ACTA MATHEMATICA VIETNAMICA
Tom 6, No 2 (1981)

SHAPE THEORY IN THE CATEGORY OF METRIC SPACES AND UNIFORMLY CONTINUOUS MAPS

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INTRODUCTION. In 1968 Borsuk [2] introduced the notion of shape for compact metric spaces. Since then the notion of shape has been extended into several categories [4] [6] [9]. It is known that the interesting basic results in shape theory which have been proved for compact metric spaces can not be extended to non-compact metric spaces. However these results can be extended into the category of metrie spaces and uniformly continuous maps as shown recently by Nguyen To Như [14]. The aim of this paper is to investigate shape theory in the category of metric spaces and uniformly continuous maps. For this purpose we introduce the notions of FANRU —, FARU — and FAEU-spaces. The basic properties of these spaces are established

§1. UNIFORM RETRACTS AND UNIFORM NEIGHBOURHOOD RETRACTS

In this section we fix notation and for onvenience in references, we formulate some facts needed throughout the paper.

In the sequel write $Z \supset Y$ iff Z is a metric space containing the metric space Y isometrically and the relation of homotopy between uniformly continuous maps denoted by $\cong$.

A metric space Y is called an $ANRU$ — space (written $Y \in$ ANRU) [7] iff for every metric space $Z \supset Y$, there exist a uniform neighbourhood U of Y in Z and a uniformly continuous map r (called a *uniform retraction*) from U onto Y such that $r(y) = y$ for every $y \in Y$.

If we always can take $U = Y$ in the situation above then Y is called an ARU-space. Here U is called a *uniform neighbourhood* of a subset Y in the metrie space (Z,d) if $U \supset B_\varepsilon(Y) = \{x \in Z : d(x, Y) \leqslant \varepsilon\}$ for some $\varepsilon > 0$.

It is clear that every ARU — space is an ANRU-space and every ANRU-space is complete.

The following is known [11].

1.1. PROPOSITION ([11]). *Let Y is an ANRU — space. Then for every uniformly continuous map f from a subset A of a metric space into Y there exist a neighbourhood U of A in X and a uniformly continuous map* $\tilde{f} : U \to Y$ *such that* $\tilde{f} \mid A = f$.

A metric space Y is called an *AEU — space* iff for every space X, every uniformly continuous map f from a subset A of a space X into Y can be extended to a uniformly continuous map $\tilde{f}$ from X into Y.

We have the following [10]

1.2. PROPOSITION ([10]). *A metric space* $Y \in AEU$ *if and only if* Y *is an ANU — space and diam* $Y < \infty$.

Given any set D, let l^∞ (D) denote the Banach space of all bounded real functions on D with the supremum norm. By a theorem of Isbell [7], $l^\infty (D) \in ANRU$. Thus by the Kuratowski-Wojdyslawski theorem (see, [11]) and the Katětov theorem [8], we have the following.

1.3. PROPOSITION. Every metric space X can be imbeded isometrically into an ANRU-space E. Moreover, if diam $X < \infty$ then one can take E $\in$ AEU.

1 – 4. HOMOTOPY EXTENSION LEMMA ([14]). Let A be a subset of a metric space X and B be a subset of an ANRU-space Y. Let V be a uniform neighbourhood of B in Y and $f, g : A \to B$ be uniformly continuous maps homotopic in B. If f has an extension $\tilde{f} : X \to V$ then g has an extension $\tilde{g} : X \to V$ homotopic to $\tilde{f}$ in V.

1.5. LEMMA ([13]). *Let A be a subset of a metric space X and B be a subset of an ANRU-space Y. Let f, g: X→ Y be uniformly continuous maps such that* $f|_A \cong g|_A$ *in B Then for every uniform neighbourhood V of B in Y there exists a uniform neighbourhood U hf A in X such that* $f|_U \cong g|_U$ *in V.*

§ 2. FUNDAMENTAL SEQUENCES AND RETRACTS.

Let X and Y be metric spaces lying in ANRU-spaces P and Q respectively. By a *fundamental sequence* from X to Y we understand an ordered triple $f = \{f_k, X, Y\}_{P,Q}$ consisting of X, Y and a sequence of uniformly continuous maps $f_k : U_o \to Q$, where U_o is a uniform neighbourhood of X in P, satisfiying the condition (2—1). For every uniform neighbourhood V of Y in Q there exists a uniform neighbourhood U of X in P such that.

$$f_k|U \simeq f_{k+1}|U \quad \text{in } V \text{ for almost all } k.$$

If $P = Q$ then we write $f = \{f_k, X, Y\}_P$ instead of $f = \{f_k, X, Y\}_{P,P}$

Two fundamental sequences $f = \{f_k, X, Y\}_{P,Q}$ and $g = \{g_k, X, Y\}_{P,Q}$ are said to be *homotopic* (denoted $f \simeq g$) iff for every uniform neighbourhood V of Y in Q there exists a uniform neighbourhood U of X in P such that

(2—2) $\qquad f_k|U \simeq g_k|U$ in V for almost all k.

