ACTA MATHEMATICA VIETNAMICA
Vol.7 No2 (1982)

MULTIVALUED QUASI-MARTINGALES AND UNIFORM AMARTS

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INTRODUCTION.

Vector-valued asymptotic martingales (amarts) and multifunctions have been extentively studied in recent year by Ronnov [26], Chatterji [5], Uhl [26], Rao [22], Bellow [2], Luu [15], Aumann [1], Debreu [8], Rockafellar [25], Himmelberg [13], Castaing and Valadier [4], among others. The main purpose of this paper is to extend some results in [2] and [22] to multi-valued quasi-martingales and uniform amarts. For the terminology and fundamental properties of multivalued conditional expectations, the reader is referred to [11]. In Section 1 a brief summary of the notions of measurability, integrability and conditional expectations of multifunctions is given. In Section 2 we consider the class of multivalued quasi-martingales and prove some representation theorems for this class. It is, worth noting that from these results one could derive the main theorems in [16]. In Section 3 we give some characterizations of the class of multivalued uniform amarts. Finally, in Section 4 we discuss some applications of the previous results to the study of the Radon-Nikodym property (RNP) in Banach spaces.

1. MEASURABILITY, INTEGRALS AND CONDITIONAL EXPECTATIONS OF MULTIFUNCTIONS.

Throughout this paper **B** will denote a real separable Banach space, $(\Omega, \mathcal{A}, P)$ a probability space and $L_1(\Omega, \mathcal{A}, P, \mathbf{B}) = L_1(\mathbf{B}, \mathcal{A})$ the Banach space of all (equivalence classes of) Bochner integrable functions $f : \Omega \to \mathbf{B}$ with the norm

$$\| f \|_1 = E(\| f \|) = \int_\Omega \| f(\omega) \| \, dP.$$

We shall consider the class $\mathbb{K}$ of all closed bounded non-empty subsets of $\mathbf{B}$. For $X \in \mathbb{K}$, cl(X) denotes the colsure of X and $|X| = h(X, \{o\})$, where h is the Hausdorff metric in $\mathbb{K}$.

A multifunction $X: \Omega \to \mathbf{K}$ is called weakly measurable (briefly, measurable), if for every open subset V of $\mathbf{B}$ the set $\{\omega; X(\omega) \cap V \neq \phi\}$ is measurable. If this occurs, we write $X \in \mu(\mathbf{K}, \mathcal{A})$ and

$$S_X(\mathcal{B}) = \{f \in \mu(\mathbf{K}, \mathcal{B}); f(\omega) \in X(\omega), \text{ a.e.}\},$$

where $\mathcal{B}$ is a sub-6-field of $\mathcal{A}$. The following result is due to Castaing /3/:

LEMMA 1.1. *$X \in \mu(\mathbf{K}, \mathfrak{B})$ iff there is a sequence $< f_n > \subset S_X(\mathfrak{B})$ such that*

$$X(\omega) = cl\,\{f_n(\omega);\ n \geqslant 1\},\ a.\ e..$$

If this occurs X we write

$$X \overset{\|\cdot\|}{\longleftrightarrow} < f_n >_{n=1}^{\infty} \qquad (\text{w . r . t } \mathfrak{B})$$

A multifunction $X: \Omega \to \mathbf{K}$ is called integrably bounded if the real-valued function $\omega \to |X(\omega)|$ is integrable. If this occurs then we write $X \in L_1(\mathbf{K}, \mathcal{A})$. It is known that in this case $S_X(\mathcal{A})$ is closed in $L_1(\mathbf{B}, \mathcal{A})$ and the integral of X is defined by

$$\int_\Omega X\, d\,P = \{\int_\Omega f\ d\,P;\ f \in S_X(\mathcal{A})\}$$

where $\int f\ d\,P$ is the usual Bochner integral of f. This concept was introduced by Aumann |1| as a natural generalization of the Bochner integration of vector-valued functions. For $A \in \mathcal{A}$, $\int_A X$ dP is the integral of the restriction of X to A.

In connection with *Lemma 1 . 1*, if there is a sequence $< f_n >$ of $S_X(\mathfrak{B})$ such that $S_X(\mathfrak{B}) = cl\,\{f_n;\ n \geqslant 1\}$ then we say that X *admits a representation* $< f_n >$ *in L_1-norm and* we write $X \underset{L_1}{\longleftrightarrow} < f_n >_{n=1}^{\infty}$ (w . r . t. $\mathfrak{B}$).

It is easy to see that $X \underset{L_1}{\longleftrightarrow} < f_n >_{n=1}^{\infty}$ implies $X \overset{\|\cdot\|}{\longleftrightarrow} < f_n >_{n=1}^{\infty}$. The converse statement, in general, is not true. For $X, Y \in L_1(\mathbb{K}, \mathcal{A})$ let define

$$H(X, Y) = \int_\Omega h(X(\omega), Y(\omega))\, dP.$$

Then, as is known, $< L_1(\mathbb{K}, \mathcal{A}), H >$ is a complete metric space.

Now for $X \in L_1(\mathbb{K}, \mathcal{A})$, define $M = cl\,\{E^{\mathcal{B}}(f);\, f \in S_X(\mathcal{A})\}$, where $E^{\mathcal{B}}(f)$ denotes the conditional expectation of vector-valued $f \in L_1(\mathbb{B}, \mathcal{A})$.

It is easy to check that M becomes a closed bounded decomposable and non-empty subset of $L_1(\mathbb{B}, \mathcal{B})$. Thus there is a unique multifunction $E(X, \mathcal{B})$ of $L_1(\mathbb{K}, \mathcal{B})$ such that

$$S_{E(X, \mathcal{B})}(\mathcal{B}) = M$$

Such a function $E(X, \mathcal{B})$ will be called a conditional expectation of X (given B). This concept was introduced by Hiai an Umegaki in [11] is a natural generalization of the conditional expectations for vector-valued functions.

