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ON THE CONVERGENCE OF POINT PROCESSES IN A MODEL OF NO SPACE – TIME CLUSTERING

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Necessary and sufficient conditions are given for the weak convergence of point processes to the Poisson processs under the hypothesis of no space — time clustering.

1. INTRODUCTION

Let us consider n observations (S_i, T_i) $(i = 1, 2, \ldots, n)$ of some random phenomenon where $S_i \in R^2$ and $T_i \in R^1$ is the space and the time coordinate of the i — th observation, respectively,

One is interested in detecting space — time clustering. Presumably, if there is some space — time clustering, points in a cluster will be close both in space and time, while unrelated points will tend to have a larger average separation in space and time.

We define the hypothesis of no clustering to be equivalent to the one that the coordinates in time are matched at random with the coordinates in space, there being a total of n! equiprobable sets of maichings. Thus, one can consider the space coordinates as fixed, while the time coordinates are random variables T_i $(i = 1, 2, \ldots, n)$ with the uniform distribution on the probability space Д of all permutations of the numbers $(1, 2, \ldots n,)$.

In epidemiological application, e.g., when n cases of some disease are considered, the existence of space — time clustering is regarded as evidence for contagion of the disease.. The idèas of Knox and some results of other authors for detecting clustering of patients are summarized in [1].

Throughout this paper, our problems are considered under the hypothesis of no elustering.

Now, for fixed n, let N_n denote the point process in R^3 with realizations defined as follows:

$$N_n(\pi) = \sum_{i=1}^{n} \varepsilon_{(S_i, T_{\pi(i)})} \quad \text{for all } \pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in \Pi$$

where $\varepsilon(ST)$ denotes the unit mass placed at the point (S,T) The author (c.f. [2]) earlier gaved necessary and sufficient conditions for the convergence of law of N_n $(A_n \times B_n)$ $(A_n \subset R^2$ $B_n \subset R_1$ to the degenerate, Poissonian or binomial law.

K. Krickeberg posed the more general question: Can N_n converge weakly to a Poisson process as n tends to infinity? In this paper, the problem is solved in the affimative.

2. THE MAIN RESULT

THEOREM: *The sequence of point processes N_n $(n = 1, 2, \ldots)$ converges weakly to the Poisson process with intensity measure λ in R^3 if and only if for each bounded half open cuboid $\prod_{L=1}^{3} [\alpha_i, \beta_i)$ having the property that*

$\lambda(x_i = \alpha_i) = 0 = \lambda(x_i = \beta_i)$ $(i = 1, 2, 3)$ the following convergence relation is satisfied:

$$\frac{N_n^1(A) N_n^2(B)}{n} \rightarrow \lambda(A \times B) \text{ and, moreover, for } \lambda(A \times B) > 0, \text{ we have } N_n^1(A) \rightarrow$$

$$\rightarrow \infty \text{ and } N_n^2(B) \rightarrow \infty \; (n \uparrow \infty)$$

where $A = \prod_{i=1}^{2} [\alpha_i, \beta_i)$, $B = [\alpha_3, \beta_3)$, $N_n^1(A) = \sum_{i=1}^{n} 1_A(S_i)$, $N_n^2(B) = \sum_{i=1}^{n} 1_B(I_i)$.

First, we prove some Lemmas.

Let us consider any finite sequence of pairwise disjoint bounded half-open cuboids $A_S \times B_S$ $(s = 1, 2, \dots, k)$.

For all $i', j' \in C = \{\text{all subsets of } (1, 2, \dots, k)\}$, denote

$$A^*_{i'} = A_{i'} - \bigcup_{i''\in C, i''\supset i'} A_{i''} \ (A_{i'} = \bigcap_{S\in i'} A_s),\ B^*_{j'} = B_{j'} - \bigcup_{j''\in C, j''\supset j'} B_{j''} \ (B_{j'} = \bigcap_{S\in j'} B_s).$$

Then, for $I = \{ i' : i'\in C,\ A^*_{i'} \neq \phi \}$ and $j = \{ j' : j'\in C,\ B^*_{j'} \neq \phi \}$,

$\{ A^*_{i'} \}_{i'\in I}$ and $\{ B^*_{j'} \}_{j'\in J}$ constitutes a decomposition of $\bigcup_{S=1}^{k} A_s$ and $\bigcup_{S=1}^{k} B_s$, respectively.

