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INVARIANT PROPERTIES OF MOMENT SPACES

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The geometrical shape of the moment space associated with a Tchebycheff system of functions (abbrev. T—system) suggests the formulation of weaker but still interesting systems of functions. We introduce in this paper the notion of an invariant system of functions (abbrev. I—system, see Def. 1) and use geometrical and inductive arguments to obtain the description of the momênt space associated with such system (i.e. the canonical representation for points on the associated moment space). We refer to [1] and [2] for the literature on the deseription of the moment space associated with a T—system.

Our study of an I—system was motivated by an attempt to construct invariant estimators for polynomial regressions on simplexes (see [3]): roughly speaking, an I—system can be considered as a finite family of linearly independent and continuous functions whose moment space posses a boundary which is similar to that of the moment space associated with a system of power polynomials with even order. However from a geometrical point of view the notion of I—systems is different from that of T—systems in the sense that mayexist a hyperplane which cuts the curve generated by a given I—system of order n at an infinite number of points (see the example in §2).

The paper is divided into two sections. In the first section it is shown theorems that: Given an I—system of order n,

— in order to describe the upper boundary of its momenl space it suffices to know the description of the lower boundary of its $(n-1)$—th moment space,

— in order to describe the lower boundary of its moment space it suffices to know the description of the interior of its $(n-1)$—th moment space and,

— in order to describe the singleton part of its moment space it suffices to know the description of the whole boundary of its $(n-1)$—th moment space.

The last section is devoted to the application of the results of the first section to the study of invariant properties of moment spaces associated with symmetric systems.

1. DESCRIPTION OF THE MOMENT SPACE ASSOCIATED WITH AN I—SYSTEM

Let $X = [o, a]$ be a compact interval of the real line and let $C(X)$ be the vector space of real continuous functions defined on X. By a syslem of functions of order n we always mean in this paper a set of n linearly independent vectors in $C(X)$.

We denote by $M^1(X)$ the set of all probability measures on X equipped with its weak topology, so $M^1(X)$ is a compact convex set. For a given system of functions $U = (u_i)_o^n$ we can associate a compact convex set $\Gamma^n(X)$ in R^{n+1}, which is called the n—th moment space of the system U, by the following way: The relation

$$(1) \qquad \begin{cases} \mu \xrightarrow{\mathcal{P}_n} (\mu(u_i))_o^n \in \mathbf{R}^{n+1} \\ \text{where } \mu \in m^1(X) \text{ and } \mu(u_j) = \int_X u_j(t)\, \mu(dt) \ (o \leqslant j \leqslant n). \end{cases}$$

defines a continuous and linear mapping from $M^1(X)$ into $\mathbf{R}^{n+1}$ and $\Gamma^n(X)$ is defined to be the image of this mapping. By the same way we can define the k–th moment space associated with the system $U_k = (u_i)_o^k$ $(o \leqslant k \leqslant n)$ by considering the corresponding image of the mapping:

$$\mu \overset{\mathcal{P}_k}{\longmapsto} (\mu(u_j))_o^k \in \mathbf{R}^{k+1} \qquad (o \leqslant k \leqslant n).$$

Furthermore, for $k = o,..., n$ $\Gamma^k(X)$ can be identified with the convex hull of the following compact curve:

$$t \to U_k(i) = \left\{(u_j(t))_o^k\right\},\ t \in X \qquad (o \leqslant k \leqslant n).$$

In order to represent points of moment space $\Gamma^n(X)$ let us fix a coordinate system $(e_j)_o^n$ in the Euclidian space $\mathbf{R}^{n+1}$. Every moment point $\mathbf{c} =)c_i)_o^n \in \Gamma^n_{(X)}$ can be written as follows:

$$(2) \qquad \begin{cases} c = \sum_o^n c_i e_i \\ \quad = \sum_o^n \mu(u_i)\, e_i \\ \text{where } \mu \text{ is a probability measure such that } \mu(u_i) = c_i \qquad (o \leqslant i \leqslant n) \end{cases}$$

A measure μ satisfying (2) will be called a representing measure of $\mathbf{c}$. The set of all representing measures of a given moment point c will be denoted by $V(c)$.

Let $P_{k,k'}$ $(o \leqslant k' \leqslant k \leqslant n)$ be the orthogonal projection from the subspace span $(e_j \mid o \leqslant j \leqslant k)$ onto the subspace span $(e_j \mid o \leqslant j \leqslant k')$ then it is clear that $\Gamma^{k'}(X)$ can be identified with the image of $\Gamma^k(X)$ by the projection $P_{k,k'}$. Let $c = (c_i)_0^k (o \leqslant k \leqslant n-1)$ be a fixed moment point in $\Gamma^k(X)$, then by fiber $\mathcal{G}_c \subseteq V^{k+1}(X)$ associated with the given point c we mean the one dimensional compact convex set

