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AN APPROXIMATE SOLUTION TO THE FREE BOUNDARY VALUE PROBLEM FOR A FLUID FLOW THROUGH A DAM WITH VERTICAL LAYERS

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1. INTRODUCTION

In this paper, we consider the free boundary value problem related to a stationary flow between two water reservoirs of different levels which are seprated by a dam consisting of two vertical layers. A numerical analytic solution of this problem is received by the method of right lines, and an approximate free frontier is directly constructed. Our problem is more general than that of [3] and the proposed solution method is quite different from [1] and [2].

The problem is stated as follows.

Let

$$D = \{(x, y) \in R^2 \mid 0 < x < 1,\ 0 < y < Y_1\}$$

be a plane rectangle representing the section of a dam, where Y_1 is the water level of the left reservoir. Denote by Y_2 $(Y_2 < Y_1)$ the water level of the right reservoir. The dam consists of two vertical layers separated by the line $x = a$ $(0 < a \leqslant 1)$. Let k be the coefficient of permeability of the dam, which is assumed to be constant in each layer. Denote by Ω the part of the dam that is saturated with liquid and by $\mathcal{G}$ the inter–face (the so–called free boundary) between the wet region and the dry region of the dam. The equation of the free boundary is

$$y = \varphi(x), \quad 0 < x < 1,$$

where $\varphi(x)$ is a strictly decreasing function and

$$\varphi(0) = Y_1, \quad \varphi(1) \geqslant Y_2.$$

It has been shown that all characteristic quantities of our filtration problem can be found by solving the following non – linear boundary value problem (cf. [1]):

$$\operatorname{div}(k \operatorname{grad} u) = k\chi_\Omega, \quad (x, y) \in D, \tag{1}$$

$$\begin{gathered} u = g, \qquad (x, y) \in \partial D, \\ u \mid_{a-0} = u \mid_{a+0} \\ k_1(D_x u) \mid_{a-0} = k_2(D_x u) \mid_{a+0} \\ u > 0 \text{ if } (x, y) \in \Omega ; \; u = 0 \text{ if } (x, y) \in D \setminus \Omega, \end{gathered} \tag{2}$$

where χ_Ω is the characteristic function of the region Ω; k is the coefficient of permeability of the dam, which has the following form

$$k = k(x) = \begin{cases} k_1 \text{ if } 0 < x \leqslant a, \; 0 < y < y_1 \\ k_2 \text{ if } a < x < 1, \; 0 < y < y_1 \end{cases}$$

where k_1 and k_2 are positive constants. The function g is determined by the formula

$$g(x, y) = \begin{cases} \frac{1}{2} Y_1^2 - \frac{q}{k_1} x & \text{if } 0 \leqslant x \leqslant a, \; y = 0 \\ \frac{1}{2} Y_2^2 + \frac{q}{k_2}(1 - x) & \text{if } a < x \leqslant 1, \; y = 0 \\ \frac{1}{2}(Y_1 - y)^2 & \text{if } x = 0, \; 0 \leqslant y \leqslant Y_1 \\ \frac{1}{2}(Y_2 - y)^2 \chi(Y_2) & \text{if } x = 1, \; 0 \leqslant y \leqslant Y_1 \\ 0 & \text{if } y = Y_1, \; 0 \leqslant x \leqslant 1, \end{cases}$$

where q is the discharge of the dam, which is calculated by the expression

$$q = \frac{k_1 k_2 (Y_1^2 - Y_2^2)}{2[a k_2 + (1-a) k_1]}$$

and $\chi(\xi)$ is the characteristic function for the variable y:

$$\chi(\xi) = \begin{cases} 1 & \text{if } 0 < y < \xi \\ 0 & \text{if } \xi < y < Y_1 \end{cases}$$

2. APPROXIMATE SOLUTION TO THE NON LINEAR BOUNDARY VALUE PROBLEM (1) (2)

Now we apply the method of right lines to find an approximate solution of the problem (1), (2). To this end let us introduce the family of lines which are parallel to the axis y:

$$x_i = ih, \quad h = \frac{1}{n+1}, \quad i = 0, 1, \ldots, n + 1$$

where n is an integer number. We suppose that the separating line of two vertical layers coincides with the right line $x = x_{n_1}$, e.g.

