) = \begin{cases} \sup\limits_{x \in T^{-1}(y)} A(x), & \text{if } T^{-1}(y) \neq \emptyset \\ 0 & \text{else} \end{cases}$$

where

$$T^{-1}(y) = \{x \in X \mid y \in Tx\}$$

$$(T_F^{-1}[B])(x) = \sup_{y \in Tx} B(y), \qquad \forall x \in X.$$

In the case when T is point-valued, the notions of image, inverse image and F-continuity introduced above, coincides with the ones, given by C. L. Chang in [3].

Now, let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two f.t.s., let $T : X \to Y$ be a point-valued mapping, then T_F is a fuzzy point-valued mapping. The F-continuity of T_F defined by Chang can be stated as follows:

DEFINITION 2. 3.

T_F is F-continuous if $T_F^{-1}[B]$ is an open fuzzy set in X for any open fuzzy set B in Y.

This notion coincides with that of F-upper semicontinuity defined above. This is illustrated by the following proposition:

PROPOSITION 2. 3.

Let $(X, \mathfrak{F})$ and $(Y, \mathcal{G})$ be two f.t.s., $T : X \to Y$ be a point-valued mapping. Then T_F is F-upper semicontinuous if and only if $T_F^{-1}[B]$ is an open fuzzy set in X for any open fuzzy set B in Y.

Proof. We shall show that $T_F^{-1}[B] = T_F^{+}[B]$ for any fuzzy set B in Y. First, we observe that

1. $T_F[T_F^{-1}[B]] \subseteq B, \ \forall\ B \in L(Y)$,
2. $T_F^{-1}[T_F[A]] \subseteq A, \ \forall\ A \in L(X)$.

Indeed, let $y \in Y$ if $T^{-1}(y) = \phi$ then

$$(T_F[T_F^{-1}[B]])(y) = 0.$$

If $T^{-1}(y) \neq \emptyset$ then

$$(T_F[T_F^{-1}[B]])(y) = \sup_{x \in T^{-1}(y)} (T_F^{-1}[B])(x) = \sup_{x \in T^{-1}(y)} B(Tx) = B(y).$$

So $(T_F\ [\ T_F^{-1}\ [B]\ ])\ (y) \leqslant B\ (y)$, $\forall$ y $\in$ Y whence 1/ follows.

On the other hand:

$$(T_F^{-1}\ [\ T_F\ [A]\ ])\ (x) = (T_F\ [A])\ (Tx) = \sup_{z \in T^{-1}(Tx)} A(z) \geqslant A(x),\ \forall\ x \in X$$

whence 2/ follows.

Now let $\widetilde{x} \subseteq T_F^{-1}\ [B]$. We have $T_F\ [\widetilde{x}] \subseteq T_F\ [T_F^{-1}\ [B]\ ] \subseteq B$, hence $\widetilde{x} \in T_F^{+}\ [B]$. Conversely, if $\widetilde{x} \in T_F^{+}\ [B]$ then $T_F^{+}\ [\widetilde{x}] \subseteq B$, hence $\widetilde{x} \in T_F^{-1}\ [\ T_F\ [\widetilde{x}]\ ] \subseteq T_F^{-1}\ [B]$.

Consequently, $T_F^{+}\ [B] = T_F^{-2}\ [B]$.

Our assertion follows now from Theorem 2. 2.

Let X and Y be two topological spaces. As fuzzy topologies on X and Y we consider the induced ones. In [7], M. Weiss has prove l that a point - valued mapping $T : X \longrightarrow Y$ is continuous if and only if T_F is F - continuous. So by Proposition 2. 1, T is continuous if and only if T_F is F—upper semicontinuous.

In the case X and Y are topological spaces equipped with the local fuzzy topologies, Butnariu have defined the Sup semicontinuity for a fuzzy set-valued mapping as follows (see [2]):

DEFINITION 2. 4.

A fuzzy set-valued mapping R *from* X *into* Y *is called* (X, Y) — *Sup semicontinuity if for any point* $x \in X$, *any open fuzzy set* G *with* $R_x \subseteq G$, *there exists a neighbourhood* V *of* χ_x *such that* $R[V] \subseteq G$.

Remark 2.2.

If X and Y are two topological space equipped with the local fuzzy topologies, the F-upper semicontinuity defined above reduces to the (X, Y) — Sup semicontinuity defined by Butnariu.