$$\mathcal{G}_c = \{v \in \Gamma^{k+1}(X) \mid P_{k+1,k}(v) = c\} \tag{3}$$

Thus $\mathcal{G}_c$ can be identified with an oriented segment $[\underline{c}, \overline{c}]$ of the $(k+1)^{th}$ — coordinate axis. Since $\Gamma^{k+1}(X)$ is a convex body in $\mathbf{R}^{k+1}$, only two cases can happen: either $[\overline{c}, \underline{c}]$ is a real segment (i. e. $\overline{c} \neq \underline{c}$) hence both points $\overline{c}$ and $\underline{c}$ belong to the boundary $\partial\Gamma^{k+1}(X)$ of $\Gamma^{k+1}(X)$ or $[\overline{c}, \underline{c}]$ is reduced to the a single point $\{\underline{c}\}$(i. e. a singleton) of $\partial\Gamma^{k+1}(X)$. In both cases we will call respectively $\underline{c}$ a lower boundary point and $\overline{c}$ an upper boundary point of $\Gamma^{k+1}(X)$; with this convention $\partial\Gamma^{k+1}(X)$ can be divided into two disjoint parts: the lower boun-dary $\underline{\partial}\Gamma^{k+1}(X)$ consisting of all lower boundary points of $\Gamma^{k+1}(X)$ and the upper boundary $\overline{\partial}\Gamma^{k+1}(X)$ consisting of all upper boundary points of $\Gamma^{k+1}(X)$. Thus,

$$\left\{ \begin{array}{l} \partial\Gamma^{k+1}(X) = \underline{\partial}\Gamma^{k+1}(X) \cup \overline{\partial}\Gamma^{k+1}(X) \\ \qquad\qquad (0 \leqslant k \leqslant n). \end{array} \right. \tag{4}$$

Let $\in : X \to \{1, 2\}$ be a function such that

$$\in(t) = \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{otherwise,} \end{cases}$$

then by index of a discrete measure $\mu = \sum_1^m \alpha_i \partial t_i$ in $\mathcal{M}^1(X)$ we mean that:

$$\operatorname{Ind}(\mu) = \sum_1^m \in(t_i). \tag{5}$$

The index of a moment point c in $\Gamma^k(X)$ is by definition,

$$\operatorname{Ind}(c) = \operatorname{Inf} \{\operatorname{Ind}(\mu) \mid \mu \in V(c) \text{ and } \mu \text{ is discrete}\} \tag{6}$$

Note that by the Caratheodory theorem the set of measure μ defined by the right hand side of (6) is always non-empty.

Given a system of functions $\mathsf{U} = (u_i)_0^n$ in $C(X)$, every element of the subspace span$(u_i \mid o \leqslant i \leqslant n)$ will be called a polynomial or more precisely an U– polynomial.

DEFINITION 1. A system of continuous functions $U = (u_i)_0^n$ on the compact interval $X = [0, a]$ is called an invariant system (abbrev. I — system) of order n provided that,

1) U is linearly independent and the constant function 1 is an U-polynomial,

2) Every moment point of index smaller or equal to n must belong to the boundary $\partial\Gamma^n(X)$,

3) If $\mu = \sum_0^m \alpha_i \delta_{t_i}$ is a representing measure for some point Υ of index n and if a'' is a real such that, $a' = \sup \{t \mid u_i(t) = u_i(t_j) \text{ for a } 0 \leqslant j \leqslant m\} \leqslant a'' < a$ then the open interval $]\Upsilon, M[$ where $M = (u_i(a'))_0^n$ is contained in the interior of the moment space associated with the restricted system $(u_i \mid [o, a''])_0^n$.

4) If $\bar{c}$ is an upper boundary point of $\Gamma^n(X)$ and if $\mu \in V(\bar{c})$ then the support of $\bar{\mu}$ must contain the point a.

Remarks.

1) The condition 2) of the above definiton can be stated analytically as follows: Precribed points $(t_i)_0^m \subseteq [o, a]$ are zeros of some non negative polynomial of the system provided that $\sum_1^m \in (t_i) \leqslant n$.

2) Intituitively the condition 3) means that the lower boundary $\partial\Gamma^n(X)$ must not be locally flat, we will give a carefull treatment of this question in the next lema 1.

3) Although the condition 2), 3) and 4) of the definition 1) are verified by any T—system of oder n, however, there exist I—systems which are not T—systems (see the example in § 2)

4) It is clear from the geometrical characterization of a given T—system defined in the interval X, say, $V = (v_i)_0^n$, that for every $0 < a' \leqslant a$ the restricted system $V_{a'} = (v_i \mid X_{a'})_0^n$ where $X_{a'} = [0, a']$ turns out to be also a T—system. It is worthwhile to note that this fundamental property of a T—system is not true in general for an I—system and that it seems to be interesting to ask whether one can determine the set of all points $o < a' \leqslant a$ such that the restricted system $V_{a'} = (v_j \mid X_{a'})_0^n$ will be al so an I—system.

Définition 2. A system of continuous functions $U = (u_i)_0^n$ defined on the compact interval $X = [o, a]$ is called a complete I—system if $U_k = u_i)_0^k$ is I—system of order k for each $k = 0, 1, \ldots, n$.

For $i \in \{0,1,\ldots n\}$ let us define an equivalent relation $\overset{i}{\sim}$ on the interval $[o, a]$ as follows:

$$t \overset{i}{\sim} t' \quad \text{iff} \quad u_i(t) = u_i(t')$$

We denote by $A_i(t)$ the equivalent class w.r.t. $\overset{i}{\sim}$ which contains the point t.

