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THE MOD 2 EQUIVARIANT COHOMOLOGY ALGEBRAS OF CONFIGURATION SPACES

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1. INTRODUCTION

Let the symmetric group Σ_m of degree m act on the configuration space

$$F(\mathbf{R}^q, m) = \{(x_1,\dots, x_m) : x_i \in \mathbf{R}^q,\ x_i \neq x_j \text{ if } i \neq j,\ 1 \leqslant i, j \leqslant m\}$$

by permutations of the factors. The direct limit $F(\mathbf{R}^\infty, m)/\Sigma_m = \varinjlim_q F(\mathbf{R}^q, m)/\Sigma_m$ becomes a classifying space of Σ_m. So we have the commutative diagram

$$\text{(1.1)}\qquad \begin{array}{ccc} H^*(\Sigma_\infty) & \xrightarrow{\mathrm{Res}_1} & H^*(\Sigma_m) \\ \downarrow{\scriptstyle i(F,q)} & & \downarrow{\scriptstyle i(F,q)} \\ H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty) & \xrightarrow{\mathrm{Res}_2} & H^*(F(\mathbf{R}^q, m)/\Sigma_m) \end{array}$$

where $H^*(\Sigma_\infty) = \varprojlim_m H^*(\Sigma_m)$ and $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty) = \varprojlim_m H^*(F(\mathbf{R}^q, m)/\Sigma_m)$ are the cohomology Hopf algebras introduced essentially by Nakaoka [6]. Here and in what follows, the ring of coefficients is always assumed to be $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. According to the classical Steenrod's decomposition theorem, one knows that the homomorphisms Res_1, Res_2 are surjective, and according to Huỳnh Mùi [3; 10.8] so are the homomorphisms $i(F, q)$.

In [8] we have determined the (Hopf) algebra structure of $H^*(\Sigma_\infty)$ and given what we called the *Dickson generator system* for this algebra. As a consequence, the algebra $H^*(\Sigma_m)$ is computed by means of the epimorphism Res_1. The purpose of this paper is to determine the Hopf algebra structure of $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$, and further, the algebra structure of $H^*(F(\mathbf{R}^q, m)/\Sigma_m)$ by use of Res_2. As it is already observed in 1.1, we need to focus our study on the epimorphisms $i(F, q)$, more concretely, on the images of the Dickson generators of $H^*(\Sigma_\infty)$ under $i(F, q)$. Here we remind the reader that the *modules* (but not the *algebras*) $H^*(F(\mathbf{R}^q, m)/\Sigma_m)$, $1 \leqslant m \leqslant \infty$, are computed previously by a geometrical approach in [4], [3], and from [3], the epimorphisms $i(F, q)$ can be concretely expressed by means of the Nakamura's cellular decomposition of

$F(\mathbf{R}^q, m)$. Technically, the study of the epimorphisms $i(F, q)$ based on the study of the following commutative diagram

$$\begin{array}{ccc} H^*(\Sigma_{2^n}) & \xrightarrow{\text{Res}} & H^*(\Sigma_{2^n,2}) \\ \downarrow & & \downarrow \\ H^*(F(\mathbf{R}^q, 2^n)/\Sigma_{2^n} & \longrightarrow & H^*(M(q, n)) \end{array} \tag{1.2}$$

where $M(q, n)$ is the n—iterated wreath product of the projective spaces, which is embedded canonically in $F(\mathbf{R}^q, 2^n)/\Sigma_{2^n}$, and when $q \to \infty$ which becomes the classifying of $\Sigma_{2^n,2}$, the Sylow 2—subgroup of Σ_{2^n}.

The main tools to study this diagram will be the Nakamura's cellular decomposition, the Steenrod theory on cohomology of the wreath products and the Dickson and Huỳnh Mùi modular invariant theory (see [2]).

From our study on the algebra $H^*(F(\mathbf{R}^q, m)/\Sigma_m)$ we obtain the results of M. Nakaoka [7] when $q = \infty$, $m = 4$, of D. B. Fuks [1] when $q = 2$ and of the author [8] when $q = \infty$.