Indeed, let R be a Sup semicontinuous fuzzy set-valued mapping from X into Y. Let $\widetilde{x}$ be any fuzzy point with support x, G be an open fuzzy set in Y with $R[\widetilde{x}] \subseteq G$. Since G is a characteristic function we have

$$R_x = R[\chi_x] \subseteq G.$$

Hence, there exists a neighbourhood V of χ_x such that $R[v] \subseteq G$. But evidently, V is a neighbourhood of $\widetilde{x}$ and so R is F-upper semicontinuous at $\widetilde{x}$.

DEFINITION 2. 5.

Let X *and* Y *be two topological spaces,* R *be a fuzzy set-valued mapping from* X *into* Y. *The set-valued mapping* $T : X \longrightarrow 2^Y$ *defined* by

$$T(x) = Supp\ R_x, \ \forall x \in X,$$

is called the associated set-valued mapping of R.

We observe that for any set-valued mapping $T : X \longrightarrow 2^Y$, the associated set-valued mapping of the fuzzy set-valued mapping T_F is just T.

In what follows, we shall deal with the relationship between the F-continuity of a fuzzy set-valued mapping R and the continuity of its associated set-valued mapping T.

THEOREM 2. 4.

Let X *and* Y *be two topological spaces equipped either with the associated fuzzy topologies of the local fuzzy topologies (or any fuzzy topologies stronger than the local and weaker than the associated).*

If R *is a F-upper semicontinuous fuzzy set-valued mapping from* X *into* Y *then its associated set-valued mapping* T *is upper semicontinuous.*

Proof. It suffices to consider the case when X is equipped witth the associated fuzzy topology and Y is equipped with the local one.

Let $x \in X$, G be an open set in Y with $Tx \subseteq G$. We have then

$$R[\chi_x] = R_x \subseteq \chi_{Tx} \subseteq \chi_G = B.$$

Since R is F-upper semicontinuous and B is an open set, there exists a neighborhood V of χ_x such that

$$R\,[V] \subseteq B.$$

Evidently, $U = Supp\ V$ is a neighborhood of x. Moreover, let $y \in T(U)$ i.e., $y = Tx'$ for some $x' \in U$, we have

$$R(x', y) = R_{x'}(y) > 0.$$

But $V(x') > 0$ and $(R[V])(y) = \sup_{x \in X} \{R(x, y).\ V(x)\}$.

Hence $R[V](y) > 0$, which implies $B(y) > 0$, i.e., $y \in G$. So we have shown that $T(U) \subseteq G$. The theorem is proved.

In the special case when Y is equipped with the local fuzzy topology, the F-upper semicontinuity of R is equivalent with the upper semicontinuity of T.

THEOREM 2.5.

Let X and Y be two topological spaces. If Y is equipped with the local fuzzy topology, X is equipped either with the local fuzzy topology or with the associated fuzzy topology (or any fuzzy topology stronger than the local and weaker than the associated). Then a fuzzy set-valued mapping R from X into Y is F-upper semicontinuous if and only if its associated set-valued mapping T is upper semicontinuous.

Proof. By Theorem 2.4, it remains for us to prove that the upper semicontinuity of T implies the F-upper semicontinuity of R.

Let $\tilde{x}$ be a fuzzy point with support x, let $G = \chi_B$ be an open fuzzy set (i.e. B is open) such that $R[\tilde{x}] \subseteq G$. Then we have $Tx \subseteq B$, hence there is a neighborhood V of x such that $T(V) \subseteq B$.

Therefore $R[\chi_V] \subseteq \chi_{T(V)} \subseteq \chi_B = G$

aud χ_V is a neighborhood of $\tilde{x}$. So R is F-upper semicontinuous.

3. FIXED POINT THEOREMS AND APPLICATIONS.

The notion of fixed point for fuzzy set-valued mappings has a great importance in fuzzy games (see [3], [1] and [2]). In this section we deal with the fixed point problem in locally convex spaces and its application to two-persons fuzzy games.

DEFINITION 3.1.

Let X be a set, Q be a subset of X, R be a fuzzy set-valued mapping from Q into X.

A point $x_o \in Q$ is called a fixed point of R if $R(x_o, x_o) > 0$.

It is called a well-fixed point of R if

$$R(x_o, x_o) \geqslant R(x_o, y) \qquad \forall\, y \in X.$$

REMARK 3.1.

a) Let R be a fuzzy set-valued mapping from Q into X with nonempty values. Then a well-fixed point of R is also fixed point. The inverse assertion holds in the case when R is point-valued or a characteristic function of a set in $X \times Y$.

b) Let $T: Q \to 2^X$ be a set-valued mapping with nonempty values. Then a point x_o is a fixed point of T, i. e., $x_o \in T x_o$, if and only if x_o is a fixed point of the fuzzy set-valued mapping T_F.