Thus, for all non – negative integers L_s $(s = 1, 2, \dots, k)$,

$$(1)\ P\left\{ N_n(A_s \times B_s) = L_s,\ s = 1, 2, \dots, k \right\} = \sum_{p,q} \left\{ \prod_{i'\in I} \begin{bmatrix} N^1_n(A^*_{i'}) \\ \sum_{s\in i'} p_{si'} \end{bmatrix} \frac{(\sum_{s\in i'} p_{si'})!}{\prod_{s\in i'} p_{si'}!} \times \right.$$

$$\times \prod_{j'\in J} \begin{bmatrix} N^2_n(B^*_{j'}) \\ \sum_{s\in j'} q_{sj'} \end{bmatrix} \cdot \frac{(\sum_{s\in j'} q_{sj'})!}{\prod_{s\in j'} q_{sj'}!} \times \prod_{s=1}^{k} L_s! \times \sum_{v} \left[\prod_{i'\in I} \begin{bmatrix} N^1_n(A^*_{i'}) - \sum_{s\in i'} p_{si'} \\ \sum_{j\in J_{i'}} r_{i'j'} \end{bmatrix} \cdot \frac{(\sum_{i'\in J_{i'}} r_{i'j'})!}{\prod_{j'\in J_{i'}} r_{i'j'}!} \times \right.$$

$$\times \prod_{j'\in J} \begin{bmatrix} N^2_n(B^*_{j'}) - \sum_{s\in j'} q_{sj'} \\ \sum_{i'\in I_{i'}} r_{i'j'} \end{bmatrix} \cdot (\sum_{i'\in I_{j'}} r_{i'j'})! \times \begin{pmatrix} n - \sum_{i'\in I} N^1_n(A^*_{i'}) \\ \sum_{j'\in J} N^2_n(B^*_{j'}) - \sum_{s=1}^{k} L_s - \sum_{i'\in I, j'\in J_{i'}} r_{i'j'} \end{pmatrix}$$

$$\left.\left. \times \left(\sum_{j\in J} N^2_n(B^*_{j'}) - \sum_{s=1}^{k} L_s - \sum_{i'\in I, j'\in j_{i'}} r_{i'j'} \right)! \right] \right\} \cdot \left(n - \sum_{j'\in J} N^2_n(B^*_{j'}) \right)!/n!$$

where $\Sigma_{p,q}$ is over all sets of non-negative integers $p_{s_{i'}}$ and $q_{s_{i'}}$ $(s = 1, 2, \dots, k;$ $i' \in I,\ j' \in J)$ for which

$$\sum_{i'\in I^s} p_{s_{i'}} = L_s\ (I^s = \{ i' \cdot i' \in I,\ s \in i' \}),\ \sum_{s\in i'} p_{s_{i'}} \leqslant N^1_n\ (A^*_{i'})$$

and $\sum_{j'\in J^s} q_{s_{i'}} = L_s (J' = \{j' : j' \in J, s \in j'\}), \sum_{s\in j'} p_{s_j} \leqslant N_n^2 (B_j^*)$, respectively.

$I_{j'} = \{ i' : i' \in I, s \neq t \text{ for all } s \in i', t \in j' \}$ for all $j' \in J$,

$J_{i'} = \{ j' : j' \in J, t \neq s \text{ for all } i' \in I, s \in i' \}$ for all $i' \in I$

$\sum_v$ is over all sets of non-negative integers $r_{i'j'}$ ($i' \in I$, $j' \in J_{i'}$) for which

$$\sum_{i'\in I_{j'}} r_{i'j'} \leqslant N_n^1 (A_{i'}^*) - \sum_{i'\in I_{j'}} p_{s_i} \text{ and } \sum r_{i'j'} \leqslant N_n^2 (B_{j'}^*) - \sum_{s\in j'} q_{s_{i'}}.$$