$$x_{n_1} = n_1 h = a,$$

where n_1 is an integer number, such that

$$0 < n_1 \leqslant n + 1$$

Using the integral interpolation for the equation (1) with respect to the variable y we have the difference homogeneous scheme cf. [4])

$$\frac{1}{h}\left(a_{i+1}\frac{u_{i+1}-u_i}{h} - a_i \frac{u_i - u_{i-1}}{h}\right) + d_i \frac{d^2u_i}{dy^2} = d_i \chi(y_i), \quad (3),$$

where

$u_i = u_i(y) = u(x_i, y)$, $y_i = \varphi(x_i)$,

$$a_i = \left(\frac{1}{h}\int_{x_i-1}^{x_i} \frac{1}{k(x)}dx\right)^{-1} d_i = \frac{1}{h}\int_{x_i-1/2}^{x_i+1/2} k(x)dx,$$

with

$$x_{i\pm\frac{1}{2}} = x_i \pm \frac{1}{2}h$$

Using (3) and (2') we obtain the following system of difference differential equations:

$$k_1 \frac{d^2u_1}{dy^2} + \frac{k_1}{h^2}(-2u_1 + u_2) = k_1 \chi(y_1) - \frac{k_1}{h^2}u_o(y),$$

$$k_1 \frac{d^2u_2}{dy^2} + \frac{k_1}{h^2}(u_1 - 2u_2 + u_3) = k_1 \chi(y_2)$$

. . . .

$$k_1 \frac{d^2u_{n_1-1}}{dy^2} + \frac{k_1}{h^2}(u_{n_1} - 2u_{n1-1} + u_{n_1}) = k_1 \chi(y_{n_1-1})$$

$$\frac{k_1+k_2}{2}\frac{d^2u_{n1}}{dy^2} + \frac{1}{h^2}k_1 u_{n_1-1} - (k_1+k_2)u_{n_1} + k_2 u_{n_1+1} = \frac{k_1+k_2}{2}\chi(y_{n_1})$$

$$k_2 \frac{d^2 u_{n_1+1}}{dy^2} + \frac{k_2}{h^2}(u_{n_1} - 2u_{n_1+1} + u_{n_1+2}) = k_2 \chi(y_{n_1+1})$$

. . . .

$$k_2 \frac{d^2u_{n-1}}{dy^2} + \frac{k_2}{h^2}(u_{n-2} - 2u_{n-1} + u_n) = k_2 \chi(y_{n-})_1$$

$$k_2 \frac{d^2u_n}{dy^2} + \frac{k^2}{h^2}(u_{n-1} - 2u_n) = k_2 \chi(y_n) - \frac{k_2}{h^2}u_{n+1}(y),$$

where

$$u_o(y) = \frac{1}{2}(Y_1 - y)^2, \qquad u_{n+1}(y) = \frac{1}{2}(Y_2 - y)^2 \chi(Y_2).$$

Let us introduce two vectors

$$\vec{u} = \{u_1, u_2, \ldots, u_n\}, \qquad \vec{f} = \{f_1, f_2, \ldots, f_n\}$$

with

$$f_1 = \chi(y_1) - \frac{1}{h^2} u_o(y), \; f_n = \chi(y_n) - \frac{1}{h^2} u_{n+1}(y), \; f_i = \chi(y_i), \; i = 2, 3, \ldots, n.1.$$

and two matrices of order n:

$$\rho = \operatorname{diag}\left[k_1, k_1, \ldots, k_1, \frac{k_1 + k_2}{2}, k_2, \ldots, k_2\right]$$

$$\overset{\displaystyle n_1^1 -}{}$$

$$T = \begin{bmatrix} O & k_1 & & & & & & & \\ k_1 & O & k_1 & & & & & O & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & k_1 & O & k_1 & & & & \\ & & & k_1 & O & k_2 & \cdot & \cdot & \cdot \\ & O & & & k_2 & O & k_2 & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & k_2 & O & k_2 & \\ & & & & & & k_2 & O & \end{bmatrix} n_1^{\,1}-$$

Then we can write the system (4) in the vector form

$$\frac{d^2 \vec{u}}{dy^2} + \frac{1}{h^2} (\rho^{-1} T - 2E) \vec{u} = \vec{f}. \tag{5}$$

We can show that (cf. [5])

$$\rho^{-1} T = P \Lambda P^*, \; P^*_\rho = P^{-1} \tag{6}$$

where p^* (resp. p^{-1}) is the transposition (resp. the inversion) of matrix p, and

$$\Lambda = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \; P = [a_{ij}]^n_{i,j=1},$$

$$a_{ij} = C_j \begin{cases} \sin i\theta_j, & i = 1, 2, \ldots, n_1 \\ \sigma_j \sin (n + 1 - i)\theta_j) & i = n_1, n_1+1, \ldots, n, \end{cases}$$

$$\lambda_j = 2 \cos \theta_j, \; j = 1, 2, \ldots, n,$$

and θ_j ($j = 1, 2, \ldots, n$) are solutions of the equation

$$\sin (n + 1)\theta + \frac{k_1 - k_2}{k_1 + k_2} \sin (n + 1 - 2n_1)\theta = 0$$

in the interval (o, π). Those coefficients σ_j and C_j are determined by formulas

$$\sigma_j = \begin{cases} \dfrac{\sin n_1 \theta_j}{\sin(n+1-n_1)\theta_j} & \text{if } \sin (n+1-n_1)\theta_j \neq 0 \\ -\dfrac{k_1 \cos n_1 \theta_j}{k_2 \sin(n+1-n_1)\theta_j} & \text{if } \sin (n+1-n_1)\theta_j = 0 \end{cases}$$

$$C_j = \left\{k_1\left[\frac{n_1}{2} - \frac{\cos(n_1-1)\theta_j \sin n_1\theta_j}{2\sin\theta_j}\right] + \frac{k_1+k_2}{2}\sin^2 n_1\theta_j + \right.$$

$$\left. + \sigma_j^2 k_2\left[\frac{n+1-n_1}{2} - \frac{\cos(n-n_1)\theta_j \sin(n+1-n_1)\theta_j}{2\sin\theta_j}\right]\right\}^{-\frac{1}{2}}$$