The details and further developments of this paper will appear elsewhere.

2. THE CONFIGURATION SPACES AND THE ITERATED WREATH PRODUCTS OF PROJECTIVE SPACES

Motivated by the Steenrod theory on cohomology of the wreath products of finite groups, we define the n-iterated wreath products $M(q, n)$ of projective spaces by induction on n as follows.

(2.1) $M(q, 0) = \{*\}$, the space consisting of exactly one point,

$$M(q, n) = M(q, n-1) \int \mathbf{P}^{q-1} = M(q, n-1)^2 \underset{E}{\times} S^{q-1},$$

where the group $E \cong \mathbf{Z}_2$ operates on $M(q, n-1)^2$ by permutations of the factors and on S^{q-1} by the antipodal map.

We are going to define the continuous embedding

$$i(q, n): M(q, n) \to F(\mathbf{R}^q, 2^n)/\Sigma_{2^n}. \tag{2.2}$$

To this end, we fix a positive number $\varepsilon < \frac{1}{3}$ and always consider S^{q-1} as the unit sphere of the Euclidean space $\mathbf{R}^q$. The map $i(q, 0) : M(q, 0) = \{*\} \to F(\mathbf{R}^q, 2^0)/\Sigma_{2^0} = \mathbf{R}^q$ is given by $i(q, 0)(*) = 0$. Suppose that $i(q, n-1)$ has been defined, further for $x \in M(q, n-1)$ we have

$$i(q, n-1)(x) = [i(q, n-1)_1(x), \ldots, i(q, n-1)_{2^{n-1}}(x)],$$

where the right side denotes the non-ordered collection of the distinct points $i(q, n-1)_1(x), \ldots, i(q, n-1)_{2^{n-1}}(x)$ in $\mathbf{R}^q$. Now we define $i(q,n) = [i(q, n)_1, \ldots, i(q, n)_{2^n}]$ by the formula

$$i(q, n)_j(t) = \begin{cases} \varepsilon i(q, n-1)_j(x) + z & \text{if} \quad j \leqslant 2^{n-1} \\ \varepsilon i(q, n-1)_{j-2^{n-1}}(y) - z & \text{if} \quad j > 2^{n-1}, \end{cases}$$

or $t = [x, y, z] \in M(q, n)$, where $x, y \in M(q, n-1)$, $z \in S^{q-1}$.

As easily seen, this map is well defined. When $q=2$, the subspace $i(2, n)$ $I(2, n)$ of $F(\mathbb{R}^2, 2^n)/\Sigma_{2^n}$ has been used in Fuks [1] by a different language.

.3 THEOREM. $i^*(q, n) : H^*F(\mathbb{R}^q, 2^n)/\Sigma_{2^n} \to H^*(M(q, n))$ *is a monomorphism or* $q \geqslant 1$, $n \geqslant 0$.

dea of the proof. Let us consider the restrictions $H^*(\Sigma_{2^n}) \to H^*(E^n)$, $H^*(\Sigma_{2^n,2}) \to H^*(E^n)$ from the groups Σ_{2^n} and $\Sigma_{2^n,2}$ to their special maximal elementary belian 2-subgroup E^n (see [2], [8]). Using Diagram 1.2, we have proved the heorem by means of invariants in $H^*(E^n)$ of the Weyl groups of E^n in Σ_{2^n} and $\Sigma_{2^n,2}$. These Weyl groups are respectively $GL_n = GL(n, Z_2)$ and $GL_{n,2}$, he subgroup consisting of all upper triangular matrices in GL_n with 1 in the liagonals (cf. [2]).