Define

$$\mathbb{K}_c = \{X \in \mathbb{K};\ X \text{ is convex}\},$$

$$\mathbb{K}_{cc} = \{X \in \mathbb{K}_c;\ X \text{ is compact}\},$$

$$\mathbb{K}_b\ \{X \in \mathbb{K}_c;\ X \text{ is a closed ball}\}.$$

Then the following result has been established in [11, 18]

LEMMA 1. 2. (1) $L_1(\mathbb{K}_c, \mathcal{A})$, $L_1(\mathbb{K}_{cc}, \mathcal{A})$ *and* $L_1(\mathbb{K}_b, \mathcal{A})$ *are closed subspaces* of $< L_1(\mathbb{K}, \mathcal{A}), H >$

(2) *If* $X \in L_1(\mathbb{K}_c, \mathcal{A})$, $Y \in L_1(\mathbb{K}_{cc}, \mathcal{A})$, and $Z \in L_1(\mathbb{K}_b, \mathcal{A})$ *then* $E(X, \mathcal{B}) \in L_1(\mathbb{K}_c, \mathcal{B})$; $E(Y, \mathcal{B}) \in L_1(\mathbb{K}_{cc}, \mathcal{B})$ *and* $E(Z, \mathcal{B}) \in L_1(\mathbb{K}_b, \mathcal{B})$

Furthermore, Hiai has observed in [10] that there is a (separable) seal Banach space $\mathbb{B}$ such that one can embed $L_1(\mathbb{K}_{cc}, \mathcal{A})$ as a closed convex cone in $L_1(\widehat{\mathbb{B}}, \mathcal{A})$ in such a way that

(i) the embedding is isometric

(ii) addition in $L_1(\widehat{\mathbb{B}}, \mathcal{A})$ induces a co 2 responding operation in $L_1(\mathbb{K}_{cc}, \mathcal{A})$

(iii) multiplication by nonnegative real L_∞-functions in $L_1(\widehat{\mathbf{B}}, \mathcal{A})$ induces a corresponding operation in $L_1(\mathbf{K}_{cc}, \mathcal{A})$

Using this embedding, Hiai and Umegaki [11] extended some results in [5] and /27/, to martingales in $L_1(\mathbf{K}_{cc}, \mathcal{A})$. In the recent paper [18] we have proved that one can also embed $L_1(\mathbf{K}_b, \mathcal{A})$ as a closed convex cone in the Banach space $L_1(\mathbf{B}, \mathcal{A}) \otimes L_1(\mathbf{R}, \mathcal{A})$. Therefore centain results in [5],[27] and [15] can be extended to the corresponding sequences in $L_1(\mathbf{K}_b, \mathcal{A})$. But neither our embedding nor the embedding of Hiai and Umegaki can be applied to $L_1(\mathbf{K}_c, \mathcal{A})$, if $\mathbf{B}$ is infinitely-dimensional. We propose thus the following problems: in $L_1(\mathbf{K}_c, \mathcal{A})$ has (P) then there exists a measurable P-selection $\langle f_n \rangle$ of $\langle X_n \rangle$ with the property (P). Such a sequence $\langle f_n \rangle$ will be denoted by $\langle f_n \rangle \in P - S(\langle X_n \rangle)$.

PROBLEM II. Suppose that (P) is a property such that if $\langle X_n \rangle$ in $L_1(K_c, \mathcal{A})$ has proper (P) then $P - S(\langle X_n \rangle)$ is nonempty. Give general representations of $\langle X_n \rangle$ in terms of $P - S(\langle X_n \rangle)$

Example 1. 3. Let $\langle \mathcal{A}_n \rangle$ be an increasing sequence of sub σ-fields $\mathcal{A}$. A sequence $\langle X_n \rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ is said to have the property (AD) (it is adapted to $\langle A_n \rangle$) if X_n is $\mathcal{A}_n$-measurable for all n. From [14] and [3] we know that as (P) one can take the property (AD) (see Lemma 1. 1)

Example 1. 4. A sequence X_n in $L_1(\mathbf{K}_c, A)$ is said to have property (M) (it is a martingale w. r. t. $\langle \mathcal{A}_n \rangle$) if it has property (AD) and the condition $X_n = X_n(m)$ holds for all $m > n \geq 1$, where

$$X_n(m) = E(X_m, \mathcal{A}_n) \qquad (m \geqslant n \geqslant 1)$$

Similarly, a sequence X_n in $L_1(\mathbf{K}_c, \mathcal{A})$ is said to have property (RM) (it is a regular martingale), if it has property (AD) w. r. t. $\langle A_n \rangle$ and the condition $X_n = E(X, \mathcal{A}_n)$ holds for all n and some $X\ L_1\ \mathbf{K}_c, \mathcal{A}$. Thus our results in [16] show that as (P) one can also take (M) or (RM).

2. MULTI-VALUED QUASI-MARTINGALES.

Throughout this and the next sections we shall suppose that we are given an increasing sequence $\langle \mathcal{A}_n \rangle$ of sub б-fields of with $\mathcal{A}_n \uparrow \mathcal{A}$. All our sequen--ces are assumed to be taken from $L_1(\mathbf{K}_c, \mathcal{A})$ and adapted to $\langle \mathcal{A}_n \rangle$. The following notion of multivalued quasi-martingales is a natural exten-ion of that of real-valued quasi-martingales given by Rao /22/.

DEFINITION 2. 1. *A sequence $\langle X_n \rangle$ is said to be a quasi-martingale (it has property (QM)), it the following condition holds*

$$\sum_{n \geqslant 1} H(X_n, X_n(n+1)) < \infty \qquad (2.\ 1)$$

It is easy to see that if $\langle X_n \rangle$ is a martingale then it is a quasi-martingale. But the converse statement is not true. The main purpose of this section is to consider Problems I—II for the property (QM) defined above.

PROPOSITION 2. 2. *Let $\mathcal{E} \subset \mathfrak{B}$ be two sub σ-fields of $\mathcal{A}$. Suppose that $X \in \mu(\mathbf{K}, \mathcal{E})$, $Y \in \mu(\mathbf{K}, \mathfrak{B})$ and $\varphi \in \mu(\mathbf{R}, \mathfrak{B})$ with $\varphi(\omega) > o$ for all $\omega \in \Omega$. Then $\forall f \in S_X(\mathcal{E})$ $\exists g \in S_Y(\mathfrak{B})$ $\| f(\omega) - g(\omega) \| \leqslant h(X(\omega), Y(\omega)) + \varphi(\omega)$,* **a. e.** *Thus, in particular, if X and Y are integrably bounded then*

$$\forall f \in S_X(\mathcal{E})\ \forall \varepsilon > o\ \exists g \in S_Y(\mathfrak{B}) \quad E(\| f - g \|) \leqslant H(X, Y) + \varepsilon \qquad (2.\ 2)$$

Proof. Let $\mathcal{E}$, $\mathfrak{B}$, X, Y, and φ be as in the assumptions of Proposition 2. 2. Fix $f \in S_X(\ )$. Since $Y \in \mu(\mathbf{K}, \mathfrak{B})$, view of Lemma 1. 1. there is a sequence

$\langle g_n \rangle$ of $S_Y(\mathfrak{B})$ such that $Y \xleftrightarrow{\|\cdot\|} \langle g_n \rangle_{n=1}^{\infty}$. Define

$\tau : \Omega \to \mathbf{N}$ (the set of all positive integers) by

$\tau(\omega) = \inf \{ n ;\ \| f(\omega) - g_n(\omega) \| \leqslant d(f(\omega), Y(\omega)) + \varphi(\omega) \}$