In view of (6) the equation (5) can be written as follows

$$\frac{d^2\vec{u}}{dy^2} + \frac{1}{h^2} P(\Lambda - 2E)P_\rho^*\vec{u} = \vec{f}. \tag{7}$$

Now multiplying both sides of the equation (7) by P and using the following notations

$$\vec{v} = (v_k)_{k=1}^n = P_\rho^*\vec{u},\ \vec{g} = P_\rho^*\vec{f},$$

we obtain

$$\frac{d^2v_k}{dy^2} - \nu_k^2 v_k = g_k\ (k = 1, 2, ..., n), \tag{8}$$

where

$$\nu_k = \frac{2}{h}\sin\frac{\theta_K}{2},$$

$$g_k(y) = \frac{C_k \sin\theta_K}{2h^2}[(k_1+k_2\sigma_k\chi(Y_2))y^2 - 2(k_1Y_1+k_2\sigma_kY_2\chi(Y_2))y + (k_1Y_1^2 +$$

$$+ k_2\rho_k Y_2^2\chi(Y_2)] + \sum_{i=1}^n a_{ik}\rho_i\chi(y_i). \tag{9}$$

In order to solve the equation (8) we need the following lemma

LEMMA 1. *Consider the differential equation*

$$\frac{d^2v}{dy^2} - a^2v = g(y), \qquad 0 < y < b, \tag{10}$$

where

$g(y) = \alpha_i y^2 + \beta_i y + \gamma_i,\ y_i - 1 < y < y_i,\ i = 1, 2, ..., M+1,\ y_0 = 0,\ y_{M+1} = b.$

The solution of the equation (10) in the class $C^1(0, b) \cap W_2^2(0, b)$ is

$$v(y) = Ae^{ay} + Be^{-ay} - \frac{1}{2a}R(y), \tag{10'}$$

where A and B are arbitrary constants and the function R(y) is determined by the formula

$$R(y) = 2\alpha_i y^2 + 2\beta_i y + \frac{4\alpha_i}{a^2} + 2\gamma_i +$$

$$+\sum_{j=1}^{i-1} e^{a(y-y_j)}\left[\left(\alpha_j-\alpha_{j+1}\right)\left(y_j^2+\frac{2y_j}{a}+\frac{2}{a^2}\right)(\beta_j-\beta_{j+1})\left(y_j+\frac{1}{a}\right)+\gamma_j-\gamma_{j+1}\right]+$$

$$+ e^{-a(y-y_j)}\left[(\alpha_j-\alpha_{j+1})\left(y_j^2-\frac{2y_j}{a}-\frac{2}{a^2}\right)+(\beta_j-\beta_{j+1})\left(y_j-\frac{1}{a}\right)+\gamma_j-\gamma_j+1\right] \tag{10''}$$

on the interval $y_i \leqslant y \leqslant y_i$, *and sum in* (10") *is* taken to be zero if i = 1.

Proof. Note that $Ae^{at}+Be^{-at}$ is a general solution of the homogeneous equation and $-\frac{1}{2a^2}R(y)$ is a particular solution of the non-homogeneous equation (10). Now we verify the continuity of R(y) at the points $y=y_i$, $i=1, 2, \ldots, M$.:

$$R\Big|_{y_i-0}=2\alpha_i y_i^2+2\beta_i y_i+\frac{4\alpha_i}{a^2}+2\gamma_i+\sum_{j=1}^{i-1}\{\ldots\}_{y=y_i},$$

$$R\Big|_{y+0}=2\alpha_{i+1}y_i^2+2\beta_{i+2}y_i+\frac{4\alpha_{i+1}}{a^2}+2\gamma_{i+1}+\sum_{j=1}^{i-1}\{\ldots\}_{y=y_j^+}$$

$$+2(\alpha_i-\alpha_{i+1})\left(y_i^2+\frac{2}{a^2}\right)+2(\beta_i-\beta_{i+1})\,x_i+2(\gamma_i-\gamma_{i+1})$$

$$=2\alpha_i y_i^2+2\beta_i y_i+\frac{4\alpha_i}{a^2}+2\gamma_i+\sum_{j=1}^{i-1}\{\ldots\}_{y=y}=R\big|_{y_i-0}$$

The continuity of $\frac{dR}{dy}$ at the points $y=y_i$, $i=1, 2, \ldots, M$ is verified analogously. Finally, due to (10) we have

$$\frac{d^2R}{dy^2}\in L^2(0, b). \qquad \text{Q.E.D}$$

We remark that the sipecial case of Lemma 1, when $M=1$, has been considered in [3].