3. THE HOPF ALGEBRA $H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)$

Note that $F(\mathbb{R}^q, \infty)/\Sigma_\infty = \varinjlim_m F(\mathbb{R}^q, m)/\Sigma_m$ is not an H-space. However ts homology admits a structure of Hopf algebras. Besides this, the algebra $H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)$ is equipped with multiplicity such that

$$_mH_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty) = H_*(F(\mathbb{R}^q, m)/\Sigma_m;\ F(\mathbb{R}^q, m-1)/\Sigma_{m-1}).$$

The reader, who is not familiar with the notion of algebra with multiplicity, can refer to Nakamura [5]. Further, if $A = \bigoplus_{n \geqslant 0} {}_nA$ is such an algebra, then we put $A(m) = \bigoplus_{n \leqslant m} {}_nA$. Particularty, we have

$$H^*(F(\mathbb{R}^q, m)/\Sigma_m) = H^*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)(m).$$

Let $W_{n,s}$, $0 \leqslant s < n$, denote the Stiefel–Whitney characteristic class of dimension $2^n - 2^s$ of the vector fibre bundle

$$\mathbb{R}^{2^n} \times_{\Sigma_{2^n}} F(\mathbb{R}^q, 2^n) \to F(\mathbb{R}^q, 2^n)/\Sigma_{2^n},$$

where the group Σ_{2^n} operates on $\mathbb{R}^{2^n}$ by permutations of the coordinates. Obviously, the canonical homomorphism $i(F, q) : H^*(F(\mathbb{R}^\infty, 2^n)/\Sigma_{2^n}) \to H^*(F(\mathbb{R}^q, 2^n)/\Sigma_{2^n})$ brings $W_{n,s}$ to $W_{n,s}$ for $0 \leqslant s < n$, $q \geqslant 2$

According to Huỳnh Mùi [2], (see also [8 ; §2], we have $H^*(F(\mathbb{R}^\infty, 2^n)/\Sigma_{2^n}) = H^*(\Sigma_{2^n}) = \operatorname{Ker} \operatorname{Res}(E^n, \Sigma_{2^n}) \oplus Z_2[W_{n,0}, \dots, W_{n,n-1}]$. By means of Theorem 2.3 we get

$$(3.1) \qquad H^*(F(\mathbb{R}^q, 2^n)/\Sigma_{2^n}) = X \oplus \frac{Z_2[W_{n,0}, \dots, W_{n,n-1}]}{\left(\prod_{s=0}^{n-1} W_{n,s}^{h_s};\ h_0 > 0,\ \sum_s h_s = q\right)}$$

where $X = i(F, q)(\operatorname{Ker} \operatorname{Res}(E^n, \Sigma_{2^n}) \oplus Z_2[W_{n,1}, \dots, W_{n,n-1}])$.

This allows us to define (compare with [8; 2.4 and 2.5].

3.2 DEFINITION. (i) For each $K = (k_0, \ldots, k_{n-1}) \in J^+(q) = \{(h_0, \ldots, h_{m-1});\ m > 0, h_0 > 0, h_i \geqslant 0, \sum_i h_i < q\}$ we define the Dickson element $D_K = D_{k_0, \ldots, k_{n-1}} \in H_*(F(\mathbb{R}^q, 2^n)/\Sigma_{2^n}$ by the conditions

$$\langle D_{k_0, \ldots, k_{n-1}}, X \rangle = 0$$

$$\langle D_{k_0, \ldots, k_{n-1}}, \prod_{s=0}^{n-1} W_{n,s}^{h_s} \rangle = \begin{cases} 1 & (k_0, \ldots, k_{n-1}) = (h_0, \ldots, h_{n-1}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. The image of D_K under the canonical monomorphism $H^*(F(\mathbb{R}^q, 2^n)/\Sigma_{2^n}) \to H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)$ will be denoted simplyby D_K.