Assume that $U = (u_i)_o^n$ is a complete I−system such that $u_o \equiv 1$ then,

$$(7) \quad A_i(t) \supseteq A_{i-1}(t) \quad \forall t \in [o, a] \qquad (2 \leqslant i \leqslant n).$$

In fact let t and t' be two points in $[o, a]$ such that $u_1(t) = u_1(t')$, from the condition 3) of the definition 1 both points $(1, u_1(t), u_2(t))$ and $(1, u_1(t'), u_2(t'))$ must belong to the boundary $\partial\Gamma^2(X)$. From the condition 4) of the definition the two above mentionned points must coincide with the lower boundary point of the fiber $\mathcal{F}_c$ associated with the point $c = (1, u_1(t)) = (1, u_1(t'))$, thus $u_2(t) = u_2(t')$ or in other words $A_2(t) \supseteq A_1(t)$ for all $t \in [o, a[$. (7) can be obtained by the same arguments.

Note that the set $A_1(a) = \{t \in X \mid u_1(t) = u_1(a)\}$ must be reduced to the point set $\{a\}$, otherwise it is possible to construct a reprenting measure $\overline{\mu}$ for an upper boundary point $\overline{c} \in \Gamma^1(X)$ such that $\sup P(\overline{\mu}) \not\subseteq \{a\}$ and this contradicts the condition 4) of the definition 1.

Definition 3. Let $U = (u_i)_o^n$ be an I − system on the compact interval $X = [o, a]$ and let $P = \sum_o^n \alpha_i u_i$ be an U − polynomial, if $t_o \in [o, a[$ is a zero of P then the set $A_1(t_o)$ will be called an equivalent class of zero for **P**.

According to (7) if $t \in A_1(t_o)$ where t_o is an zero of P then t is also a zero of P thus the zeros set $Z(P) = \{t \mid P(t) = 0\}$ of P is divided into disjoint equivalence classes of zeros,

LEMMA 1. *Let* $U = (u_i)_n^o$ *be an* I−*system, then* U *satisfies the three equivalent properties:*

a) *Let an* U−*polynomial,* $P = \sum_o^n \alpha_i u_i$ *be non negative on the interval* $[o, a']$ $(a' < a)$ *if* P *admits* m *disjoint equivalence classes of zeros* $A_1(t_i)$ $(1 \leqslant i \leqslant m)$ *such that* $\sum_1^m \in (t_i) = n$ *then these classes are the unique equivalence classes of zero of* P,

b) *If* $\mu = \sum_1^m \beta_j \delta_{t_j}$ *is a representing measure for some point* $\Upsilon \in \partial\Gamma^n(X)$ *of index* n *and let* **a''** *be a real such that* $a' = \sup\{\bigcup_1^m A_1(t_j)\} \leqslant a'' \leqslant a$ *then there is no hyperplan which at same time supports the moment space* $\Gamma(X_{a'})$ *and contains the line* $\overline{\Upsilon M}$, *where* $M = (u_i(a''))_o^n$.

c) *Under the same hypotheses as in b) then the open interval* $]\Upsilon M[$ *is contained in the interior of* $\Gamma^n(X_{a''})$.

Proof. We first remark that the assertion c) is nothing else than the condition 3) of the definition 1. Let us prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ *(b):* suppose that b) is false, then there is a hyperplane $H = \{m = (m_i)_o^n \mid \sum_o^n \alpha_i m_i = 0\}$ which at same time supports the convex set $\Gamma^n(X_{a'})$ and contains the line $\overline{\Upsilon M}$. Since the index of Υ is n, this boundary point can be represented by a probability measure $\xi = \sum_1^e \alpha_i \delta_{t_j'}$ such that $(t_j')_1^l \subseteq [o, a']$, $\sum_1^l \in (t_j') = n$ and that $A_1(t_i') \cap A_1(t_i') = \phi (1 \leqslant i \neq j \leqslant l)$.

Consequently the polynomial $P = \sum_o^m \alpha_i u_i$ is on the one hand non negative on $X_{a'} = [o, a']$ and on the other hand possesses (m + 1) disjoints classes of zero $\{(A_1(t_j'))_o^m, A_1(a'')\}$, but this contradicts (a).

(b) $\Rightarrow$ *(c)*: Let $c = (c_i)_o^n$ be any arbitrary point of the open interval $]\Upsilon, M[$. In order to prove that $c \in \mathrm{Int}\ (\Gamma^n(X_{a''}))$ it suffices to show that any hyperplane H passing through the point c should determine two open halfspaces which both contain points of $\Gamma^n(X_{a''})$. Only two cases can happen:

– If H does not contain the line $\overline{\Upsilon M}$ then it is clear that Υ and M are two points of $\Gamma^n(X_{a''})$ which belong respectively to the two open halfspaces determined by H,

– If H contains the line $\overline{\Upsilon M}$ then since H does not support the moment space $\Gamma^n(X_{a'})$ there always exist points of $\Gamma^n(X_{a''})$ which lie on both sides of H.