THEORME 3. *The numerical analytic solution of the problem* (1), (2) *by the method of right lines has the following form:*

$$u_k(y)=\sum_{j=1}^{n}\frac{\operatorname{Sh}\nu_j y}{\operatorname{Sh}\nu_j Y_1}a_{kj}\sum_{i=1}^{n}q_i a_{ji}-\sum_{j=1}^{n}\frac{\operatorname{Sh}\nu_j(y-\nu_1)}{\operatorname{Sh}\nu_j Y_1}a_{kj}\sum_{j=1}^{n}t_i\alpha_{ij} \tag{11}$$
$$-Q_k(y)+S_k(y),$$

where

$$Q_k(y)=\sum_{j=1}^{n}\frac{1}{\nu_j^2}a_{kj}\sum_{i=1}^{n}a_{ij}\rho_i.\begin{cases}1 & \text{if } 0\leqslant y\leqslant y_j\\ \operatorname{ch}\nu_j(y-y_i) & \text{if } y_i<y\leqslant Y_1,\end{cases} \tag{12}$$

$$S_k(y)=\frac{1}{4}k_1(y_1-y)^2\sum_{j=1}^{n}Q_{kj}C_j\operatorname{ctg}\frac{\theta_d}{2}+\frac{1}{2}\sum_{j=1}^{n}\frac{k_1+k_2\sigma_j}{\nu_j^2}a_{Kj}C_j\operatorname{ctg}\frac{\theta_i}{2}+$$

$$+\begin{cases}\frac{1}{4}k_2(Y_2-y)^2\sum_{j=1}^{n}\alpha_{Kj}C_j\sigma_j\operatorname{ctg}\frac{\theta_j}{2}, & 0\leqslant y\leqslant Y_2,\\ -\frac{k_2}{2}\sum_{j=1}^{n}\frac{\sigma_j}{\sigma_j^2}a_{Kj}c_j\operatorname{ctg}\frac{\theta_j}{2}+\frac{1}{2}\sum_{j=1}^{n}\frac{\sigma_j}{\nu_j^2}ak_jc_i\operatorname{ctg}\frac{\theta_j}{2}\frac{\theta_j}{2}\operatorname{ch}\nu_j(y-Y_2), & Y_2\leqslant y\leqslant Y_1\end{cases} \tag{13}$$

$$q_l = \sum_{j=1}^{n} \frac{1}{\nu_j^2} a_{jl} \sum_{i=1}^{n} a_{ij}\rho_i ch\nu_j(Y_1 - y_i) - \frac{1}{2} \sum_{j=1}^{n} \frac{C_j}{\nu_j^2} a_{jl} \rho_j ctg \frac{\theta_j}{2} \times$$

$$\times [k_1 + k_2\sigma_j ch\nu_j(Y_1 - Y_2)], \quad (14)$$

$$t_l = -\frac{1}{4} \sum_{j=1}^{n} a_j l \rho_j C_j ctg \frac{\theta_j}{2} \Big(k_1 Y_1^2 + k_2 Y_2^2 + \frac{2}{\nu_j^2}(k_1 + k_2\sigma_j) +$$

$$+ \sum_{j=1}^{n} \frac{1}{\nu_j^2} a_{ji}\rho_j \sum_{i=1}^{n} a_{ij}\rho_j + \sum_{j=1}^{n} a_{ji}\rho_j \sum_{i=1}^{n} a_{ij}\rho_i u_i(0). \quad (15)$$

Proof. Applying Lemma 1 to the equation (8) yields

$$v_k(y) = A_{kl}\ e^{\nu_k y} + B_{kl}\ ^{-\nu_k y} - \frac{1}{\nu 2_K^2} R_k(y),\ k = 1, 2, \ldots, n, \quad (16)$$

where A_K and B_K are constants and

$$-\frac{C_k \sin\theta_k}{h^2}\left[k_1(Y_1 - y)^2 + k_2\sigma_k(Y_2 - y)^2 + \frac{2}{\nu_k^2}(k_1 + k_2\sigma_k)\right] + 2\sum_{i=1}^{n} a_{ik}\rho_i,$$

$$\text{if } 0 \leqslant y \leqslant Y_2,$$

$$-\frac{k_1 C_k \sin\theta_k}{h^2}\left[(Y_1 - y)^2 + \frac{2}{\nu_k^2}\right] - \frac{2k_2 S_k \sin\theta_k}{h^2 \nu_k^2} ch\nu_k(y - Y_2) +$$

$$+ 2\sum_{i=1}^{n} a_{ik}\rho_i. \begin{cases} 1 \text{ if } Y_2 < y \leqslant y_i, \\ ch\ \nu_k(y - y_i) \text{ if } y_i < y \leqslant Y_1. \end{cases}$$