Indeed, $x_o \in T x_o \leftrightarrow \chi_{Graph\, T}(x_o, x_o) = T_F(x_o, x_o) = 1$

hence $T_F(x_o, x_o) \geqslant T_F(x_o, y)$, $\forall\, y \in X$.

Now, let X be a Hausdorff locally convex space, Q be a nonempty convex compact subset of X. We denote by $I_Q(x)$ the set

$I_Q(x) = \{ y \in X \mid \exists\, u \in Q,\ \exists\, a \geqslant 0,$ *such that* $y = x + a(u - x)\}$ *and by* $\overline{I}_Q(x)$ the closure of $I_Q(x)$.

Let X be a topological space. By Remark 1. 2, a fuzzy set in X is Ass-compact in the associated fuzzy topology if and only if it is Ass-compact in the local topology (or, in any fuzzy topology on X stronger than the local and weaker than the associated). For brevity, we shall say Ass-compact instead of Ass-compact in the associated fuzzy topology.

THEOREM 3.1.

a) Let X be a Hausdorff locally convex space, Q be a nonempty convex Ass-compact fuzzy set in X. Let R be a F-upper semicontinuous fuzzy set-valued *mapping from X into X (with respect to the associated fuzzy topology on X and the local fuzzy topology on Q, respectively), with nonempty convex ass-closed values. If for each $x \in K = \text{Supp } Q$ we have*

$$\text{Supp } R_x \cap \overline{I}_K(x) \neq \phi,$$

then R has a fixed point.

b) In addition, if R is point-valued or a characteristic function of a set in $X \times Y$ then R has a well-fixed point.

Proof. Since R is F-upper semicontinuous the associated set-valued mapping T of R is upper semicontinuous (Theorem 3. 4). Moreover $T_{|K}: K \to 2^X$ satisfies the boundary condition of Halpern ([5]). Hence there exists a point $x_o \in K$ such that $x_o \in T x_o$, or equivalently, $R(x_o, x_o) > 0$.

If R is point-valued or a characteristic function of a set in $X \times Y$ then

$$R(x_o, x_o) \geqslant R(x_o, y), \quad \forall y \in X, \text{ i. e}$$

x_o is a well-fixed point, of R.

Let R be a fuzzy set-valued mapping from X into X. We define a fuzzy set-valued R' from X into X as follows:

$$(3.1.) \quad R'_x(y) = \begin{cases} R_x(y) \text{ if } R_x(y) \geqslant R_x(z), \forall z \in X \\ O \quad \text{else} \end{cases}$$

Then any fixed point of R' is an well-fixed point of R. So we have:

THEOREM 3.2.

Let (X, $\mathcal{T}$) *and* Q *be as in Theorem* 3.1. *Let* R *be a fuzzy set-valued mapping from* X *into* X.

If R' *defined by* (3.1.) *satisfies the hypothesis* a) *of Theorem* 3.1, *then* R *has a well-fixed point.*

COROLLARY 3.1.

Let X *and* Q *be as in* ***Theorem*** 3.1.

Let T *be a point-valued mapping from* X *into* X *such that*

1. T_F *is F-upper semicontinuous in the induced fuzzy topology,*
2. $T_F[Q] \subseteq Q$.

Then T *has a fixed point in* Supp Q.

Proof. *Since* T_F satisfies all the conditions of Theorem 3.1, the proposition follows.

In [7] Weiss has proved an analogous theorem (Theorem 4.1) but the notion of compactness there is different.

PROPOSITION 3.1.

Let K *be a nonempty convex compact subset of* $R^n (n \geqslant 1)$. *Let* R *be a fuzzy set-valued mapping from* K *into* X. *If the fuzzy set-valued mapping* R' *from* K *into* X *defined by* (3.1) *is* F-*upper semicontinuous (with respect to the associated fuzzy topology on* K *and the local fuzzy topology on* Y, *respectively), if for each* $x \in K$ *the set* Supp R'_x *is nonempty convex closed and* .

$$Supp\ R'_x \cap \bar{I}_K(x) \neq \emptyset,$$

then R *has a well-fixed point.*

Proof. Since the associated set-valued mapping T' of R' has a fixed point, R has a well-fixed point.