By denoting

$$E_1(n) = \left[\prod_{s=1}^{k} \binom{N_n^1 (A_s)}{L_s} \binom{N_n^2 (B_s)}{L_s} L_s! \Bigg/ \left[\binom{n}{\sum_{s=1}^{k} L_s} \left(\sum_{s=1}^{k} L_s \right)! \right] \right],$$

$$E_2(n, p, q, s) = \left[\prod_{i' \in I^s} \binom{N_n^1 (A_{i'}^*)}{p_{s_{i'}}} \times \prod_{j'\in J^s} \binom{N_n^2 (B_{j'}^*)}{q_{s_{j'}}} \right] \Bigg/ \left[\binom{N_n^1 (A_s)}{L_s} \binom{N_n^2 (B_s)}{L_s} \right]$$

$$E_3(n,p,i') = \left[\binom{N_n^1 (A_{i'}^*)}{\sum_{s\in i'} P_{s_{i'}}} \left(\sum_{s\in i'} P_{s_{i'}}\right)! \right] \Bigg/ \left[\prod_{s\in i'} \binom{N_n^1 (A_{i'}^*)}{P_{s_{i'}}} P_{s_{i'}}! \right],$$

$$E_4(n, q, i') = \left[\binom{N_n^2 (B_{j'}^*)}{\sum_{s\in i'} q_{s_{i'}}} \left(\sum_{s\in j'} q_{s_{j'}}\right)! \right] \Bigg/ \left[\prod_{s\in i'} \binom{N_n^2 (B_{j'}^*)}{q_{s_{j'}}} q_{s_{j'}}! \right],$$

$$E_5(n, r, p, q) = \frac{\left[\prod_{i'\in i, j'\in J_{i'}} \binom{N_n^1 (A_{i'}^*) - \sum_{s\in i'} P_{si'}}{r_{i'j'}} \binom{N_n^2 (B_{j'}^*) - \sum_{s\in i'} q_{s_{j'}}}{r_{i'j'}} r_{i'j'}! \right]}{\left[\binom{N - \sum_{s\times 1}^{k} L_s}{\sum_{i'\in i, j'\in J_{i'}} r_{i'j'}} \left(\sum_{i'\in i, j\in J_{i'}} r_{i'j'} \right)! \right]},$$

$$E_6\,(n,\, r,\, p,\, i') = \frac{\left[\binom{N_n^1\,(A^*_{i'}) - \sum\limits_{S\in i'} p_{si'}}{\sum\limits_{i'\in J_{i'}}} \left(\sum\limits_{j\in J_{i'}} r_{i'j'}\right)!\right]}{\left[\prod\limits_{j'\in J_{i'}} \binom{N_n^1\,(A^*_{i'}) - \sum\limits_{S\in i'} p_{si'}}{r_{i'j'}} r_{i'j'}\,!\right]}$$

$$E_7\,(n,\, r,\, y,\, j') = \frac{\left[\binom{N_2^1\,(B^*_{j'}) - \sum\limits_{S\in j'} q_{sj'}}{\sum\limits_{i'\in I_{j'}} r_{i'j'}} \left(\sum\limits_{i'\in I_{j'}} r_{i'j'}\right)!\right]}{\left[\prod\limits_{i'\in I_{j'}} \binom{N_n^2\,(B^*_{j'}) - \sum\limits_{S\in j'} q_{sj'}}{r_{i'j'}} r_{i'j'}\,!\right]}$$

$$E\,(n,\, r) = \frac{\binom{n - \sum\limits_{i'\in I} N_n^1\,(A^*_{i'})}{\sum\limits_{i'\in J} N_n^2\,(B^*_{j'}) - \sum\limits_{S=1}^{k} L_S \sum\limits_{i'\in I,\, j'\in I_{i'}} r_{i'j'}}}{\binom{n - \sum\limits_{S=1}^{k} L_S - \sum\limits_{i'\in I,\, j'\in J_{i'}} r_{i'j'}}{\sum\limits_{j'\in J} N_n^2\,(B^*_{j'}) - \sum\limits_{S=1}^{k} L_S - \sum\limits_{i'\in I,\, j'\in J_{i'}} r_{i'j'}}}$$

and rearranging terms of (1) we obtain:

LEMMA 1: $P\ \{ N_n\,(A_s \times B_s) = L_s\,,\, s = 1,2,\ldots k\} = \sum\limits_{p,q} \{E_1\,(n) \times \prod\limits_{S=i}^{k} E_2\,(n,\, p,\, q,\, s) \times$

$\times \prod\limits_{i'\in I} E_3\,(n,\, p,\, i') \times \prod\limits_{j'\in J} E_4\,(n,\, q,\, j') \times \sum\limits_{r} [\, E_5\,(n,\, r,\, p,\, q) \times \prod\limits_{i'\in I} E_6\,(n,\, r,\, p,\, i') \times$

$\times \prod\limits_{j'\in J} E_7\,(n,\, r,\, q,\, j') \times E_8\,(n,\, r)\,]\}.$

For $A_s \times B_s$ (s $\times$ 1,2,..., k) satisfying the convergence relation in the Theorem, we shall prove the

LEMMA 2: $E_1(n) \times \prod_{s=1}^{k} E_2(n, p, q, s) \times \prod_{i \in J} E_3(n, p, i') \times \prod_{j' \in J} E_4(n, q, j')$

$$\rightarrow \prod_{s=1}^{k} \frac{[\lambda(A_s \times B_s)]^{L_s}}{L_s!} \prod_{s \in (1,2,\dots,k) : \lambda(A_s \times B_s) > 0} \left[\frac{L_s}{\prod_{i' \in I} p_{s_{i'}}!} \cdot \prod (\alpha_{s_{i'}})^{p_{s_{i'}}}\right]$$

$$\left[\frac{L_s!}{\prod_{j' \ni J^s} p_{s_{j'}}!} \cdot \prod_{j' \in J^s} (\beta_{s_{i'}})^{q_{s_{j'}}}\right] \qquad n \rightarrow \infty$$

where $\alpha_{s_{i'}} = \lim_{n\rightarrow\infty} N_n^1(A_{j'}^*)/N_n^1(A_s)$, $\beta_{s_{j'}} = \lim_{n\rightarrow\infty} N_n^2(B_{j'}^*)/N_n^2(B_s)$ *and for convenience,* $0^0 = 1$.

LEMMA 2: $\sum_r [E_5(n, r, p, q) \times \prod_{i' \in I} E_6(n, r, p, i') \times \prod_{j' \in J} E_7(n, r, q, j') \times$

$\times E_8(n, r)] \rightarrow e(-\sum_{s=1}^{k} \lambda(A_s \times B_s))(n\rightarrow\infty)$.

Proof of Lemma 2: Firstly, by denoting that $L_o = 0$,

$$E_1(n) = \begin{cases} 1 & \text{if } \sum_{s=1}^{k} L_s = 0 \\ \prod_{s \in 1,2,\dots,k): L_s \geqslant 1} (1/L_s!) \prod_{s=0}^{L_s - 1} \dfrac{[N_n^1(A_s) - i][N_n^2(B_s) - i]}{n - \sum_{t-1}^{s-1} L_t - i} & \text{if } \sum_{s=1}^{k} L_s \geqslant 1 \end{cases} \rightarrow$$

$$\longrightarrow \prod_{s=1}^{k} [\lambda(A_s \times B_s)]^{L_s} / L_s!.$$

Secondly, by induction,

$$E_2(n,p,q,s) \rightarrow \frac{L_s!}{\prod_{i' \in I^s} p_{s_{i'}}!} \cdot \prod_{i' \in I^s} (\alpha_{s_{i'}})^{p_{si'}} \times \frac{L_s!}{\prod_{j' \in J^s} q_{s_{j'}}!} \cdot \prod_{j \in J^s} (\beta_{s_{j'}})^{q_{s_{j'}}} \qquad \text{if } \lambda(A_s \times B_s) > 0$$