Since $Y \xleftrightarrow{\|\cdot\|} \langle n_n \rangle_{n=1}^{\infty}$ and $\varphi(\omega) > 0$ for all $\omega \in \Omega$, the function τ is well-defined. Fix $n \in \mathbf{N}$. We have

$$\{\tau = n\} = \bigcap_{j=1}^{n-1} \{ \| f - g_j \| > d(f, Y) + \varphi \} \cap \{ \| f - g_n \| \leqslant d(f, Y) + \varphi \}.$$

Since all functions f, g_i, g_n, φ and $\omega \to d(f(\omega), Y(\omega))$ are measurable, it follows that $\{\tau = n\} \in \mathfrak{B}$. So the function τ is itself $\mathfrak{B}$-measurable. This implies that the function $g : \Omega \to \mathbf{B}$ defined by $g(\omega) = g_{\tau(\omega)}(\omega)$ is a $\mathfrak{B}$-measurable selection of Y. Moreover

$$\| f(\omega) - g(\omega) \| \leqslant d(f(\omega), Y(\omega)) + \varphi(\omega) \leqslant h(X(\omega), Y(\omega)) + \varphi(\omega), \text{ a. e.}$$

Thus in particular, if X and Y are integrably bounded then

$$\forall f \in S_X(\mathscr{F}) \ \forall \varepsilon > 0 \ \exists g \in S_Y(\mathfrak{B}) \ E(\| f - g \|) \leqslant H(X, Y) + \varepsilon$$

The proof is completed.

The following proposition solves Problem *1* for the property $(Q\ M)$

PROPOSITION 2. 3 *Let $\langle X_n \rangle$ be a quasi-martingale. Then*

$\forall k \geqslant 1 \ \forall f_k \in S_{X_k}(\mathscr{A}_k) \ \forall \varepsilon > 0 \exists \langle f_n \rangle \in QM - S(\langle X_n \rangle)$ such that

$$\forall_{n \geqslant 1} \ E(\| f_n - f_n(n+1) \| \leqslant H(X_n, X_n(n+1)) + \frac{\varepsilon}{2_n} \qquad (2.3)$$

Proof. Let $\langle X_n \rangle$ be a quasi-martingale in $L_1(\mathbf{K}_c, \mathscr{A})$

Fix $k \in \mathbf{N}$, $f_k \in S_{X_k}(\mathscr{A}_k)$ and $\varepsilon > 0$. Sin X_k and $X_k(k+1)$ are both $\mathscr{A}_k$ measurable then by Proposition 2. 2 (2. 2) there is some

$g_k \in S_{X_k(k+1)}(\mathscr{A}_k)$ such that

$$E(\| f_k - g_k \|) \leqslant H(X_k : X_k(k+1)) + \frac{\varepsilon}{2^{k+1}}$$

Further, since $g_k \in S_{X_{k(k+1)}}(\mathscr{A}_k) = cl\ \{ E\mathscr{A}k_{(f)} ; f \in S_X(\mathscr{A}) \}$, by Theorem 5. 3 (2) in /II/ there is some $f_{k+1} \in S_{X_{k+1}}(\mathscr{A}_{k+1})$

such that $E(\| g_k - f_k(k+1) \|) \leqslant \frac{\varepsilon}{2^{k+1}}$

It follows that for a given $f_k \in S_{X_k}(\mathcal{A}_k)$, one can choose

$$f_{k+1} \in S_{X_{k+1}}(\mathcal{A}_{k+1})$$

Such that $E(\| f_k - f_k(k+1) \|) \leqslant H(X_k, X_k(k+1)) + \frac{\varepsilon}{2k}$.

Thus, by induction, we can contruct a sequence $< f_n >_{n=k}^{\infty}$

such that $f_n \in S_{X_n}(\mathcal{A}_n)$ and

$$E(\| f_n - f_n(n+1) \|) \leqslant H(X_n, X_n(n+1)) + \frac{\varepsilon}{2^n}$$

for all $n \geqslant k$

Again, since X_{k-1} and $X_{k-1}(k)$ are both $\mathcal{A}_{k-1}$ - measurable in view of Proposition 2.2 (2.2). there is some $f_{k-1} \in S_{X_{k-1}}(\mathcal{A}_{k-1})$ such that

$$E(\| f_{k-1} - f_{k-1}(k) \|) \leqslant H(X_{k-1}, X_{k-1}(k)) + \frac{\varepsilon}{2k-1}$$

Henceby a finite number of steps we can construct $f_{k-1}, f_{k-2}, \ldots, f_1$ such that $f_m \in S_{X_m}(\mathcal{A}_m)$ and

$$E(\| f_m - f_m(m+1) \|) \leqslant H(X_m, X_m(m+1) \|) + \frac{\varepsilon}{2m} \quad (1 \leqslant m \leqslant k-1)$$

which proves (2.3). This completes the proof.

The following results give us several representations of multi-valued quasi-martingales in terms of their quasi-martingale selections.

THEOREM 2. 4. *Let $< X_n >$ be a sequence in $L_1(\mathbf{K}_c, \mathcal{A})$.*

Then $<X_n>$ is a quasi-martingale if there is a sequence α_n of nonnegative real numbers with $\sum_{n \geqslant 1} \alpha_n < \infty$

d such that

$$k \in \mathbf{N}\ S_{X_k}(\mathcal{A}_k) = \{f_k \,;\, <f_n> \in QMS\,(<X_n>), \tag{2.4}$$

$$\| f_n - f_n(n+1) \|_1 \leqslant \alpha_n \ \forall_n \geqslant 1\}$$

Proof. (⇒) Let $<X_n>$ be a quasi-martingale and $\varepsilon > 0$ any but fixed positive real number.

Put $\alpha_n = H(X_n, X_n(n+1)) + \frac{\varepsilon}{2n}$. In view of (2.1) and Proposition .3 (2.3) we get (2.4) for this sequence $<\alpha_n>$.

(⇐) Conversely, suppose that (2.4) holds for some sequence $<X_n>$. Then,

$$\sum_{n\geqslant 1} H(X_n, X_n(n+1)) \leqslant \sum_{n\geqslant 1} \alpha_n < \infty$$

Thus, $<X_n>$ is a quasi-martingale.

THEOREM 2.5. *Let $<X_n>$ be a quasi-martingale in $L_1(\mathbf{K}_c, \mathcal{A})$*

Then there is a sequence $<\alpha_n>$ of nonnegative real numbers with $\sum_{n\geqslant 1} \alpha_n < \infty$

and a sequence $\{<f_n^i>\}_{i=1}^{\infty}$ of $QMS\,(<X_n>)$ such that

(1) $\forall n, 1 \in \mathbf{N}$ $\quad E(\| f_n^i - f_n^1(n+1) \|) \leqslant \alpha_n$ and

(2) $\forall k \in \mathbf{N}$ $\quad X_k \xleftrightarrow{\|\cdot\|} <f_k^i>_{i=1}^{\infty}$ $\quad$ (w.r.t $\mathcal{A}_k$)

Proof. It follows from Theorem 2.4 and Lemma 1.1.