(c) $\Rightarrow$ *(b)* : Let $c = (c_i)_o^n$ be a point of $]\Upsilon, M[$ then by hypotheses c is an interior point of $\Gamma^n(X_{a''})$. Thus, for any polynomial $P = \sum_o^n \alpha_i u_i$ which is non negative on $X_{a''} = [o, a'']$ the form $C(P) = \sum_o^n \alpha_i c_i$ is strictly positive (see for instance [2], Chap. II). Suppose that a) is false then there is a polynomial $P(t) = \sum_o^n \alpha_i u_i(t)$ which on the one hand is non negative on [o, a'] and on the other hand admits (m + 1) zeros, say, $t_1 < t_2 < \ldots < t_{m+1}$ such that $\sum_1^n \in (t_i) =$ n and such that $A_1(t_j) \cap A_1(t_i) = \phi\ (1 \leqslant j \neq i \leqslant m + 1)$. Now let a' be a real

such that $\sup \{ \bigcup_1^m A_1(t_j) \} < a'$ and that $a' \in A_l(t_{m+1})$ and let $\Upsilon = (\Upsilon_i)_0^n$ be the moment point in $\Gamma^n(X_{a''})$ which has a representing measure of the form $\sum_1^m \delta_j \delta_{t_j}$.

Choose a point $c = (c_i)_0^n \in]\Upsilon, M[$, where $M = (u_i(t_{m+1}))_0^n$, thus c can be represented as,

$$c = \beta\Upsilon + (1-\beta) M \qquad (0 < \beta < 1)$$

or equivalently,

$$(8) \qquad \begin{cases} c_i = \beta\Upsilon_i + (1-\beta)\ m_i \\ \quad = \beta(\sum_0^n \delta_j u_i(t_j)) + (1-\beta)\ u_i(t_{m+1})\ (0 \leqslant i \leqslant n). \end{cases}$$

On the other hand if we compute c(P), then

$$C(P) = \sum_0^m \alpha_i c_i = \sum_0^n \alpha_i (\beta (\sum_1^m \delta_j u_i(t_j) + (1-\beta) u_i(t_{m+1})$$

$$= \beta \sum_1^m \delta_j (\sum_0^m \alpha_i u_i(t_j)) + (1-\beta)(\sum_0^n \alpha_i u_i(t_{m+1})) = 0$$

and this is absurd, since

$$c \in \operatorname{int}(\Gamma^n(X_{a''})).$$

Q.E.D.

Consider the moment space $\Gamma^n(X)$ associated with an I-system $U = (u_i)_0^n$ defined on $X = [o, a]$ and put:

$$(9) \qquad \begin{cases} \partial\Gamma^n(X) = \{ v \in \Upsilon^n(X) \mid \operatorname{Ind}(v) = n \} \\ \tilde{\partial}^b\Gamma^n(X) = \{ v \in \Upsilon^n(X) \mid \operatorname{Ind}(v) < n \} \\ \tilde{\partial}\Gamma^n(X) = \partial\Upsilon^n(X) \setminus \{ \underset{\sim}{\partial}\, \Upsilon^n(X) \cup \partial^b\Upsilon^n(X) \} \end{cases}$$

Note that by the definition of an I-system $\partial\Gamma^n(X)$ and $\partial^b\Gamma^n((X)$ are two disjoint subsets of $\partial\Gamma^n(X)$, hence (9) forms a partition of $\partial\Upsilon^n(X)$ into three disjoint parts This partition will be justified later on since we will identify in the next corollary 3 the set $\underset{\sim}{\partial}^n\Gamma(X) \cup \partial^b\Gamma^n(X)$ (resp. $\tilde{\partial}\Gamma^n(X)$) with the lower boundary $\underline{\partial}^n\Gamma(X)$ (resp. the upper boundary $\bar{\partial}\Gamma^n(X)$. For any $o < a' < a$ let us denote by $U_{a'} = (u_i \mid X_{a'})_0^n$ the restricted system of the givén I-system on the interval $X_{a'} = [o, a']$. The moment space associated with $U_{a'}$ will be denoted by $\Gamma^n(X_{a'})$ and by using the partition (9) for $\Gamma^n(X_{a'})$ we can also divīded its boundary $\partial^n\Gamma(X_{a'})$ into three disjoint parts.

THEOREM 2. *Let $U = (u_i)_0^{m+1}$ (0,1...) be a complete I-system defined on $X = [o, a]$ and let n be an arbitrary fixed number such that $o \leqslant n \leqslant m$. For each moment point $v \in I_n t(\Upsilon^n(X))$ let $\mathcal{G}_v = [\underline{c}, \bar{c}]$ be thê fiber in $\Gamma^{n+1}(X)$ associated with the point v, then:*

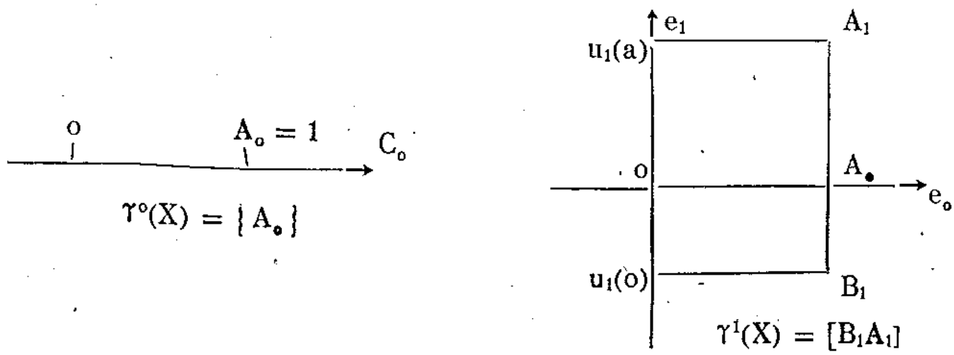
(a) *a measure* $\overline{\mu} \in V(v)$ *is a representing measure of the upper boundary point* $\overline{c}$ *provided that* $Ind(\overline{\mu}) = n+2$ *and that* $supp\ (\overline{u}) \supseteq \{a\}$, *furthermore the set of the just mentionned measures is not empty,*