(where it is obvious that $\lambda(A_s \times B_s) > 0$ implies

$N_N^1(A_s) \to \infty, N_n^2(B_s) \to \infty$ and the existence of the limits $\alpha_{s_{i'}}, \beta_{s_{j'}}$),

$$E_3(n, p, i') \to 1 \text{ if } N_n^1(A^*_{i'}) \to N^1(A^*_{i'}) = \infty \text{ and}$$

$$E_4(n, q, j') \to 1 \text{ if } N_n^1(B^*_{j'}) \to N^2(B^*_{j'}) = \infty.$$

Therefore, if at least one of the conditions:

(i) $\exists\, s \in (1, 2, \ldots, k): L_s \geqslant 1, \lambda(A_s \times B_s) = 0,$

(ii) $\exists\, s \in (1, 2, \ldots, k), \exists\, i' \in I^s: p_{s_{i'}} \geqslant 1, \lambda(A_s \times B_s) > 0, \alpha_{s_{i'}} = 0,$

(iii) $\exists\, s \in (1, 2, \ldots, k), \exists\, j' \in J^s: q_{s_{i'}} \geqslant 1, \lambda(A_s \times B_s) > 0, \beta_{s_{j'}} = 1$

is satisfied, then

$$E_1(n) \times \prod_{s=1}^{k} E_2(n, p, q, s) \times \prod_{i' \in I} E_3(n, p, i') \times \prod_{j' \in J} E_4(n, q, j') \to 0.$$

If (i) — (iii) are not satisfied, then

$$\prod_{s \in (1,2,\ldots,k):(A_s \times B_s) = 0} E_2(n, p, q, s) \times \prod_{i' \in I: N^1(A_{i'}) < \infty} E_3(n, p, i') \times \prod_{j' \in J: N^2(B^*_{j'}) < \infty} E_4(n, q, j') = 1 \text{ because}$$

all $p_{si'}$, $q_{sj'}$ were equal to 0, therefore

$$E_1(n) \times \prod_{s=1}^{k} E_2(n, p, q, s) \times \prod_{i' \in I} E_3(n, p, i') \times \prod_{j' \in J} E_4(n, q, j') \to$$

$$\prod_{s \in (1,2,\ldots,k): \lambda(A_s \times B_s) > 0} \left\{ \frac{[\lambda(A_s \times B_s)]^{L_s}}{L_s!} \times \frac{L_s!}{\prod_{i' \in I^s: \alpha_{s_{i'}} > 0} p_{s_{i'}}!} \prod_{i' \in I^s: \alpha_{s_{i'}} > 0} (\alpha_{s_{i'}})^{p_{s_{i'}}} \times \right.$$

$$\left. \times \frac{L_s!}{\prod_{j' \in J^s: \beta_{s_{j'}} > 0} q_{s_{i'}}!} \cdot \prod_{j' \in J^s\, \beta_{s_{j'}} > 0} (\beta_{s_{j'}})^{q_{s_{j'}}} \right\}.$$

Thus Lemma 2 is proved.

Proof of Lemma 3: On the one hand in the same way as in the proof of Lemma 2,

$E_5(n, r, p, q), \to \prod_{i'\in I, j'\in J i'} [\lambda(A^*_{i'} \times B^*_{j'})]^{r_{i'j'}} / r_{i'j'}!$ where $\lambda(A^*_{i'} \times B^*_{j'}) =$

$= \lim_{n\to\infty} N^1_n(A^*_{i'}) N^1_n(B^*_{j'}) / n,$

$E_6(n, r, p, i') \to 1$ if $N^1(A^*_{i'}) = \infty.$

$E_7(n, r, q, j') \to 1$ if $N^2(B^*_{j'}) = \infty,$

and, by denoting $a_n = \sum_{i'\in I} N^1_n(A^*_{i'})$, $b_n = \sum_{j\in J} N^2_n(B^*_{j'})$, $L = \sum_{S=1} L_s +$