THEOREM 2.6. *Let $<X_n>$ be a sequence in $L_1(\mathbf{K}_c, \mathcal{A})$. Suppose furthermore that $\mathcal{A}$ is σ-generated. Then the sequence $<X_n>$ is a quasi-martingale if there is a sequence $<\alpha_n>$ of nonnegative numbers with $\sum_{n\geqslant 1} \alpha_n < \infty$ and a sequence $\{<f_n^i>\}_{i=1}^{\infty}$ of QM-S $(<X_n>)$ such that*

(1) $\forall n, i \in \mathbb{N}\ E(\| f_n^i - f_n^i(n+1) \|) \leqslant \alpha_n$ and

(2) $\forall k \in \mathbb{N}\ X_k \langle \xrightarrow[L_1]{\|\cdot\|} \rangle < f_k^i >_{i=1}^{\infty}$ $(w. r. t. \mathcal{A}_k)$.

Proof. (⇒) This follows from Theorem 2.4 and the assumption that $\mathcal{A}$ is 6-generated.

(⇐) This follows from the same arguments as those given in the proof of (⇐) of Theorem 2.4, by noting that if both conditions (1) and (2) in Theorem 2.6 hold then

$$\forall k \in \mathbb{N}\ H(X_k, X_k(k+1)) \leqslant \alpha_k .$$

DEFINITION 2, 7. *A sequence* $< X_n >$ in $L_1(\mathbb{K}_c, \mathcal{A})$ *is said to be regular if there si an* $X \in L_1(\mathbb{K}_c, \mathcal{A})$ *such that*

$$\lim_{n \to \infty} H(X_n, E(X, \mathcal{A}_n)) = 0 \tag{2.5}$$

It is easy to see that if $< f_n >$ is a sequence in $L_1(\mathbb{B}, \mathcal{A})$ then $< f_n >$ is regular if it is convergent in L_1. But in general, this statement fails to be valid for a sequence $< X_n >$ in $L_1(\mathbb{K}_c, \mathcal{A})$. Indeed, there is a regular martingale in $L_1(\mathbb{K}_c(l_2), \mathcal{B}_{[0,1)})$ which fails to be convergent ([11], Example (3.3)).

COROLLARY 2. 8. *Let* $\mathbb{B}$ *be a separable Banach space with the (RNP) (see the definition in Section 4). Suppose that a sequence* $< X_n >$ *in* $L_1(\mathbb{K}_c, \mathcal{A})$ *is a regular quasi-martingale (it has Property RQM) then there is a sequence* $\{< f_n^i >\}_{i=1}^{\infty}$ *of* $RQMS(< X_n >)$ *such that*

$$\forall \quad \ldots \quad \xrightarrow{\|\cdot\|} < f_k^i >_{i=1}^{\infty}.$$

Proof. Let $< X_n >$ be a regular quasi-martingalein L_1 (K_c, $\mathcal{A}$) i.e. there is an $X \in L'_1$ (K_c, $\mathcal{A}$) such that (2.5) holds. Thus in view of [10], the regular martingale $< E(X, \mathcal{A}_n) >$ is uniformly integrable and L_1 — bounded, hence by (2.5) the sequence $< X_n >$ is itself uniformly integrable and L_1 — boun ded. It follows that if $\langle f_n \rangle \in QMS$ ($\langle X_n \rangle$) then $\langle f_n \rangle$ is uniformly integrable and L_1 — bounded. Now, if we suppose that a Banach space **B** has the (RNP) then in view of /15/, $\langle f_n \rangle$ is a regular quasi-martingale. Thismeans that, QM-S ($\langle X_n \rangle$) = RQM-S ($\langle X_n \rangle$). Therefore Theorem 2.5 implies Corollary 2. 8.

Note that in the case where **B** does not have the (RNP), problems (I — II remain open for Property *(RQM)! The* author should like to know *that* the representation theorem for multivalueed martingales given in [16] can be estab lished from Theorem 2.4, noting that $\langle X_n \rangle$ is a martingale iff

$$\sum_{n \geqslant 1} H(X_n, X_n\ (n+1)) = 0.$$

3. MULTI-VALUED UNIFORM AMARTS

The notion of vector-valued uniform amarts has been recently introduced by Bellow/51 as a special one of vector-valued amarts /9/, for which the strong almost sure convergence obtains. This idea is clear but the Bellow's definition is very complicated. In fact, it is very hard to check whether a sequence $\langle f_n \rangle$ in L_1 (B, $\mathcal{A}$) is a uniform amart. However, the Below's definition is equiva- lent to the following:

A sequence $\langle f_n \rangle$ in L_1(**B**, $\mathcal{A}$) is a uniform amart iff

$$\forall \varepsilon > 0\ \exists k\ \forall \sigma \in T\ (\sigma \geqslant k) \text{ implies } \| \mu\sigma - (\mu \mid \mathcal{A}_\sigma) \| \leqslant \varepsilon \quad (3.1)'$$

where $\mu\sigma(A) = \int_A f_\sigma \, dP \quad (A \in \mathcal{A}_\sigma);$

$\mu(A) = \lim \int f_\tau \, dP \quad (A \in \bigvee \mathcal{A}_n)$ and T is the set of all bounded stopping times (w. r. t. $\langle \mathcal{A}_n \rangle$) with the usual order. It is not hard to check that (3. 1)' is equivalent to the following condition:

$$\forall_{\varepsilon > 0} \ \exists k \in \mathbf{N} \ \forall_{\eta \geqslant \sigma \geqslant k} \ (\eta, \sigma \in T) \ E(\| f_\sigma - f_{\sigma(\eta)} \|) \leqslant \varepsilon \qquad (3.\ 1)''$$

This remark suggests the following definition of multivalued uniform amarts:

DEFINITION 3. 1. *A sequence $\langle X_n \rangle$ in L_1 ($\mathbf{K}_c$, $\mathcal{A}$ is a uniform amart if the following condition holds*

$$\forall_{\varepsilon > 0} \ \exists \ k \in \mathbf{N} \ \forall_{\eta > \sigma \geqslant k} \ (\eta, \sigma \in T) \ H(X_\sigma, X_\sigma(\eta)) \leqslant \varepsilon \qquad (3.\ 1)$$

where $E(X_\eta, \mathcal{A}_\sigma) = X_\sigma(\eta) \qquad (\eta \geqslant \sigma \in T)$

Example 3. 2. Asequence $\langle X_n \rangle$ in L_1 $\mathbf{K}_c$, $\mathcal{A}$) is a uniform amart. Indeed, if $\langle X_n \rangle$ is a martingale in L_1 $\mathbf{K}_c$, $\mathcal{A}$, then in view of [16], the sequence $\langle X_\tau, \tau \in T \rangle$ is also a martingale in L_1 ($\mathbf{K}_c$, $\mathcal{A}$) (w. r. t. $\langle \mathcal{A}_\tau \rangle$). Thus the condition (3. 1) is automatically satisfied.