For determining those constants A_k and B_k we use the condition in horizontal lines $y = 0$ and $y = y_1$:

$$\begin{aligned} A_k + B_k &= \frac{1}{2\nu_k^2} K_k(o) + \sum_{i=1}^{n} a_{ik}\ \rho_i\ u_i\ (o), \\ A_k e^{\nu_k Y_1} + B_K e^{-\nu_k Y_1} &= \frac{1}{2\nu_K^2} R_k(Y_1), \end{aligned} \quad (17)$$

From this it follows

$$\begin{aligned} A_k &= \frac{1}{4\nu_k^2\, Sh\nu_k Y_1}[R_k(Y_1) - e^{-\nu_k Y_1} R_k(o)] - \frac{e^{\nu_k Y_1}}{2Sh\ \nu_k Y_1} \sum_{i=1}^{n} a_{ik}\ \rho_i\ u_i(o), \\ B_k &= -\frac{1}{4\nu_k^2\, Sh\nu_k\ Y_1}[R_k(Y_1) - e^{\nu_k Y_1} R_k(0)] + \frac{e^{\nu_k Y_1}}{2Sh\nu_k\ Y_1} \sum_{i=1}^{n} a_{ik}\ \rho_i\ u_i(o), \end{aligned} \quad (18')$$

where

$$u_i(o) = \begin{cases} \frac{1}{2} Y_1^2 - \frac{q}{k_1} ih & \text{if } \ i = 1, 2, \ldots, n_1 \\ \frac{1}{2} Y_2^2 + \frac{q}{k_2}(1 - ih) & \text{if } \ i = n_1 + 1, \ldots, n, \end{cases}$$

Substituting (18) into (16) we find that

$$vk(y) = \frac{Sh\,\nu_k\,y}{Sh\,\nu_k\,y_1}\cdot\frac{1}{2v_k^2}R_k(Y_1) - \frac{Sh\,\nu_k\,(y-y_1)}{Sh\,\nu_k\,y_1}\left[\frac{1}{2v_k^2}Rk(o) + \right.$$

$$\left. + \sum_{i=1}^{n} a_i k\,\rho_i\,u_i\,(o)\right] - \frac{1}{2v_k^2}Rk(y). \qquad (19)$$

Finally, using the inverse transformation $\vec{u} = P\vec{v}$ we obtain the formula (11).

PROPOSITION 3. *In the special case of Theorem 2, when the dam is homogeneous,* e.g.

$$k_1 = k_2 = 1, \qquad n_1 = n + 1, \qquad (20)$$

the formula (11) *takes the following form;*

$$uk(y) = 2h\sum_{j=1}^{n}\frac{Sh\,\nu_j\,y}{Sh\,\nu_j\,y_1}\sin\pi\,kjh\sum_{i=1}^{n} q_i\,a_{ji} - Qk(y) + Sk(y), \qquad (21)$$

where

$$Qk\,(y) = 2h\sum_{j=1}^{n}\frac{1}{v_j^2}\mathrm{Sin}\pi kjh\sum_{i=1}^{n}\sin\pi ijh.\begin{cases} 1, & 0 \leqslant y \leqslant y_i, \\ ch\nu_j\,(y-y_i) & y_i < y \leqslant Y_1, \end{cases} \qquad (22)$$

$$Sk(y) = \frac{1}{2}\left[(Y_1-y)^2 + kh\right](1-kh) +$$

$$+\begin{cases} \frac{1}{2}(Y_2-y)^2\,kh, & 0 \leqslant y \leqslant Y_2 \\ -\frac{1}{6}(1-k^2h^2)kh - h\sum_{j=1}^{n}\frac{(-1)^j}{v_j^2}\sin\pi\,kjh\,ctg\,\frac{\pi jh}{2}ch\nu_j\,(y-Y_2), & Y_2 < y \leqslant Y_1, \end{cases} \qquad (23)$$

$$q_1 = 2h\sum_{j=1}^{n}\frac{1}{v_j^2}\sin\pi\ jlh\sum_{i=1}^{n}\sin\pi\ ijhch\nu_j\,(Y_1-y_i) -$$

$$-h\sum_{i=1}^{n}\frac{1}{v_j^2}\sin\pi\,jlhctg\,\frac{\pi jh}{2}\left(1-(-1)^j ch\nu_j\,(Y_1-Y_2)\right) \qquad (24)$$

Proof. In this case it is easy to verify that

$$\theta_j = \pi\,jh, \quad \nu_j = \frac{2}{h}\sin\frac{\pi jh}{2} \quad C_j = \sqrt{2h},$$

$$Q_{ij} = \sqrt{2h}\sin\pi\,ijh; \quad \rho_j = 1, \quad j = 1, 2, \ldots, n. \qquad (25)$$

Due to (25) and the relation

$$\sum_{j=1}^{n}\sin\frac{\pi kj}{n+1} = \frac{1+(-1)^{j-1}}{2}\,ctg\,\frac{\pi k}{2(n+1)} \qquad (26)$$

we can verify the correctness of formulas (22), (23) and (24). From equalities (25), (26) and usingsome trigonometrical relations in [3] it follows that

$$t_l = 0,\ l = 1, 2, \ldots, n \tag{27}$$

Substituting (22) — (25) and (27) into (11) we obtain (21).