$\sum_{i'\in I, j'\in J i'} r_{i'j'},$

$$E_8(n, r) = \begin{cases} \prod_{i=L}^{b_n - 1} \left(1 - \frac{a_n - L}{n - i}\right) & \text{if } a_n > L,\ b_n > L \\ 1 & \text{otherwise} \end{cases}$$

$$E_8(n, v) \to e\left(-\lim_{n\to\infty} a_n b_n / n\right) = e\left(- \sum_{i'\in I, j\in J i'} \lambda(A^*_{i'} \times B^*_{j'})\right)$$

because $\prod_{i=L}^{b_{n-1}} \left(1 - \frac{a_n - L}{n - i}\right)$ is bounded from below and above by

the sequence $\left(1 - \frac{a_n - L}{n - b_{n+1}}\right)^{b_n - L - 1}$ and $\left(1 - \frac{a_n - L}{n - L}\right)^{b_n - L - 1}$,

respectively.

which have the same limit $e(-\lim_{n\to\infty} a_n b_n / a)$.

Thus, new line $E(n,r,p,y) = E_5(n,r,p,q) \times \prod_{i'\in I} E_6(n,r,q,i') \times \prod_{j'\in J} E_7(n,v,q,j') \times$

$\times E_8(n,v).$

$$\to \begin{cases} 0 \text{ if } \exists\, i' \in I, \exists\, j' \in J_{i'} : r_{i'j'} \geqslant 1,\ \lambda(A^*_{i'} \times B^*_{j'}) = 0 \\ e\left(- \sum_{i'\in I, j'\in J_{i'}} \lambda(A^*_{i'} \times B^*_{j'})\right) \cdot \prod_{i'\in I, j'\in J_{i'} : \lambda(A^*_{i'} \times B^*_{j'}) > 0} [\lambda(A^*_{i'} B^*_{j'})]^{v_{i'j'}} / r_{i'j'}! \text{ otherwise.} \end{cases}$$

On the other hand, since $E_6(n, r, p, i') \leqslant 1$, $E_7(n, r, q, j') \leqslant 1$ and $E_8(n, r) \leqslant 1$, we have $E(n, r, p, q) \leqslant E_5(n, r, p, q) =$

$$=\begin{cases} 1 \text{ if} & \sum\limits_{\substack{i'\in I,\, j'\in J_{i'}}} r_{i'j'} = 0 \\ \left[\prod\limits_{i'\in I,\, j'\in J_{i'}:\, r_{i'j'}\geqslant 1} (1/\nu_{i'j'}!) \prod\limits_{t=0}^{r_{i'j'}-1} \left(N_n^1(A_{i'}^*) - \sum\limits_{s\in i'} Ps_{i'} - t\right)\left(N_n^2(B_{j'}^*) - \sum\limits_{s\in j'} q_{s\,j'} - t\right)\right] \Big/ \prod\limits_{t=0}^{\sum\limits_{i'\in I,\, j'\in J_{i'}} r_{i'j'}-1} \left(n - \sum\limits_{s=1}^{k} L_s - t\right) & \text{otherwise} \end{cases}$$

$$\leqslant \prod_{i'\in I,\, j'\in J_{i'}} \left(\frac{1}{r_{i'j'}!}\right)\left[\frac{N_n^1(A_{i'}^*)N_n^2(B_{j'}^*)}{n - \min\left(\sum\limits_{i'\in I} N_n^1(A_{i'}^*),\ \sum\limits_{j'\in J} N_n^2(B_{j'}^*)\right) + 1}\right]^{\nu_{i'j'}} \leqslant \prod_{i'\in I,\, j'\in J_{i'}} (A_{i'j'})^{\nu_{i'j'}} \Big/ r_{i'j'}!$$

where $A_{i'j'}$ $(i' \in I,\ j' \in J_{i'})$ is an upper-bound of the sequence in the last bracket above having the limit: $\lambda(A_{i'}^* \times B_{j'}^*)$.