Example 3. 3. A sequence $\langle X_n \rangle$in L_1 ($\mathbf{K}_c$, $\mathcal{A}$) is called a uniform potential if the sequence $\langle | X_n | \rangle$ is a uniform potential, i. e.

$$\lim_{\tau \in T} \int_\Omega | X_\tau | \, d P = 0 \qquad (3.\ 2)$$

It follows that if $\langle X_n \rangle$ is a uniform potential in L_1 ($\mathbf{K}_c$, $\mathcal{A}$) then it is a uniform amart. Indeed since $H(X_\sigma, X_\sigma(\eta)) \leqslant H(X_\sigma, X_\eta)$ $(\eta \geqslant \sigma \in T)$ then (3. 2) implies (3. 1).

These examples lead to the following theorem which generalizes Theorem 3 of Bellow [2].

THEOREM 3.4. *A sequence $\langle X_n \rangle$ in L_1 ($\mathbf{K}_c$, $\mathcal{A}$) is a uniform amart iff there is a (unique) martingale $\langle M_n \rangle$ in L_1 ($\mathbf{K}_c$, $\mathcal{A}$ such that the sequence $\langle P_X(n) \rangle$ defined by $P_X(n) = h(X_n, M_n)$ is a nonnegative uniform potential,* i.e.

$$\lim_{\tau \in T} E(P_X(\tau)) = 0 \qquad (3.3)$$

Proof. ($\Rightarrow$) Let $\langle X_n \rangle$ be a uniform amart in $L_1(\mathbf{K}_c, \mathcal{A})$. Thus by (3.1), $\{\langle x_\sigma(\eta)\rangle\}_{\eta \geq \sigma}$ is a generalized Cauchy sequence in $L_1(\mathbf{K}_c, \mathcal{A}_\sigma)$ for every $\sigma\in T$. Hence, by Lemma 1.2, there is a generalized sequence $\langle L_\tau \rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ such that

$$\lim_{\eta\in T} H(X_\sigma(\eta), L(\sigma)) = 0 \qquad (\sigma\in T).$$

It is not hard to check that in this case, $\langle L_\tau\rangle$ is a martingale (w.r.t. $\langle \mathcal{A}_t \rangle$). In particular, there is a martingale $\langle \mathbf{M}_n\rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ such that

$$\lim_{\eta\in T} H(X_n(\eta), M_n) = 0 \qquad (n\in\mathbf{N}).$$

Hence in view of /16/ the sequence $\langle M_\tau \rangle$ is also a martingale in $L_1(\mathbf{K}_c, \mathcal{A})$. But for each $n\in\mathbf{N}$, $L_n = M_n$, a.e. therefore $L_\tau = M_\tau$, a.e. for each $\tau\in T$. Finally, if $\varepsilon > 0$ is any but fixed positive real number then by (3.1) there is some $k\in\mathbf{N}$ such that

$$\begin{aligned} H(x_\sigma, M_\sigma) &= H(X_\sigma, L_\sigma) \\ &\leqslant H(X_\sigma, X_\sigma(\eta)) + H(X_\sigma(\eta), L_\sigma) \\ &\leqslant \varepsilon + H(X_\sigma(\eta), L_\sigma) \qquad (\eta \geqslant \sigma \geqslant k) \end{aligned}$$

Therefore,

$$H(X_\sigma, M_\sigma) \leqslant \varepsilon + \lim_{\eta\in T} H(X_\sigma(\eta), L_\sigma) = \varepsilon \qquad (\sigma \geqslant k).$$

It follows that if we put $P_X(n) = h(X_n, M_n)$ then $\langle P_x(n)\rangle$ is a nonnegative uniform potential.

($\Leftarrow$) Let $\langle X_n \rangle$ be a sequence in $L_1(\mathbf{K}_c, \mathcal{A})$. Suppose that there is a martingale $\langle M_n \rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ such that (3.3) holds. Thus,

$$\forall_{\varepsilon > 0}\ \exists_{k\in\mathbf{N}}\ \forall_{\tau\in T, \tau \geqslant k}\ H(X_\tau, M_\tau) \leqslant \frac{\varepsilon}{2}$$

Therefore, if $\eta \geqslant \sigma \geqslant k$, then we get

$$H(X_\sigma, X_\sigma(\eta)) \leqslant H(X_\sigma, M_\sigma(\eta)) + H(M_\sigma(\eta), X_\sigma(\eta))$$

$$\leqslant H(X_\sigma, M_\sigma(\eta)) + H(M_\eta, X_\eta)$$

$$\leqslant H(X_\sigma, M_\sigma(\eta)) + \frac{\varepsilon}{2}$$

But as $\langle M_n \rangle$ is a martingale, by/16/ So is $\langle M_\tau \rangle$ Consequently,

$$H(X_\sigma, X_\sigma(\eta)) \leqslant H(X_\sigma, M_\sigma) + \frac{\varepsilon}{2}$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This proves (3. 1). In other words, $\langle X_n \rangle$ is a uniform amarts The proof is completed.

The following proposition shows that every quasi-martingale is a uniform amart.

PROPOSITION 3. 5, *Let $\langle X_n \rangle$ be a quasi-martingale in $L_1(\mathbf{K}_c, \mathcal{A})$. Then there is a (unique) martingale $\langle M_n \rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ such that*

$$\forall_n \in \mathbf{N} \; H(X_n, M_n \leqslant \sum_{k \geqslant n} H(X_k, X_k(k+1)) \qquad (3.4)$$

Hence $\lim_{n\to\infty} H(X_n, M_n) = 0$.