(b) *a measure* $\underline{\mu} \in V(v)$ *is a representing measure of the lower boundary point* $\overline{c}$ *provided that* $Ind\ (\overline{\mu}) = n+1$ *and that* $supp(\overline{\mu}) \supseteq \{a\}$ *furthermore the set of the just mentionned measurês is not empty,*

(c) *for any given interior point* $c' \in]\underline{c}, \overline{c}[$ *there is a measure* $\mu' \in V(v)$ *which represents c' such that* $Ind\ (\mu') = n+2$ *and* $supp\ (\mu') \supseteq \{a\}$,

Proof. The proof will be treated induction. Without loss of generality we can assume that $u_0 \equiv 1$ and próve the theorem only for the case $n = m$.

First let $m = 0$ and consider the system $(u_i)_0^1$. The corresponding moment space associated with the system (u_o) and $\{u_o, u_1\}$ are respectively the one point set $\{A_o\}$ and the compact interval $[B_1, A_1]$ (see fig. 1).



(Fig. 1)

Hence in the considered case the given point v can only be chosen to be the point A_1 and the fiber $\mathcal{F}_v$ is identified with the interval $[B_1, A_1]$. Since $\{u_0, u_1\}$ is an I-system the assertions a) b) and c) of the theorem can be essily verified. It should be noted that $B_1 = (1, u_1(0))$ and that $A_1(1, u_1(1))$. Now suppose that the theorem is true for the system $(u_i)_0^k$ with $k = 0, 1, \ldots, m-1$ and let us prove that the theorem is also true for $k = m$.

(a) Let $\overline{\mu} \in V_{(v)}$ such that $Ind\ (\overline{\mu}) = m+2$ and that $supp\ (\overline{\mu}) \supset \{a\}$ then $\overline{\mu}$ can be written as:

$$(10) \left\{ \begin{array}{l} \mu = \beta\mu' + (1-\beta)\,\delta_a \qquad (0 < \beta < 1) \\ \text{where } \mu' \in m(X) \text{ such that } Ind\ (\mu') = m \text{ and that } supp\ (\mu') \supseteq \{a\}. \end{array} \right.$$

Bythe inductive hypotheses the point $\mathcal{P}_m(\mu') = \Upsilon$ must belong to the boundary $\partial\Upsilon^m(X)$ of $\Gamma^m(X)$. Hence if we put $A_m = \mathcal{P}_m(\delta_a)$ and draw the open half-line $]\overline{A_m v}$ then since $\overline{\mu}$ represents v the point Υ must be identified with the unique intersection point of $)\overline{A_m v}$ with $\partial\Gamma^m(X)$ (see Fig, 2).

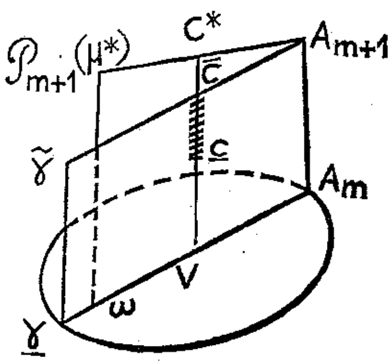
Furthermore $\underline{\Upsilon}$ must belong to the lower boundary $\partial\Gamma^m(X)$. Indeed suppose that $\underline{\Upsilon} \in \overline{\partial}\Gamma^m(X)$ then by the inductive hypotheses this point admitsa representing measure μ'' such that Ind $(\mu'') = m + 1$ and that supp $(\mu'') \supset \{a\}$. it follows that the measure $\widetilde{\mu} = \beta\mu'' + (1 - \beta)\delta_a$ belong to $V(v)$. But on the other hand since Ind $(\widetilde{\mu}) = \text{Ind}(\mu'') = m + 1$ and since supp $(\widetilde{\mu}) \supset \{a\}$ the point v, by the inductive hypotheses, turns out to belong to $\partial\Gamma^m(X)$. This contradicts the fact that $v \in \text{Int}(\Gamma^m(X))$.