Thus, if $E(n, r, p, q)$ is regarded to be equal to 0 for all sets $r_{i'j'}$ $(i' \in I,\ j' \in J_i)$ not contained in $\sum\limits_{\nu}$, the series $\sum\limits_{\nu}^{*} E(n, r, p, q)$ is majorized by the convergent sequence

$$\sum_{r}^{*} \prod_{i' \in I,\, j' \in J_{i'}} (A_{i'j'})^{r_{i'j'}} \Big/ r_{i'j'}!$$ where Σ^* is over all sets of non-negative integers $r_{i'j'}$ $(i' \in I,\ j' \in J_{i'})$.

Consequently,

$$\lim_{n\to\infty} \sum_{\nu} E(n, r, p, q) = \sum_{\nu}^{*} \lim_{n\to\infty} E(n, r, p, q)$$

$$= e\left(-\sum_{s=1}^{k} \sum_{i'\in I^s,\, j'\in J^s} \lambda(A_{i'}^* \times B_{j'}^*)\right) \cdot \prod_{i'\in I,\, j'\in J_{i'}:\, \lambda(A_{i'}^*\times B_{j'}^*)>0}$$

$$\cdot \sum_{\nu_{i'j'}=0}^{\infty} e(-\lambda(A_{i'}^* \times B_{j'}^*))[\lambda(A_{j'}^*)]^{r_{i'j'}} \Big/ \nu_{i'j'} = e\left(-\sum_{s=1}^{k} \lambda(A_s \times B_s)\right).$$

because $\sum\limits_{i'\in I^s,\, j'\in J^s} \lambda(A_{i'}^* \times B_{j'}^*) = \lambda(A_s \times B_s)\ (s = 1, 2, \ldots, k)$.

Thus, Lemma 3 is proved.

LEMMA 4: *If for a bounded half-open cuboid* $A \times B$,

$\frac{N_n^1(A) N_n^2(B)}{n} \to \lambda (0 < \lambda < \infty$ *and either* $N_n^1(A) \to N < \infty$ *or* $N_n^2(B) \to N < \infty (n \to \infty)$ *is satisfied, then* $N_n(A \times B)$ *has an asymptotic binomial law of order* N *with parameter* λ / N.

Proof: As a special case of Lemma 1, for k = 1, we have

$$P(N_n(A \times B) = L) = \frac{\binom{N_n^1(A)}{L}\binom{N_n^2(B)}{L} L!}{\binom{n}{L} L!} \cdot \left[\binom{n - N_n^1(A)}{N_n^2(B) - N} \Big/ \binom{n - L}{N_n^2(B) - N} \right].$$

On the one hand,

$$\frac{\binom{N_n^1(A)}{L}\binom{N_n^2(B)}{L} L!}{\binom{n}{L} L!} = \begin{cases} 1 & \text{if } L = 0 \\ (1/L!) . \prod_{i=0}^{L-1} \frac{[N_n^1(A) - i][N_n^2(B) - i]}{n - i} & \text{if } L \geqslant 1 \end{cases}$$

$$\to \binom{N}{L}\left(\frac{\lambda}{N}\right)^L$$

because $\frac{[N_n^1(A) - i][N_n^2(B) - i]}{n - i} \to \frac{\lambda}{N}(N - i) \qquad (i = 0, 1, 2, \ldots, L - 1).$

On the other hand,

$$\binom{n - N_n^1(A)}{N_n^2(B) - L} \Big/ \binom{n - L}{N_n^2(B) - L} = \begin{cases} \prod_{i=L}^{N_n^1(A) - 1} \left(1 - \frac{N_n^2(B) - L}{n - i}\right) = \\ = \prod_{i=L}^{N_n^2(B) - 1} \left(1 - \frac{N_n^1(A) - L}{n - i}\right) \text{ if } N_n^1(A) > L, N_n^2(B) > L. \\ 1 \text{ otherwise.} \end{cases}$$

$$\to \left(1 - \frac{\lambda}{Y}\right)^{N - L}.$$

Thus, $P(N_n(A \times B) = L \to \binom{N}{L}\left(\frac{\lambda}{N}\right)^L \left(1 - \frac{\lambda}{N}\right)^{N-L}$ for $L = 0, 1, \ldots, N$

which completes the proof of Lemma 4.