Proof. Let $\langle X_n \rangle$ be a martingale in $L_1(\mathbf{K}_c, \mathcal{A})$. Thus for $(m > k \geqslant v \geqslant 1)$ we have

$$H(X_n(m), X_n(k)) \leqslant \sum_{j=k}^{m-1} H(X_j, X_j(j+1)) \leqslant \sum_{j \geqslant k} H(X_j, X_j(j+1)).$$

Therefore, by (2. 1), $\langle X_n(m) \rangle_{m \geqslant n}$ is a Cauchy sequence in $L_1(\mathbf{K}_c, \mathcal{A}_n)$ for each $n \geqslant 1$. Consequently, by Lemma 1. 2, there is a sequence $\langle M_n \rangle$ in $L_1(\mathbf{K}_c, \mathcal{A})$ such that

$$\lim_{m\to\infty} H(X_n(m), M_n) = 0 \qquad (n \in \mathbb{N}) \qquad (3.5).$$

It is easy to check that in this case $\langle M_n \rangle$ is even a martingale. Finally, if $\geqslant n \geqslant 1$ then,

$$H(X_n, M_n) \leqslant H(X_n, X_n(m)) + H(X_n(m), M_n).$$

his with (3. 5) yields

$$H(X_n, M_n) \leqslant \sum_{j=n}^{\infty} H(X_j, X_j(j+1)) + \lim_{m \to \infty} H(X_n(m), M_1)$$

$$= \sum_{j=n}^{\infty} H(X_j, X_j(j+1)).$$

which proves (3. 4). Therefore in view of (2. 1) we get

$$\lim_{n \to \infty} H(X_n, M_n) = 0.$$

The proof is completed.

COROLLARY 3. 6. *Every quasi-martingale in $L_1(\mathbf{K}_c, \mathcal{A})$ is a uniform amart (see [2]).*

Proof. Let $\langle X_n \rangle$ be a quasi-martingale in $L_1(\mathbf{K}_c, \mathcal{A})$. Then by Proprosition 3. 2 there is a martingale $\langle M_n \rangle$ in $L_1 \mathbf{K}_c, \mathcal{A})$ such that (3. 4) and (3. 5) hold.

Now fix $k \in \mathbf{N}$, $\mathrm{r} \in T$ with $l \geqslant \mathrm{r} \geqslant k$, a. e. Thus by (3. 5), there is a positive integer $m \geqslant l$ such that

$$H(X_j(m), M_j) \leqslant \frac{1}{l \cdot 2^k} \qquad (k \leqslant j \leqslant l)$$

Moreover, we get the following estimation

$$H(X_\mathrm{r}, M_\mathrm{r}) = \sum_{j=k}^{l} \int_{\{\mathrm{r}=j\}} h(X_j, M_j)\, dP$$

$$= \sum_{j=k}^{l} \int_{\{\mathrm{r}=j\}} [h(X_j, X_j(m)) + h(X_j(m), M_j)] dP$$

$$\leqslant \sum_{j=k}^{l} \int_{\{\mathrm{r}=j\}} h(X_j, X_j(m))\, dP + \sum^{l} H(X_j(m), M_j)$$

$$\leqslant \sum_{j=k}^{l} \int_{\{\mathrm{r}=j\}} h(X_j, X_j(m))\, dP + \frac{1}{2^k}.$$

But as for each $j = k, \ldots, l$ we can write

$$\int_{\{\mathrm{r}=j\}} h(X_j, X_j(m)\, dP \leqslant \int_{\{\mathrm{r}=j\}} h(X_j, X_{m-1}(m)\, dP$$

$$\leqslant \int_{\{\mathrm{r}=j\}} \sum_{p=k}^{m-1} h(X_p, X_p(p+1)\, dP$$

$$= \sum_{p=k}^{m-1} \int_{\{\mathrm{r}=j\}} h(X_p, X_p(p+1)\, dP$$

we get so

$$H(X_{\mathrm{r}}, M_{\mathrm{r}}) \leqslant \sum_{j=k}^{l} \sum_{p=k}^{m-1} \int_{\{\mathrm{r}=j\}} h(X_p, X_p(p+1))\, dP + \frac{1}{2^k}$$

$$= \sum_{p=k}^{m-1} H(X_p, X_p(p+1) + \frac{1}{2^k}$$

$$\leqslant \sum_{p=k}^{\infty} H(X_p, X_p(p+1)) + \frac{1}{2^k}$$

Funther, by (2. 2) we get the following relation

$$\lim_{\mathrm{r} \in T} H(X_{\mathrm{r}}, M_{\mathrm{r}}) = \lim E(P_x(\mathrm{r})) = 0$$

This with Theovem 3. 4 implies Corollary 3. 6.

COROLLARY 3. 7. (See [2], Theorem 3.)

A sequence $\langle f_n \rangle$ in $L_1(\mathbf{B}, \mathcal{A})$ is a uniform amart iff $\langle f_n \rangle$ admits a Riesz decomposition

$$\mathbf{f}_n = g_f(n) + p_f(n) \qquad (n \geqslant 1)$$

where $\langle g_f(n) \rangle$ is a martingale and $\langle \| p_f(n) \| \rangle$ is a uniform potential.

COROLLARY 3. 8. Let $\langle X_n \rangle$ be a uniform amart in $L_1(\mathbf{K}_c, \mathcal{A})$ with Property (6. 4) in [10], i. e.

$$X_n \supset X_n(n+), \qquad \text{a. e.} \qquad (n \in \mathbf{N})$$

Then, there is a (unique) martingale $\langle M_n \rangle$ in L_1 ($\mathbb{K}_c$, $\mathcal{A}$) such that $M_n \subset X_n$, a. e. (n$\in$ $\mathbb{N}$) and

$$\lim_{\tau \in T} H\ (X_\tau,\ M_\tau) = 0.$$

Hence, by [16] we have

$$M.\ S\ (\langle X_n \rangle) \supset M.\ S\ (\langle M_n \rangle) \neq \phi.$$

Proof. This follows from Theorem 3. 1, noting that under the assumption $X_n \supset X_n (n+1)$, a.e. (n $\in$ $\mathbb{N}$) we have $M_n \supset X_n$, a. e. (n $\in$ $\mathbb{N}$), where $\langle M_n \rangle$ is the martingale
constructed in the proof of Theorem 3.4. for the uniform amarf $\langle X_n \rangle$.

Now if $\langle X_n \rangle$ is a uniform amart (it has Property (UA)), then the following Proposition 3. 9 solves Problem I for Property (UA):

PROPOSITION 3. 9. *Let $\langle X_n \rangle$ be a uniform amarl in L_1 ($\mathbb{K}_c$, $\mathcal{A}$) then there is a (unique) martingale $\langle M_n \rangle$ in L_1 ($\mathbf{K}_c$, $\mathcal{A}$) such that for any but fixed positive real numbe r $\varepsilon > 0$ we have*

(1) $\forall_k \in \mathbf{N}\ S_{X_k}(\mathcal{A}_k) = \{ f_k ; \langle f_n \rangle \in UAS\ (\langle X_n \rangle) ; \| p_f\ (n) \| \leqslant P_n, \phi.$ *a. e.* $\forall_n \in \mathbf{N} \}$ *and*

(2) $\forall_k \in \mathbf{N}\ S_{M_k}\ (\mathcal{A}_k) = \{ g_f\ (k) ; \langle f_n \rangle \in UAS\ (\langle X_n \rangle) ; \| p_f (n) \| \cdot \mathcal{A}\ P_n$

a. e $\forall_n \in \mathbf{N} \}$ where $P_n = P_X\ (n) + \dfrac{\varepsilon}{2^n}$, a. e. ($n \in \mathbf{N}$).

