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INTEGRABILITY OF POWER SERIES WITH QUASI-MONOTONE COEFFICIENTS*

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In the present note a generalization of a theorem of Jain [1] is obtained.

1. A non-decreasing, continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin is called an Orlicz function (OF). Function $\Phi \in$ OF is said to satisfy Δ_2 − condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that

$$\Phi(2u) \leq C\ \Phi(u), \text{ for } u \geq u_0.$$

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone if for some $\alpha > 0$,

$$a_{n+1} \leq a_n\left(1 + \frac{\alpha}{n}\right).$$

An equivalent definition of quasi-monotone sequence is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

We write

$$F(x) = \Sigma a_k x^k \ (0 \leq x < 1);$$

and

$$A_n = a_0 + a_1 + \dots\dots + a_n.$$

2. We take our start from the work of Woyezynski [4] who proved the following theorem.

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* This is based on Chapter III of the author's Ph. D. Thesis entitled «Integrability of function represented by Trigonometric series» submitted under the supervision of Dr. Z. U. Ahmad at Aligarh Muslim University, Aligarh in 1980.

THEOREM A. If, $a_n \geq a_{n+1} \geq 0$ $(n = 1, 2, \ldots\ldots)$ *then* $F(x) \in L_\Phi(0,1)$, *if and only if*

$$\{A_n\} \in L_\Phi(N, V),$$

where Φ *is an Orlicz function, satisfying* Δ_2 — *condition,* $d\mu = dx$, N *stands for the set of all positive integers and* V *is the measure of* N *concentrating the mass* n^{-2} *at all point* $n \in N$.

Recently, Jain [1] generalized Theorem A by replacing the condition $a_n \downarrow 0$ by the less stringent condition that $\{a_n\}$ is quasi-monotone in the following form.

THEOREM B. *Let* Φ *be an Orlicz function satisfying* Δ_2 — *condition. If* $\{a_n\}$ *is quasi-monotone sequence such that* $0 < B_1 \leqq n^\beta a_n \leqq B_2$ *with some* $\beta > 0$, $(n = 1,2, \ldots\ldots)$ *then, for* $0 \leqq v < 1$.

$$(1 - x)^{-v}\, \Phi(\,|\,F(x)\,|\,) \in L\,(0,1).$$

if and only if

$$\sum_{n=1}^{\infty} n^{v-2}\, \Phi(A_n) < \infty.$$

We prove the following theorem.

THEOREM. *Let* Φ *be an Orlicz function satisfying* Δ_2 — *condition, and let* $\lambda(t)$ *be a positive, non-increasing, integrable function on the interval* $0 < t \leqslant 1$. *If* $\{a_n\}$ *is quasimonotone sequence such that* $0 < B_1 \leqslant n^\beta a_n \leqslant B_2$ *with some* $\beta > 0$, $(n = 1, 2, \ldots)$, *then we have*

$$\lambda(1 - x)\, \Phi\,(\,|\,F(x)\,|\,) \in L(0,\,1),$$

if and only if

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\, \Phi\,(A_n) < \infty\,.$$

In the case when $\lambda(t) = t^{-v}$ $(v < 1)$, our theorem reduces to Theorem B.

4. Proof of the theorem. Set $y = 1 - x$. Since, $\left(1 - \dfrac{1}{n}\right)^n$ is an increasing sequence, we have, for $\dfrac{1}{n+1} \leqslant y \leqslant \dfrac{1}{n}$, $n \geqq 2$:

$$F(1 - y) \leqslant \sum_{k=0}^{n} a_k\,(1 - y)^k$$

$$\leqslant \left(1 - \frac{1}{n}\right)^n \sum_{k=0}^{n} a_k$$

$$\leqslant \frac{1}{4} A_n\,.$$

Thus, we get

$$F(1-y) > \frac{1}{4} A_n, \text{ for } \frac{1}{n+1} \leqslant y \leqslant \frac{1}{n}, n \geqq 2.$$

Now,

$$\sum_{n=1}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2}\, \Phi(A_n) \leqslant 2 \sum_{1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\, \Phi(A_n)\, dy$$

$$\leqslant 2 \int_{\frac{1}{2}}^{1} \lambda(y)\, \Phi(A_1)\, dy + 2 \sum_{n=2}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\, \Phi(A_n)\, dy$$

$$\leqslant 0(1) + B \sum_{n=2}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \lambda(y)\, \Phi(F(1-y))\, dy$$

$$\leqslant 0(1) + B \int_{0}^{1} \lambda(1-x)\, \Phi(F(x))\, dx < \infty,$$

by hypothesis.

Conversely, we have

$$\int_{0}^{1} \lambda(1-x)\, \Phi(F(x))dx = \sum_{n=2}^{\infty} \int_{1-\frac{1}{(n+1)}}^{1-\frac{1}{n}} \lambda(1-x)\, \Phi(F(x))dx.$$

$$= \sum_{n=2}^{\infty} \int_{\frac{1}{n}}^{\frac{1}{(n+1)}} \lambda(x)\, \Phi(F(1-x))\, dx$$

$$= \sum_{n=2}^{\infty} \int_{\frac{1}{n}}^{\frac{1}{(n-1)}} \lambda(x)\Phi\left(\sum_{k=0}^{\infty} a_k (1-x)^k\right) dx$$

$$= \sum_{n=2}^{\infty} \int_{\frac{1}{n}}^{\frac{1}{(n-1)}} \lambda(x)\, \Phi\left(\sum_{k=0}^{\infty} a_k \left(1-\frac{1}{n}\right)^k\right) dx$$

$$= O(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n}\right)^k \right)$$

$$= O(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j \left(1 - \frac{1}{n}\right)^j \right)$$

$$= (1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} \left(\sum_{j=0}^{n} a_j + \sum_{nk}^{n(k+1)} a_j \right)\right)$$

$$= O(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} \left(A_n + Bn^{1-\beta}(k+1)\right)\right)$$

$$= O(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi\left(\sum_{k=0}^{\infty} B(k+2)\, e^{-k} A_n \right)$$

$$= O(1) \sum_{n=2}^{\infty} \lambda\left(\frac{1}{n}\right) n^{-2} \Phi(A_n)$$

$$< \infty,$$

by hypothesis.

This completes the proof of the theorem.

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