Q.E.D.

We remark that Proposition 3 has been considered in [3].

3. DETERMINATION OF THE FREE FRONTIER

The representation formula for the solution (11) contains unknown parameters y_k $(k = 1, 2, \ldots, n)$ which determine the free frontier. Let us find these parameters now. Denote by $\Phi_k(y_1, y_2, \ldots, y_k)$ the value of the function $U_k(y)$ at the point y_k $(k = 1, 2, \ldots, n)$. Then we have

$$\Phi_k(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{n} \frac{\mathrm{Sh}\nu_j y_k}{\mathrm{Sh}\nu_j Y_1} a_{kj} \sum_{i=1}^{n} q_i a_{ji} +$$

$$+ \sum_{j=1}^{n} \frac{\mathrm{sh}\nu_j (Y_1 - y_k)}{\mathrm{Sh}\nu_j Y_1} a_{kj} \sum_{i=1}^{n} t_i a_{ji} - \sum_{j=1}^{n} \frac{1}{Y_j^2} a_{kj} [\sum_{i=1}^{k} a_{ij} S_i +$$

$$+ \sum_{i=k+1}^{n} a_{ij} S_i \mathrm{ch}\nu_j (y_k - y_i)] + \frac{k_2}{2} \sum_{j=1}^{n} \frac{\sigma}{\nu_j^2} a_{Kj} c_j \mathrm{ctg} \frac{\theta_j}{2} \mathrm{ch} Y_j' (y_k - Y_2) +$$

$$+ \frac{k_1}{4} (Y_1 - y_k)^2 \sum_{j=1}^{n} a_{kj} C_j \mathrm{ctg} \frac{\theta_j}{2} + \frac{k_1}{2} \sum_{j=1}^{n} \frac{1}{\nu_j^2} a_{kj} c_j \mathrm{ctg} \frac{\theta_j}{2}. \tag{28}$$

Since $u_k(y) = 0$ in the free frontier, we obtain

$$\Phi_k(y_1, y_2, \ldots, y_n) = 0,\ k = 1, 2, \ldots, n. \tag{29}$$

This is a complete system of non-linear algebraic equations for determining y_K $(k = 1, 2, \ldots, n)$. Now we show the existence of solution to the system (29) in a special case.

THEOREM 4. *We suppose that the measures of thickness of the two vertical layers are equal and Y_2 is closed enough to Y_1. Then the system of non-linear algebraical equation (29) has at least one solution which is different from* $(y_1 = \ldots = y_n = Y_1)$.

For the proof of this theorem, ve need some lemmas

LEMMA 5. The following relations are true

$$\sum_{j=1}^{n} j \sin \frac{\pi k j}{n+1} = (-1)^{k-1} \frac{n+1}{2} \mathrm{ctg} \frac{\pi k}{2(n+1)}. \tag{30}$$

$$\sum_{j=1}^{n} \sin \frac{\pi kj}{2n} = \frac{1 - \cos \frac{\pi k}{2}}{2} \operatorname{ctg} \frac{\pi k}{4n} - \frac{1}{2} \sin \frac{\pi k}{2}. \tag{31}$$

$$\sum_{j=1}^{n-2} \cos \frac{\pi kj}{2n} = -\frac{1}{2}\left(1 + \cos \frac{\pi k}{2}\right) + \frac{1}{2} \sin \frac{\pi k}{2} \operatorname{ctg} \frac{\pi k}{4n} \tag{32}$$

$$\sum_{l=1}^{n-1} \cos \frac{\pi sl}{n} = \begin{cases} n-1 & \text{if} \quad s = 2\nu n \\ -\left(\frac{1+(-1)^{s-1}}{2}\right) & \text{if} \quad s \neq 2\nu n \end{cases} \tag{33}$$

where ν, s, k, l, n are integers.

The verification of the equalities is straightforward.