Proof. Let $\langle X_n \rangle$ be a uniform amart in L_1 ($\mathbf{K}_c$, $\mathcal{A}$). By Theorem 3. 4. here is a (unique) martingale $\langle M_n \rangle$ in L_1 ($\mathbf{K}$, $\mathcal{A}$) such that $\langle P_X\ (n) \rangle$ is a uniform potential. Now fix $\varepsilon > 0$. Define $P_n = P_X + \dfrac{\varepsilon}{2^n}$ ($n \in \mathbb{N}$). It is clear that the sequence $\langle p_n \rangle$ is a uniform potential we show first that (1) holds.

Indeed, fix $(1 \leqslant) k \in \mathbf{N}$ and $f_k \in S_{X_k}(\mathcal{A}_k)$ Since $\langle M_n \rangle$ is a martingale in $L_1(\mathrm{K}_c, \mathcal{A})$ by [16] there is a sequence $\{ \langle g_n^i \rangle \}_{i=1}^{\infty}$ of $MS(\langle M_n \rangle)$ such that $M_m \langle \xrightarrow{\| \cdot \|} \rangle \langle g_m^i \rangle \rangle_{i=1}^{\infty}$ (w. r. f. $\mathcal{A}_m$) $m \in \mathbf{N}$). Define $\tau(\omega) = \inf \{ i ;$ $\| f_k(\omega) - g_k^i(\omega) \| \leqslant d(f_k(\omega), M_k(\omega)) \frac{\varepsilon}{2^k} \}$ and $g_k(\omega), \sum_{i \in \mathbf{N}} \mathbf{1}_{k \{ \tau = i \}} g_k^i(n) =$ $= g_k^{\tau(\omega)}(\omega)$ then by, the same arguments a in the proof of Proposition 2.2, the function τ is $\mathcal{A}_k$ — measurble and $g_k \in S_{M_k}(\mathcal{A}_k)$.

Moreover $\| f_k(\omega) - g_k(\omega) \| \leqslant P_k(\omega)$, a. e.

Now, put $g_n = \sum_{i \in \mathbf{N}} \mathbf{1}_{\{\tau = i\}} g_n^i \quad (n \geqslant k)$ and

$$g_m = E^{\mathcal{A}m}(g_k) \quad (1 \leqslant m \leqslant k).$$

It is easy to check that $< g_n > \in MS(<M_n>)$.

Again, given a martingale $< g_n >$ we can construct a sequence $< f_n >$ as in Proposition 2.2, such that $f_n \in S_{X_n}(\mathcal{A}_n)$ and $\| f_n(\omega) - g_n(\omega) \| \leqslant P_n(\omega)$, a. e. $(n \in \mathbf{N})$. Finally, since $< P_n >$ is a uniform potential, $< g_n >$ is a martingale, then in view of Corollary 3.7 $< f_n > \in UAS(< X_n >)$. Moreover, $g_f(n) = g_n$; $P_f(n) = f_n - g_n$

and

$$\| P_f(n) \| \leqslant P_n, \quad \text{a.e.} \quad (n \in \mathbf{N}).$$

Thus the first assertion of the proposition is proved. Note that the above argument simultineously proves (2). Therefore the proof is complete.

The following result gives a solution of Problem II for Property (UA).

THEOREM 3.10. *Let $< X_n >$ be a sequence in L_1 $(\mathbb{K}_c, \mathcal{A})$. Then $< X_n >$ is a uniform amart iff there is a (unique) martingale $< M_n >$ in L_1 $(\mathbb{K}_c, \mathcal{A})$ and a nonnegative uniform potential $< P_n >$ such that both conditions (1), (·) in Proposition 3.9 hold.*

THEOREM 3.11. *Let $< X_n >$ be a uniform amart in L_1 $(\mathbf{K}_c, \mathcal{A})$. then there is sequence $\{< f_n^i >\}_{i=1}^{\infty}$ of UAS $(< X_n >)$ such that*

$$\forall_{k \in \mathbb{N}} \; X_k \overset{\|\cdot\|}{\longleftrightarrow} < f_k^i >_{i=1}^{\infty} \qquad (\text{w.r.t. } \mathcal{A}_k)$$

THEOREM 3.12. *Let $\langle X_n$ be a regular uniform amart in L_1 $(\mathbf{K}_c, \mathcal{A})$ it has property (RUA)). Suppose further that $\mathbb{B}$ has the (RNP). Then there is a sequence $\{\langle f_n^i \rangle\}_{i=1}^{\infty}$ of RUAS $(\langle X_n \rangle)$* such that

$$\forall_{k \in \mathbb{N}} X_k \overset{\|\ \|}{\longleftrightarrow} \langle f i_k \rangle_{i=1}^{\infty} \qquad (\text{w.r.t. } \mathcal{A}_k \ )$$

4. RELATIONS BETWEEN THE REGULARITY OF MULTI-VALUED UNIFORM AMARTS AND THE RN PROPERTY IN BANACH SPACES

A Banach space **B** is said to have the (RNP) w. r. t. the probability space $(\Omega, \mathcal{A}, P)$, if for every **B**-valued measure μ defined on $\mathcal{A}$ of bounded variation and absolutely continuous w. r. t. P there is a function $f \in L_1$ $(\mathbf{B}, \mathcal{A})$ such that

$$\mu(A) = \int f \, dP \qquad (A \in \mathcal{A}).$$

In [21], A. Phillips showed that every reflexive Banach space has the (RNP). Other geometric characterizations of the (RNP) in Banach spaces are given in ([24], [7], [20], [19], [12], [17]. Especially, in ([26], [5], [2], [15]) a martingale approach to (RNP) in B-space is presented In particular in [5] Chatterji proved that a Banach space B has the RNP w . r . t.

$(\Omega, \mathcal{A}, P)$ iff every uniformly integrable and L_1-bounded martingale in $L_1(\mathbf{B}, \mathcal{A})$ is regular. The idea of extending this result to multivalued martingales is due to Hiai and Umegaki in [11]. They have proved that if a separable Banach space $\mathbb{B}$ has the (RNP) and its topological dual $\mathbf{B}^*$ is separable, then every uniformly integrable and L_1-bounded martingale in $L_1(\mathbf{K}_c, \mathcal{A})$ is regular. Recently, using the limit projective methods Costé [6] (see also, [16]) has obtained this result without the extra assumption that $\mathbf{B}^*$ is separable. But note that the limit projective method by Costé can not be applied to larger classes of multivalued amarts such as the class of multivalued quasi-martingales. Hence, for the last class, the approximation method developed in the proof of Proposition 2 . 3 is more effective.