Let us now draw the closed half-line $\overline{]A_{m+1}\overline{c}}$ where $A_{m+1} = \mathcal{P}_{m+1}(\delta_a)$. Since $\Upsilon^{m+1}(X)$ is a convex body the intersection of this moment space with $\overline{]A_{m+1}\overline{c}}$ must be a compact interval, says $[A_{m+1}\ \widetilde{\Upsilon}]$. I say that the fiber in $\Gamma^{m+1}(X)$ associated with the point $\underline{\Upsilon}$ must be reduced to the singleton $\{\widetilde{\Upsilon}\}$ (see Fig. 2). In fact suppose that the projection of $\widetilde{\Upsilon}$ into $\Gamma^m(X)$ is a point $\omega \in]\Upsilon, v]$ then $\widetilde{\Upsilon}$ belongs to the fiber $\mathcal{G}_\omega \subset \Gamma^{m+1}(X)$ associated with the point ω and furthermore it follows from the condition 4) of the definition 1 that $\widetilde{\Upsilon}$ is not the upper point of this fiber. Let $\mu^* \in V(\omega)$ be a representing measure of the upper point of $\mathcal{G}_\omega$ and put:

$$c^* = \beta\mathcal{P}_{m+1}(\mu^*) + (1 - \beta)A_{m+1} \tag{11}$$

then it is clear that c^* is one moment point in $\Gamma^{m+1}(X)$ which lies on a strictly higher position than $\overline{c}$ in the fiber $\mathcal{G}_v$ and this contradicts the faet $\overline{c} \in \overline{\partial}^{m+1}(X)$. Thus we proved that the fiber $\mathcal{G}_{\underline{\Upsilon}}$ in $\Gamma^{m+1}(X)$ associated with $\underline{\Upsilon}$ contains the point $\widetilde{\Upsilon}$. On the other hand if $\mu \in V(\Upsilon)$ then it is necessary that supp $(\mu) \supseteq \{a\}$ otherwise since v is interior point of $\Gamma^m(X)$ it can be shown that $\underline{\Upsilon}$ is also an interior point of the convex body $\Gamma^m(X)$ and this is absurd. Thus by the condition 4) of the definition 1 the fiber $\mathcal{G}_{\Upsilon}$ has no upper point hence it must be reduced to the singleton $\{\widetilde{\Upsilon}\}$. Consequently:

— for every $\mu \in V(\underline{\Upsilon})$ the measure $\beta\mu + (1 - \beta)\delta_a \in V(\overline{c})$ and, a fortiori, this fact must also true for the measure μ' given by (10),



(Fig. 2)

— the index of Υ must be equal to m otherwise it can be easily show that v will be a boundary point of $\Upsilon^m(X)$ and this is absurd.

Thus the assertion (a) of the theorem is proved for $k = m$.

(b) Let $\underline{\mu} \in V(v)$ such that $\text{Ind}(\underline{\mu}) = m+1$ and $\text{Supp}(\underline{\mu}) \supseteq \{a\}$ then by the ondition 3) of definition 1 $\underline{\mu}$ represents a boundary point of fiber $\mathcal{G}_v = [\underline{c}, \overline{c}]$. By (a) $\underline{\mu}$ must represent the lower boundary point $\underline{c}$.

Now since $v \in \text{Int}(\Upsilon^m(X))$ it follows from the inductive hypotheses that there is a representing measure $\underline{\mu}$ for v such that

$$\text{Ind}\ (\underline{\mu}) = m+1 \qquad \text{and}$$

$$\text{Supp}\ (\underline{\mu}) \supseteq \{a\}$$

From what we just say $\underline{\mu}$ turns out to be a representing measure for $\underline{c}$.

(c) Let $v' \in [\underline{c}, \overline{c}]$ then $v' = (v_i)_0^{m+1}$ is clearly an interior point fo $\Gamma^{m+1}(X)$. There is a real $o < a' < a$ such that v' is still an interior point of $\Gamma^{m+1}(Xa')$. For the proof of this fact we refer to the Lemma 5.1 ([2], Chap. IV). Let a_o be the infimum of the set of all real numbers $a' \in [o, a]$ for which

$$v' \in \text{Int}(\Gamma^{m+1}(X_{a'})) \text{ then } v' \in \Gamma^{1+1}(X_{a_o}).$$

Let M_o be the moment point in $\Upsilon^{m+1}(X)$ of the Dirac measure δ_{a_o} then the open half line $M_o v'$ will cut the boundary $\delta\Gamma^{m+1}(X)$ at an unique point $\underline{\Upsilon}$ We have

$$v' = \beta\underline{\Upsilon} + (1-\beta)M_o \quad (0 < \beta < 1). \tag{12}$$

Since $v' \in \Upsilon^{m+1}(X_{a_o})$ it can be shown that $\Upsilon \in \Upsilon^{m+1}(X_{a_o})$, hence by the condition 4) of the definition 1 Υ must be a lower boundary point of $\Gamma^{m+1}(X)$. By the part (6) we have $\text{Ind}(\underline{\Upsilon}) \leqslant m+1$ and thus only three cases can happen:

1) If $\text{Ind}(\underline{\Upsilon}) = m+1$ then by the Lemma 1 the open interval $]\underline{\Upsilon}, M_o[$ is contained in the interior of $\Gamma^{m+1}(X_{a_o})$ and consequently $v \in \text{Int}\Gamma^{m+1}(X_{a_o})$ Now it is not difficult to see that there is a real $\tilde{a}$ such that $a < \tilde{a} < a_o$ and that $v \in \text{Int}(\Gamma^{m+1}(X\tilde{a}))$, but this fact contradicts the definition of the number a_o itself.

2) if $\text{Ind}\ (\underline{\Upsilon}) < m$ then by (12) we see that the index of v' is smaller than $(m+2)$ and it follows from the condition 3) of the definition 1 that v' must be a boundary point of $\partial\Gamma^{m+1}(X)$ which is absurd.