LEMMA 6. *If $2n_1 = n + 1$ we have*

$$J(k) \overset{\text{def}}{=} \sum_{j=1}^{n} a_{jk} \rho_j = \frac{k_1 + k_2 \sigma_k}{2} c_k \operatorname{ctg} \frac{\pi k}{2(n+1)} \tag{34}$$

Proof. In this case we observe that

$$\theta_k = \frac{\pi k}{n+1} = \frac{\pi k}{2n_1}, \quad k = 1, 2, \dots, n \tag{35}$$

$$a_{ik} = c_k \begin{cases} \sin \frac{\pi ki}{2n_1}, & i = 1, 2, \dots, n_1 \\ \sigma_k \sin \frac{\pi ki}{2n_1}, & i = n_1 + 1, \dots, n \end{cases} \tag{36}$$

From this we obtsin

$$J(k) = c_k \left\{ k_1 \sum_{j=1}^{n_1-1} \sin \frac{\pi ki}{2n_1} + \frac{k_1 + k_i}{2} \sin \frac{\pi k}{2} + k_2 \sigma_j \sum_{j=n+1}^{n} \sin (m_1 - \sigma) \frac{\pi k}{m_1} \right\} =$$

$$= c_k \left\{ \left(k_1 - k_2 \sigma_k \cos \frac{\pi k}{2}\right) \sum_{j=1}^{n_1-2} \sin \frac{\pi kj}{2n_1} + \sin \frac{\pi k}{2} \left[\frac{k_1 + h_2}{2} + k_2 \sigma_k \sum_{j=1}^{n_1-1} \cos \frac{\pi kj}{2n_1} \right] \right\}$$

Using formulas (31) and (32) in the lemma 6 we find that

$$J(k) = c_k \left\{ \left(k_1 - k_2 \sigma_k \cos \frac{\pi k}{2}\right) \left(\frac{1 - \cos \frac{\pi k}{2}}{2} \operatorname{ctg} \frac{\pi k}{4n_1} - \frac{1}{2} \sin \frac{\pi k}{2} \right) + \right.$$

$$\left. + \sin \frac{\pi k}{2} \left[\frac{k_1 + k_2}{2} + k_2 \sigma k \left[-\frac{1}{2}\left(1 + \cos \frac{\pi k}{2}\right) + \frac{1}{2} \sin \frac{\pi k}{2} \operatorname{ctg} \frac{\pi k}{4n_1} \right] \right] \right\} \tag{37}$$

If k is an odd integer we have $\sigma_k = 1$ and $\cos\frac{\pi k}{2} = 0$. Hence J(k) takes the form

$$J(k) = c_k \frac{k_1 + k_2}{2} \operatorname{ctg} \frac{\pi k}{4n_1} \tag{38}$$

If k is an even integer we have $\sigma_k = -\frac{k_1}{k_2}$ and $\sin\frac{\pi k}{2} = 0$. It follows that

$$J(k) = 0 \tag{39}$$

By writing the equalities (38) and (39) in the general form we obtain the formula (34).

Q.E.D.

Now we take $Y_2 = \xi$ and

$$\varphi_k(\xi) = \left.\frac{du_k}{dy}\right|_{y = Y_1} \tag{40}$$

LEMMA 7. $\varphi_k(Y_1) = 0, \; k = 1, 2, \ldots, n$ (41)

Proof. At first we prove that

$$q_l \big|_{\xi = Y_1} = t_l \big|_{\xi = Y_1} = 0, \; l = 1, 2, \ldots, n \tag{42}$$

Since $\xi = Y_2 = Y_1$ we also have $y_i = Y_1$, $i = 1, 2, \ldots, n$. Hence from (14) we obtain

$$q_l \big|_{\xi = Y_1} = \sum_{j=1}^{n} \frac{1}{v_j^2} a_{jl}\rho_i \sum_{i=1}^{n} a_{ij}\rho_i - \frac{1}{2} \sum_{j=1}^{n} \frac{a_{il}}{v_j^2} \rho_j c_j \operatorname{ctg} \frac{\theta_t}{2} (k_1 + k_2\sigma_j)$$

$$= \sum_{j=1}^{m} \frac{1}{v_j^2} a_{lj}\rho_j \left[\sum_{i=1}^{n} a_{ij}\rho_i = \frac{k_1 + k_2}{2} \sigma_j c_j \operatorname{ctg} \frac{\theta_j}{2}\right].$$

Then using the lemma 6 we see that $q_l \big|_{\xi = Y_1} = 0$. Analogously we can show that $t_l \big|_{\xi = 0}$. Now according to (11) and (42) we have

$$\varphi_k(Y_1) = \left[\frac{d}{dy}(-Q_k(y) + S_k(y))\right]_{Y = \xi = Y_1} =$$

$$= \left[\frac{k_2}{2} \sum_{j=1}^{n} \frac{1}{v_j} a_{kj} c_j \sigma_j \operatorname{ctg} \frac{\theta_j}{2} \operatorname{sh} v_j (Y_1 - \xi)\right]_{\xi = Y_1} = 0.$$

Q.E.D.