In this section we shall prove that the (RNP) of Banch spaces is equivatent to the condition that every uniformly integrable and L_1-bounded uniform amart in $L_1(\mathbf{K}_c, \mathcal{A})$ is regular. For this purpose, we recall that if $X \in \mathbf{K}_c$ then the support function $\delta^*(X,.)$: $\mathbf{B}^* \to \mathbb{R}$ of X is given by

$$\delta^*(X, x^*) = \sup \{ < x, x^* >, x \in X \} \quad (x^* \in \mathbf{B}^*).$$

THEOREM 4 . 1 . *Let* $\mathbf{B}$ *be a separable (real) Banach space then the following conditions are equivalent:*

(1) $\mathbf{B}$ *has the (RNP),* w . r . t . $(\Omega, \mathcal{A}, P)$.

(2) *every uniformly integrable and L_1-bounded uniform amart in* $L_1(\mathbf{K}_c, \mathcal{A})$ *is regular.*

(3) *For every uniformly integrable and L_1-bounded uniform amart* $< X_n >$ in $L_1(\mathbf{K}_c, \mathcal{A})$ *there is a (unique) function* $X \in L_1(\mathbb{K}_c, \mathcal{A})$ *such that for any but fixed* $x^* \in \mathbf{B}^*$ *the sequence* $< \delta^*(x_n, x^*) >$ *is a real-valued regular uniform amart which converges almost every where and in* L_1 to $\delta^*(X, x^*)$.

(4) *For every uniform amart* $< X_n >$ in $L_1(\mathbb{K}_c, \mathcal{A})$ *with values contained almost everywhere in* δU *for some* $\delta > o$ *there is a (unique) multifunction*

$X \in L_1(\mathbf{K}_c, \mathcal{A})$ *such that for any but fixed* $x^* \in \mathbf{B}^*$ *the sequence* $< \partial^* (X_n, x^*) >$ *is a real valued uniform amart which converges almost surely and in* L_1 *to* $\partial^* (X, x^*)$.

Proof. $(1 \Rightarrow 2)$ Let $\langle X_n \rangle$ be a uniformly integrable and L_1 — bounded uniform amart in L_1 $(\mathbf{K}_c, \mathcal{A})$. Hence in view of Theorem 3.4. there is a (unique) martingale $< M_n >$ in L_1 $(\mathbf{K}_c, \mathcal{A})$ such that

$$\lim_{\tau \in T} H(X_\tau, M_\tau) = 0$$

In particular, $< M_n >$ is a uniformly integrable and L_1—bounded martingale in L_1 $(\mathbf{K}_c, \mathcal{A})$. Thus, if a Banach space $\mathbf{B}$ has (RNP) then by [6] (see also [16]) $< M_n >$ is a regular maringale, i.e. there is an $X \in L_1$ $(\mathbf{K}_c, \mathcal{A})$ such that

$$M_n = E(X, \mathcal{A}_n) \ (n \in \mathbf{N})$$

and $M_\tau = E(X, \mathcal{A}_\tau) \ (\tau \in T)$.

Hence, $\lim_{\tau \in T} H(X_\tau, M_\tau) = \lim_{\tau \tau T} H(X_\tau, E(X, \mathcal{A}_\tau)) = 0.$

Consequently, $< X_n >$ is a regular uniform amart.

$(2 \Rightarrow 3)$ Let $< X_n >$ be a uniformly integrable and L_1 - bounded uniform amart. By (2) there is an $X \in L_1$ $(\mathbf{K}_c, \mathcal{A})$ such that

$$\lim_{n \to \infty} H(X_n, E(X, \mathcal{A}_n)) = 0.$$

Now fix $X^* \in \mathbf{B}^*$. It is not hard to check that the sequence $< \delta^*(X_n, X^*) >$ is a real-valued uniformly integrable and L_1 — bounded uniform amart. Thus by (9), it is convergent almost surely and in L_1 to $\delta^* (X, X^*)$ which proves (3).

The implication $(3 \Rightarrow 4)$ can be deduced from the fact that every sequence in L_1 $(\mathbf{K}_c, \mathcal{A})$ with values contained almost surely in δ U for some $\delta > 0$ where U is the closed unite ball of $\mathbf{B}$ is uniformly integrable and L_1 — bounded.

Finally $(4 \Rightarrow 1)$ is a special case of Theorem 6 in [6]. Thus the proof is complete

DEFINITION 4.2 *A sequence $<X_n>$ in $L_1(\mathbb{K}_c, \mathcal{A})$ is said to satisfy the Uhl's condition, if*

$\forall_{\varepsilon>0}$ $\exists$ *a convex compact subset C of $\mathbb{B}$ such that*

$$\forall_{\delta>1}\ \exists_{n_0}\ \exists_{A_0\in\mathcal{A}_{n_0}}\ P(A)\geqslant 1-\varepsilon\ \forall_{n\geqslant n_0}\ \forall_{A\in\mathcal{A}_n}$$

$$\text{if } A\subset A_{n_0} \text{ then } \int_A X_n dP \subset P(A)\, C+\delta U.$$

For A general Banach space $\mathbb{B}$ it was shown by Uhl /27/ that a martingale $<f_n>$ in $L_1(\mathbb{B}, \mathcal{A})$ is regular iff it is uniformly integrable, L_1—bounded and satisfies the Uhl's condition. Using the embedding mentioned in Section 1, it has been shown in [18] and [14] that the Uhl's result can be extended to martigales with close dball on convex compact values. For general martingales with closed convex values, the problem is still open. However, we get the following result

THEOREM 4. 3. *Every uniform amart in $L_1(\mathbf{K}_c, \mathcal{A})$ which is uniformly integrable, L_1 — bounded and satisfies* the Uhl's condition *is regular.*

Proof. Let $<X_n>$ be a uniform amart in $L_1(\mathbf{K}_c, \mathcal{A})$ which is uniformly integrable L_1 — bounded and satisfies condition of (Uhl). Then by Theorem 3. 4 there is a martingale $<M_n>$ in $L_1(\mathbb{K}_c, \mathcal{A})$ such that

$$\lim_{\tau\in T} H(X_t, M_\tau)=0$$

It is not hard to check that in this case $<M_n>$ is also uniformly integrable L_1 — bounded and satisfies the Uhl' s condition. Thus by /16/ it is regular, i.e. there is an $X\in L_1(\mathbb{K}_c, \mathcal{A})$ such that $M_n = E(X, \mathcal{A}_n)$ $(n\in \mathbf{N})$

Thus, in particular,

$$\lim_{n\to\infty} H(X_n, M_n)=\lim_{n\to\infty} H(X_n, E(X,\mathcal{A}_n))=0.$$

It means that the uniform amart $<X_n>$ is regular.

Acknowledgment,

I am indebted to Professor Hoang Tuy for his encouragements and invaluable comments.

Received May 15, 1981.

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