COROLLARY 3.2 (see [2], Theorem 3.2)

Let Q *be a n-dimensional simplex in* R^n $(n \geqslant 1)$ *equipped with the local fuzzy topology. If* R *is a* F-*upper semicontinuous fuzzy point-valued mapping from* Q *into* Q, *then* R *has an well-fixed point.*

To conclude this section we give an application of Proposition 3.1 to fuzzy games. The notion of fuzzy games was described in [3].

Now we recall some definitions and results obtained in [1] and [2].

DEFINITION 3.2.

A two-person fuzzy game Γ *(briefly 2-FG) is a set of data as follows:*

(a) $P = \{1,2\}$ *is a set whose elements are called players.*

(b) *The finite sets* $\Sigma_k = \left\{\sigma_l^{(k)}, \ldots, \sigma_{n(k)}^{(k)}\right\}$, $k = 1,2$ *are finite sets whose elements are called pure strategies of*

(c) $Y_k \subseteq R^{n(k)}$ *is the set of regular choices of the player* k.

(d) $E_k \in L(Y_p \times Y_k)$ *with* $p \in P - \{k\}$ *is a fuzzy set in* $R^{n(1)+n(2)}$ *called the fuzzy set of possible choices of the player* k (k = 1,2 *where* $W_k = Y_k \times L(Y_p)$ *is the set of strategic conceptions of the player* k $(p \in P - \{k\})$.

Suppose that the following axioms are satisfied:

(I) *If* $A_2 \in L(Y_2)$ *and* $A_2 \neq \chi_\phi$ *then* $E_1[A_2] \neq \chi_\phi$.

(II) *If* $A_1 \in L(Y_1)$ *and* $A_1 \neq \chi_\phi$ *then* $E_2[A_1] \neq \chi_\phi$.

DEFINITION 3.3.

Let $s = (s_1, s_2) \in W_1 \times W_2$ *be a pair of strategic conceptions in the* 2—FG Γ, *where* $s_k = (\omega^k, A_p)$ *with* k, p $\in$ P, k $\neq$ p. *We say that* s *is a possible solution of* Γ *if and only if:*

$$E_1[A_2](\omega^1) = E_1[A_2](y^1) \text{ for any } y^1 \in Y_1,$$

$$E_2[A_1](\omega^2) = E_2[A_1](y^2) \text{ for all } y^2 \in Y_2.$$

Let $s^* = (s_1^*, s_2^*) \in W_1 \times W_2$, *with* $s_k^* = (\omega_*^k, A_p^*)$ p, $k = 1,2$, p $\neq$ k, *we say that* s* *is a possible cooperative solution of the gam* Γ *if and only if* s* *is a possible solution of* Γ *and* A_p^* *is the characteristic function of the singleton set* $\{\omega_*^p\}$, (p = 1,2).

THEOREM 3.3.([1], [2]).

Let Γ *be a given 2-FG and* $s^* = (s_1^*, s_2^*) \in W_1 \times W_2$ *with* $s_k = (\omega_*^k, A_p^*)$ $(k, p = 1, 2, p \neq k)$ *where* A_p^* *is the characteristic function of the singleton set* $\{\omega_*^p\}$ $(p = 1,2)$. *The following two assertions are equivalent:*

1. s* *is a possible cooperative solution of* Γ,

2. *the pair of vectors* (ω_*^1, ω_*^2) *is an well-fixed point of the fuzzy set-valued mapping* R^* *from* R^n *into* R^n *with* $n = n(1) + n(2)$ *defined by:*

(3. 1) $$R^*(\omega, \omega) =$$

$$Max\{E_1(\omega^2, \overline{\omega}^1), E_1(\omega^2, \omega^1). E_1(\overline{\omega}^2, \overline{\omega}^1)\}. Max\{E_2(\omega^1, \overline{\omega}^2),$$
$$E_2(\omega^1, \omega^2). E_2(\overline{\omega}^1, \overline{\omega}^2)\}, \forall\ \omega, \overline{\omega} \in R^{n(1)} \times R^{n(2)} = R^n$$

THEOREM 3.4.

Let Γ *be a 2-FG and* R^* *be defined by* (3.1). *If Dom* R^* *is a nonempty convex compact subset and R satisfies all the hypotheses in Proposition* 3.1 *with* $X = R^n$ *and* $Q = Dom\ R^*$. *Then* Γ *has a possible cooperative solution, where Dom* R^* *is the set*

$$Dom\ R^* = \{x \in R^n \mid \exists\ y \in R^n \text{ such that } R^*(x, y) > 0\}$$

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