LEMMA 8. *We have the following inequality*

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi = Y_1} < 0, \; k = 1, 2, \ldots, n. \tag{43}$$

Proof. We have

$$\left.\frac{dq_l}{d\xi}\right|_{\xi=Y_1} = \left[\frac{1}{2}\sum_{j=1}^{n} a_{jl}\rho_j c_j \mathrm{ctg}\,\frac{\theta_j}{2}\, k_2\sigma_j \mathrm{sh}\nu_j(Y_1-\xi)\right]_{\xi=Y_1} = 0. \tag{44}$$

$$\left.\frac{dt_l}{d\xi}\right|_{\xi=Y_1} = \sum_{j=1}^{n} a_{jl}\rho_j \left[\sum_{i=1}^{n} a_{ij}\rho_i \left.\frac{du_i(0)}{d\xi}\right|_{\xi=Y_1} - \frac{1}{2}\, k_2 Y_1 c_j \sigma_j \mathrm{ctg}\,\frac{\theta_j}{2}\right],$$

where

$$\left.\frac{du_i(0)}{d\xi}\right|_{\xi=Y_1} = \begin{cases} \dfrac{2k_2}{k_1+k_2}\, Y_1 h i, & i=1, 2,\ldots, n_1 \\ \dfrac{k_2-k_1}{k_1+k_2}\, Y_1 + \dfrac{2k_1}{k_1+k_2}\, Y_1 h i, & i=n_1+1,\ldots, n \end{cases}$$

After some calculations we find that

$$\sum_{i=1}^{n} a_{ij}\,\rho_i \left.\frac{du_i(0)}{d\xi}\right|_{\xi=Y_1} = \frac{1}{2}\, k_2 Y_1 \rho_j\, \sigma_j \,\mathrm{ctg}\,\frac{\pi j}{\varphi n_1},$$

which implies

$$\left.\frac{dt_l}{d\xi}\right|_{\xi=Y_1} = 0. \tag{45}$$

From (11), (44)), (45) we obtain

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} = -\frac{1}{2}\, k_2 \sum_{j=1}^{n} a_{kj}\,\rho_j\, \sigma_j \,\mathrm{ctg}\,\frac{\theta_j}{2}. \tag{46}$$

Let us distinguish two cases:

Case 1. $\quad 1 \leqslant k \leqslant n_1.$

In this case

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} = -\frac{1}{2}\, k_2 \sum_{1=j}^{n} c_j^2\, \sigma_j \sin\frac{\pi k j}{2u_1}\, (-1)^{j-1}\, \frac{2}{n+1} \sum_{i=1}^{n} i \sin\frac{\pi i j}{n+1}.$$

Using formulas (30), (33) in Lemma 5 we have

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} = -\frac{2k_2}{k_1+k_2}\,\frac{k}{n+1},$$

which shows that

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} < 0, \quad k=1, 2, \ldots, n_1. \tag{47}$$

Case 2. $n_1+1 \leqslant k \leqslant n$

In this case

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} = -\frac{k_2}{2}\sum_{j=1}^{n} c_j^2\sigma_j^2 \sin(n+1-k)\frac{\pi j}{n+1}\,\mathrm{ctg}\,\frac{\pi j}{2(n+1)}$$

$$= -\frac{k_2}{n+1}\sum_{i=1}^{n} i \sum_{j=1}^{n} c_j^2\sigma_j^2 \sin\frac{\pi kj}{n+1}\sin\frac{\pi ij}{n+1}. \tag{48}$$

It is easy to see that

$$c_j\sigma_j^2 = \begin{cases} \dfrac{4}{(n+1)(k_1+k_2)} & \text{if } j \text{ is an odd integer} \\ \dfrac{4}{(n+1)(k_1+k_2)}\,\dfrac{k_1}{k_2} & \text{if } j \text{ is an even integer} \end{cases}$$

Using formulas (48), (49) and Lemma 5 we obtain

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} = -\frac{2k_2}{k_1+k_2}\,\frac{1}{4n_1}\left[2n_1^2 + \frac{2k_1}{k_2}(n_1-1)(k-n_1)\right]$$

which implies

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} < 0 \text{ if } k = n_1+1,\dots, n. \tag{50}$$

The inequalities (47) and (50) complete the proof of Lemma 8.

Now we turn to the proof of Theorem 4.

Proof of Theorem 4.

Lemmas 7 and 8 show that

$$\left.\frac{d\varphi_k}{d\xi}\right|_{\xi=Y_1} < 0 \quad \text{and} \quad \left.\varphi_k\right|_{\xi=Y_1} = 0.$$

Hence, $\varphi_k(\xi) > 0$, $b = 1, 2, \dots, n$

with $\xi < Y_1$ in a neighbourhood of the point Y_1. But due to (42) and from the boundary condition of u we obtain

$$u_k(Y_1) = 0.$$

Therefore

$$\dot{u}_k(y) < 0\,, \quad k = 1, 2, \dots n$$

with $\xi < \varphi_1$ in a neigbourhood of the point Y_1.

In addition, the boundary condition in $y = 0$ guarantees the positiveness of $u_k(0)$. Hence from the continuity of the function $u_k(y)$ we conc lude that there exists a vector $(y_1, y_2 \dots y_n)$ on which

$$u_k(y_k) = 0\,, \quad k = 1, 2, \dots, n.$$

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