(ii) We define D_K for $K \in J(q) = \{(h_0, \ldots, h_{m-1}) \neq 0;\ m > 0,\ h_i \geqslant 0, \sum_i h_i < q\}$ by putting

$$D_{\underbrace{0, \ldots, 0}_{s}, k_0, \ldots, k_{n-1}} = D^{2^s}_{k_0, \ldots, k_{n-1}}.$$

Using Theorem 2.5 in Nguyễn Hữu Việt Hưng [8] we obtain

3.3 THEOREM. *The structures of* $H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)$ *considered as a Hopf algebra and as an algebra with multiplicity are described as follows.*

(i) $$H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty) = Z_2[D_K;\ K \in J^+(q)]$$

as algebras with multiplicity, where the multiplicity of $D_{k_0, \ldots, k_{n-1}}$ *is given to be* 2^n. *So we have the isomorphism of* Z_2*—modules for arbitrary* m

$$H_*(F(\mathbb{R}^q, m)/\Sigma_m) = Z_2[D_K;\ K \in J^+(q)](m).$$

The basis of this module consisting of all monomials in $Z_2[D_K;\ k \in J^+(q)]$ *of multiplicity* $\leqslant m$ *will be called the Dickson basis.*

(ii) *The comultiplication* Δ *of* $H_*(F(\mathbb{R}^q, \infty)/\Sigma_\infty)$ *satisfies the formula*

$$\Delta D_{k_0, \ldots, k_{n-1}} = \sum_{l_i + m_i = k_i} D_{l_0, \ldots, l_{n-1}} \otimes D_{m_0, \ldots, m_{n-1}}.$$

for $(k_0, \ldots, k_{n-1}) \in J^+(q)$, $l_i, m_i \geqslant 0$, $0 \leqslant i < n$.

4. THE ALGEBRAS $H^*(F(\mathbb{R}^q, m)/\Sigma_m)$

We determine these algebras here by the argument similar to that used in [8; § 3] for the algebras $H^*(\Sigma_m)$.

For each $(\mathcal{H}, T) = (H_1, \ldots, H_r) \times (t_1, \ldots, t_r) \in J(q)^r \times \mathbf{N}^r$, we set

(4.1) $$W_T^{\mathcal{H}} = W^{H_1, \ldots, H_r}_{t_1, \ldots, t_r} = (D_{H_1}^{t_1} \ldots D_{H_r}^{t_r})^* \in H^*(F(\mathbb{R}^q, \infty)/\Sigma_\infty),$$

where the dual is taken via the Dickson basis. For $(\mathcal{H}, T) = (H_1, \dots, H_r) \times (\underbrace{1, \dots, 1}_{r})$ we write symply $W^{\mathcal{H}} = W^{H_1, \dots, H_r}$ instead of $W^{\mathcal{H}}_T$. Notice that, if $H = (h_o, \dots h_{n-1}) \in J(q)$ we have

(4.2)
$$W^H \Big| F(\mathbf{R}^q, 2^n)/\Sigma_{2n} = \prod_{s=o}^{n-1} W_{n,s}^{h_s}.$$

As easily seen, $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$ admits the additive basis consisting of the elements

(4.3)
$$W^{\mathcal{H}}_T, (\mathcal{H}, T) = (H_1, \dots, H_r) \times (t_1, \dots, t_r) \in J^+(q)^r \times N^r,$$

with $H_1 < \dots < H_r$, $r \geqslant o$. Here $<$ is the order in J(q) defined by length and by lexicographic order for elements of the same length, where by a length of $H = (h_o, \dots, h_{n-1})$ we mean the number $l(H) = n$.

Again, the above basis is called the *Dickson basis* of $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$.

We state the main result of this note.

4.4 THEOREM. *We have the isomorphism of algebras*

$$H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty) = \frac{Z_2[W^H;\ H \in J_{odd}(q)]}{((W^H)^{2^{h(q,H)}};\ H \in J_{odd}(q))}$$

Here $J_{odd}(q) = \{(h_o, \dots, h_{n-1}) \in J(q) : n > o,$ *there exists* i *such that* h_i *are odd*$\}$ *and* $h(q, H) = \min \{ h \in \mathbf{N} : 2^h(h_o + \dots + h_{n-1}) \geqslant q \}$ *for* $H = (h_o, \dots, h_{n-1})$.