3) there remains only the possibility that $\text{Ind}\ (\underline{\Upsilon}) = m$ and it follows from (12) that there is a probability meascre $\mu' \in V(v)$ such that

$$\begin{cases} \text{Ind}\ (\mu') = m+2 \\ \text{Supp}\ (\mu) \supset \{a\} \\ \qquad \mu' \in V(v') \end{cases}$$

Q.E.D.

It follows imediately from the proof of the above theorem that

COROLLARY 3. *Let* $U = (u_i)_o^{n+1}$ *be a complete* I *— system defined on* $X = [o, a]$. *For* $o \leqslant l \leqslant m$ *we have:*

a) *The set* $\tilde{\partial}\Upsilon^{l+1}(X)$ *(resp.* $\underset{\sim}{\partial}\Gamma^{l+1}(X) \cup \partial^b\Gamma^{l+1}(X)$*) can be identified with the upper boundary* $\bar{\partial}\Gamma^{l+1}(X)$ *(resp. the lower boundary of* $\underline{\partial}\Gamma^{l+1}(X)$ *of the moment space* $\Gamma^{l+1}(X)$.

b) *The upper boundary* $\bar{\partial}\Gamma^{l+1}(X)$ *is a surfacre generated by the open segments* $]\underset{\sim}{\Upsilon}, A_{l+1}[$*; where* $A_{l+1} = (u_i(a))_o^{l+1}$ *and where* $\underset{\sim}{\Upsilon}$ *is running through the curve of singletons of index* l. *In other words:*

$$\bar{\partial}\Gamma^{l+1}(X) = \{\lambda A_{l+1} + (1-\lambda)\underset{\sim}{\Upsilon} \mid o < \lambda < 1 \text{ and Ind } (\underset{\sim}{\Upsilon}) = l\},$$

c) *If* l *is even the subset* $\underset{\sim}{\partial}\Gamma^{l+1}(X)$ *of the lower boundary* $\underline{\partial}\Gamma^{l+1}(X)$ *is identified with the surface generated by the open interval* $]\tilde{o}, \underset{\sim}{\Upsilon}[$ *where* $\tilde{o} = (u_i(o))_o^{l+1}$ *and where* $\underset{\sim}{\Upsilon}$ *is running through the curve of singleton of index* l. *In other words:*

$$\underset{\sim}{\partial}\Gamma^{l+1}(X) = \{\lambda\tilde{o} + (1-\lambda)\underset{\sim}{\Upsilon} \mid o < \lambda < 1 \text{ and Ind } (\underset{\sim}{\Upsilon}) = l\}$$

2. APPLICATIONS TO THE DESCRIPTION OF THE INVARIANT PART OF THE MOMGNT SPACE ASSOCIATED WITH A SYMMETRIC SYSETEM

We apply in this section the results of the previous section to describe the invariant moment points of a symmetric system defined on a compact symmetric inteval $S_a = [-a, a]$. S_a is identified as an one dimensional simplex, thus its symmetric group G consists only of the symmetry through the origin and the identity map, i. e., $G = \{S_o, id\}$. Let $\mathcal{M}^1(S_a)$ be the set of probability measures on S_a then the set $\mathcal{M}^1_{inv}(S_a)$ of all G—invariant probability measures can be written as follows,

$$\mathcal{M}^1_{inv}(S_a) = \left\{\frac{\mu(.) + \mu(S_o(.))}{2} \,\middle|\, \mu \in \mathcal{M}^1(S_a)\right\}.$$

Obviously $\mathcal{M}^1_{inv}(S_a)$ is a compact, convex subset of $\mathcal{M}^1(S_a)$ if both sets are equipped with the weak topologys.

Now consider a system $U_m = (f_i)_o^m$ of linearly independent and continuous functions on S_a and denote by $\Gamma^m(S_a)$ the moment space associated with the given system. The invariant moment space associated with U_m denoted by $\Gamma^{m}_{im}(S_a)$ is by definition the subset of $\Gamma^m(S_a)$ which is the image of $\mathcal{M}^1_{inv}(S_a)$ by the projection

$$\mu \xrightarrow{\mathcal{P}_m} (\mu(f_i))_0^m$$

Since $\mathcal{P}_m$ is continuous, it is clear that $\mathcal{M}^1_{im}(S_a)$ is a compact convex subset of $\Gamma^m(S_a)$.

For instance if we suppose furthermore that $m = 2n$ and that the function $f_i (0 \leqslant i \leqslant 2n)$ is even or odd according as its index is even or odd then since $\mu(f_{2k+1}) = 0 \ (0 \leqslant K \leqslant n-1)$, $\forall \mu \in \mathcal{M}^1_{im}(S_a)$ the invariant moment space $\Gamma^{2n}_{im}(S_a)$ is isomorphic to the moment space associated with the system $(f_{2k})_0^n$ on the compact interval $X = [0, a]$. In the remainder part of the paper we will consider only systems of functions with the above hypotheses.

The index of an invariant measure $\mu \in m^1_{inv}(S_a)$ and the index of an invariant moment point $c \in \Gamma^{2n}_{inv}(S_a)$ will be defined as follows

$\text{Ind}(\mu) = \{\text{Supp}(\mu)\ |$, where $|A|$ denotes the cardinal of the set $A\}$

$\text{Ind}(c) = \text{Inf}\ \{\text{Ind}(\mu)\ |\ \mu$ is discret and $\mu \in V_{(c)}\}$

where $V_{(c)} = \{\mu \in \mathcal{M}^1_{inv}(S_a)\ |\ P_{2n}(\mu) = c\}$.