Proof of the Theorem. Assume that the convergence relation in the Theorem is satisfied, then by Lemmas 1, 2, and 3

$$P\{N_n(A_s \times B_s) = L_s,\ s = 1,2,\dots,k\} \longrightarrow e\left(-\sum_{s=1}^{k} \lambda(A_s \times B_s)\right) \cdot$$

$$\times \prod_{s=1}^{k} [\lambda(A_s \times B_s)]^{L_s} \Big/ L_s! \times \prod_{s\in(1,2,\dots,k):\lambda(A_s\times B_s)>0}$$

$$\left[\sum_{\substack{\Sigma P_{s_{i'}} = L_s \\ i'\in I^s:\alpha_{s_{i'}}>0}} \cdot \frac{L_s}{\prod_{i'\in I^s:\alpha_{s_{i'}}>0} p_{s_{i'}}!} \cdot \prod_{i'\in I^s:\alpha_{s_{i'}}>0} (\alpha_{s_{i'}})^{p_{s_{i'}}}\right]$$

$$\cdot \left[\sum_{\substack{\Sigma q_{s_{j'}} = L_s \\ i'\in J^s:\beta_{s_{i'}}>0}} \cdot \frac{L_s}{\prod_{j'\in J^s:\beta_{s_{i'}}>0} q_{s_{j'}}!} \cdot \prod_{j'\in J^s:\beta_{s_{j'}}>0} (\beta_{s_{i'}})^{q_{s_{j'}}}\right]$$

thus,

$$(2)\quad P\{N_n(A_s \times B_s) = L_s,\ s = 1,2,\dots,k\} \to e\left(\sum_{s=1}^{k} \lambda(A_s \times B_s)\right)$$

$$\prod_{s=1}^{k} [\lambda(A_s \times B_s)]^{L_s} \Big/ L_s!$$

for each finite sequence of pairwise disjoint bounded cuboids

$A_s \times B_s = \prod_{i-1}^{3} [\alpha_{is}, \beta_{is})$ (s = 1,2,..., k) having the property

$\lambda(x_i = \alpha_{is}) = 0 = \lambda(x_i = \beta_{is})$ (i = r, 2, 3, s = 1, 2,..., k)

and all non-negative integers L_s (s = 1, 2,..., k).

Conversely, assume that (2) is satisfied, then we shall prove that the convergence relation in the theorem is also true.

Indeed, suppose that the convergence relation is not true for some $A \times B$, then one of conditions:

(i) there exists a subsequence $\dfrac{N^1_{n_k}(A)N^2_{n_k}(B)}{n_k}$ of $\dfrac{N^1_n(A)N^2_n(B)}{n}$ having the limit $\lambda^*(A \times B) \neq \lambda(A \times B)$

(ii) for $\dfrac{N^1_n(A)N^2_n(B)}{n} \to \lambda(A \times B) > 0$, the sequence $N^1_n(A)$ (or $N^2_n(B)$) has a finite limit

is satisfied.

If (i) is true, using the above proof for $k = 1$ and Lemma 4, $N_{n_k}(A \times B)$ has either an asymptotic Poisson distribution with parameter $\lambda^*(A \times B)$ or an asymptotic binomial distribution which contradicts the asumption that $N_{n_k}(A \times B)$ has the asymptotic Poisson distribution with parameter $\lambda(A \times B)$.

If (ii) is true, e.g., $N_n(A) \to N < \infty$, then again using Lemma 4, $N_n(A \times B)$ has an asymptotic binomial distribution which is contrary to the asumption.

Thus, we have proved the equivalence of the convergence relation in the Theorem to (2).

By using the well-known Theorem on weak convergence of distributions in the d-dimensional Euclidean space (cf. [3], p.159) our Theorem follows.

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