The comultiplication $\triangle$ of the Hopf algebra $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$ can be described via the Dickson basis by the formula

$$\triangle W^{H_1, \dots, H_r}_{t_1, \dots, t_r} = \sum_{u_i + v_i = t_i} W^{H_1, \dots, H_r}_{u_1, \dots, u_r} \otimes W^{H_1, \dots, H_r}_{v_1, \dots, v_r}$$

for $H_1, \dots, H_r \in J^+(q)$ and $H_1 < \dots < H_r$.

Remember that, in [8; 3.5] we have defined for $(\mathcal{H}, T) \in J(\infty)^r \times \mathbf{N}^r$ $(\mathcal{K}, U) \in J(\infty)^s \times N^s$ the subset $\mathcal{H}, T) \vee (\mathcal{K}, U) \subset \coprod_{t \geqslant 0} J(\infty)^t \times \mathbf{N}^t$. Now we put

(4.5)
$$(\mathcal{H}, T)^q_\vee (\mathcal{K}, U) = \big((\mathcal{H}, T) \vee (\mathcal{K}, U)\big) \cap \big(\coprod_{t \geqslant 0} J(q)^t \times \mathbf{N}^t\big).$$

From Theorem 3.5 we obtain (compare with [8; 3.6])

4.6 LEMMA. *For* $(\mathcal{H}, T), (\mathcal{K}, U) \in \coprod_{t \geqslant 0} J(q)^t \times \mathbf{N}^t$ *we have in* $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$

$$W^{\mathcal{H}}_T \cdot W^{\mathcal{K}}_U = \sum_{(\mathcal{X}, Y)} W^{\mathcal{X}}_Y,$$

where the summation runs over the representatives of $\Sigma_* -$ orbis of $(\mathcal{H}, T^q_v)(\mathcal{K}, U)$. *Here* $\Sigma'_* = \coprod_{r\geqslant 0} \Sigma_r$ *acts on* $\coprod_{r\geqslant 0} J(q)^r \times \mathbf{N}^r$ *by*

$$\sigma(H_1,\ldots,H_r)\times(t_1,\ldots, t_r) = (H_{\sigma^{-1}(1)},\ldots, H_{\sigma^{-1}(r)}) \times (t_{\sigma^{-1}(1)},\ldots, t_{\sigma^{-1}(r)}),\ \sigma \in \Sigma_r.$$

To determine the algebras $H^*(F(\mathbf{R}^q, m)/\Sigma_m)$ we need

4.7 DEFINITION. (i) The depth θ (z) of an element z in the Dickson basis of $H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$ is defined by the formulas

$$\theta(1) = 0; \quad \theta\begin{pmatrix}H_1,\ldots, H_r\\ W_{t_1},\ldots, t_r\end{pmatrix} = \sum_{i=1}^{r} t_i 2^{l(H_i)}.$$

(ii) Suppose $z = \sum_{(\mathcal{H},T)} W_T^{\mathcal{H}}$, the linear decomposition of $z \in H^*(F(\mathbf{R}^q, \infty/\Sigma_\infty)$

via the Dickson basis, then we put $\theta(z) = \min_{(\mathcal{H}, T)} \theta(W_T^{\mathcal{H}})$.

Notice that, by means of Lemma 4.6, we can compute the depth of arbitrary $z \in H^*(F(\mathbf{R}^q, \infty)/\Sigma_\infty)$.

4.8. THEOREM. *Let* $J_{odd}(q, m) = \{H \in J_{odd}(q) : 2^{(lH)} \leqslant m\}$, *then we have the isomorphism of algebras for arbitrary natural number m*

$$H^*(F(\mathbf{R}^q, m)/\Sigma_m) \cong Z_2[W_H; H \in J_{odd}(q, m)]/I(q, m),$$

where $I(q, m)$ *denotes the ideal gennerated by* $\{(W_H)\, 2^{(h(q,H)}; H \in J_{odd}(q, m)\}$ *and* $\{z \in Z_2[W_H; H \in J_{odd}(q, m)]; \theta(z) > m\}$.

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