All other definitions and terminologies used in the first section such as the upper boundary, the lower boundary can be translated words by words here to the invariant moment space.

Definition 4. A sytem $\widetilde{U}_{2n} = (f_i)_0^{2n}$ of $(2n+1)$ linearly independent and continuous functions defined on $S_a = [-a, a]$ called a symmetric system of order 2n provided that,

i) $f_0 \equiv 1$.

ii) the function $f_i (o \leqslant i \leqslant 2n)$ is even or odd according as its index $i (o \leqslant i \leqslant 2n)$ is even or odd,

iii) the even order subsystem $U = (f_{2K})_0^n$ defined on $X = [o, a]$ is an I—system.

Definition 5. A system $\widetilde{U}_{2n} = (f_i)_0^{2n}$ will be called a complete symmetric system if for $k = 0, 1, ..., n$ the truncated system $\widetilde{U}_{2k} = (f_i)_0^{2K}$ is a symmetric system of oder $2k (o \leqslant k \leqslant n)$ on the same interval S_a.

Let us give now an exemple of a symmetric system. Consider an arbitrary continuous and odd function $g(t)$ defined on S_a such that $|g(t)|$ attains its maximum only at the points $-a$ and a.

Put

$$\begin{cases} g_i(t) = [g(t)]^i \ (i = o, 1, ..., 2n) \\ \text{where } t \in S_a \end{cases}$$

PROPOSITION 4. *Under the above mentionned hypothêse* $\{(g_i)_o^{2n}\}$ *is a complete symmeiric system.*

Proof. For the cases n = 1, 2, the proof is straightforward.

Let $n \geqslant 2$ and put :

$X = \Gamma^2_{inv}(S_a) = [o, A_2]$ where $A_2 = f_2(a)$. Then it is not difficult to see that the moment space $\Gamma^{2n}_{inv}(S_a)$ is isomorphie with conv $\{(x^i)_o^n \mid x \in X\}$.

Since $(x^i)_o^n$ is a Markov–system on X it is easily verified that $(g_{2i})_o^n$ is a complete I–system and this completes the proof.

Example: Let $S_1 = [-1, 1]$ and let $(x_i)_o^\infty$ be an given sequence of real points on [0, 1/2]. Construct an odd and continuous function g(t) on $[-1, 1]$ such that,

$$\begin{cases} g(t) \text{ attains the maximun only at } t = \pm 1 \\ g_i(t) = o \Leftrightarrow t \in \{\pm x_i \mid t = 1, 2, ...\} \end{cases}$$

Put $\quad g_i(t) = [g(t)]^i \ (i = 0, 1, ... 2n), \ t \in [-1, 1].$

Then by Proposition 4 $(g_i(t))_o^{2n}$ is a complete symmetric system on S_1 and the restricted system $(g_{2i} \mid_{[0,1]})_o^n$ defined on [0,1] is a complete I–system.

Note that the zero set of the polynomial $g_2(t)$ is exactly the given se-sequence of number $\{\pm x_i \mid (i = 1, 2, ...)\}$.

The following representation theorem for invariant moment spaces is an immediatl corollary of Theorem 2:

THEOREM 5. *Let* $(f_i)_o^{2n}$ *be a complete symmetric system defined on* $S_a = [-a, a]$ *and let k be an arbitrary fixed number such that* $0 \leqslant k \leqslant n-1$. *For* $c \in Int(\Gamma^{2k}_{inv}(Sa))$ *let* $\mathcal{G}_c = [\underline{c}, \overline{e}] \subset \Gamma^{2(k+1)}_{inv}(S_a)$ *be the fiber associated with c then we have :*

1) There exists at least one invariant probability measure $\overline{\mu} \ni \mathcal{M}^1_{inv}(S_a)$ *such that,*

i) $\overline{\mu} \in V(c)$

ii) $Ind\ \overline{(\mu)} = k + 2$

iii) $Supp\ \overline{(\mu)} \supset \{-a, a\}$

Furthermore, $\mathcal{P}_{(2k+1)}\overline{(\mu)} = \overline{c}$ *holds for all measures* $\overline{\mu}$ *satisfying i), ii) and iii),* *2) There exists at least one invariant probability measure* $\underline{\mu} \in \mathcal{M}^1_{inv}(S_a)$ *such that,*

iv) $\underline{\mu} \in V(c)$

v) $Ind(\underline{\mu}) = k + 1$

vi) $Supp(\underline{\mu}) \supset \{-a, a\}$

Furthermore $\mathcal{P}_{2(k+1)}(\underline{\mu}) = \underline{c}$ *holds for all measures* $\underline{\mu}$ *satisfying iv), v) and vi),*

c) For an arbitrary $c' \in \mathcal{G}_c \setminus \{\overline{c}, \underline{c}\}$ *there exists one measure* μ' *such that*

vii) $\mu' \in V(c)$

viii) $Ind(\mu') = k + 2$

ix) $Supp(\mu') \supset \{-a, a\}$

x) $\mathcal{P}_{2(k+1)}(\mu') = c'$

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