A *composition* of fundamental sequences $f = \{f_k, X, Y\}_{P,Q}$ and $g = \{g_k, Y, Z\}_{Q,R}$ is the fundamental sequence $gf = \{g_k f_k, X, Z\}_{P,R}$.
The fundamental sequence $i_X = \{i_k, X, X\}_P$, where $i_k : P \to P$, $k = 1, 2, \dots$ is the identity map is called the *fundamental identity sequence*.

Let X and Y be metric spaces lying in ANRU-spaces P and Q respectively and let $f : X \to Y$ be a uniformly continuous map. Since $Q \in ANRU$, there exist a uniform neighbourhood U_o of X in P and a uniformly continuous map $\hat{f} : U_o \to Q$ such that $\hat{f}(x) = f(x)$ for every $x \in X$. Setting $f_k(x) = \hat{f}(x)$ for every point $x \in U_0$ and every $k = 1,2,\dots$ we get a fundamental sequence $f = \{f_k, X, Y\}_{P,Q}$. This fundamental sequence is called the fundamental sequence *generated by* f.

Let Y be a subset of a metric space Q. A sequence of uniform neighbourhoods $\{V_k\}$ of Y in Q is called *complete* iff it satisfies the following conditions.

(2.3) Every uniform neighbourhood V of Y in Q contains V_k for almost all k.

(2.4) V_k is a uniform neighbourhood of V_{k+1} for every k.

The following theorem was proved in [14].

2.5. THEOREM ([14].) *Let X, Y be metric spaces lying in ANRU-spaces P and Q respectively and* $f_k : X \to Q$ *be a sequence of uniformly continuous maps Then there exists a fundamental sequence* $f = \{f_k\ X,\ Y\}_{P,Q}$ *such that*

(2.6) $f_k|X = f'_k$ *for every* $k \in N$.

if and only if there exists a complete sequence of uniform neighbourhoods $\{V_k\}$ *of Y in Q such that*

(2.7) $f'_k \cong f'_{k+1}$ *in* V_k *for almost all k.*

Now let P be an ANRU-space and $A \subset X \subset P$. A fundamental sequence $r = \{r_k, X, A\}_P$ is called a *fundamental retraction* of X to A in P iff $r_k(x) = x$ for every $x \in A$ and $k \in N$.

Let us prove the following

2.8. LEMMA. *Let P be an AHRU-space and X be a uniform neighbourhood of* $A \subset P$ *in P rnd* $r = \{r_k, X, A\}_P$ *be a fundamental retraction. Then there exist a complete sequence of uniform neighbourhoods* $\{V_k\}$ *of A in P and a fundamental retraction* $r' = \{r'_k, X, A\}_P$ *such that*

(2.9) $r'_k(x) = x$ *fnr every* $x \in V_k$

2.10) $r' \cong r$.

Proof. By (2.5) there exist a complete sequence $\{W_k\}$ of uniform neighbourhoods of A in P and an index k_o such that

$$r_k|X \cong r_{k+1}|X \text{ in } W_k \text{ for every } k > k_o.$$

By (1.5) for every $k > k_o$ there exists a uniform neighbourhood V_k of A in P such that

(2.11) $r_k|V_k \cong id_P|V_k$ in W_{k+1} for every $k > k_o$.

It is clear that $\{V_k\}$ can be chosen to be a complete sequence such that $V_k \subset X$ and W_k is a uniform neighbourhood of V_k in P for every $k \in N$. By (1 — 4) for every $k > k_o$, there exists a uniformly continuous map $f_k : X \to W_k$ such that

(2.12) $f_k|V_k = Id_P|V_k$ and $f_k \cong r_k|X$ in W_k

Whence we get

$$f_k \cong r_k|X \cong r_{k+1}|X \cong f_{k+1} \text{ in } W_k \text{ for every } k > k_0$$

Put

(2.13) $f_k(x) = x$ for every $x \in X$ and $k < k_o$.

By (2 — 5) there exists a fundamental sequence $r' = \{r'_k, X, A\}$ P such that

(2.14) $\qquad r'_k \mid X = f_k$ for every $k \in N$.

From (2 — 12) —— (2 — 13) we obtain

(2.15) $\qquad r'_k(x) = x$ for every $x \in V_k$ and $k \in N$.

It remains to show that $r' \cong r$.

For every uniform neighbourhood V of A in P, let U' be a uniform neighbourhood of X in P and $k_1 > k_0$ be an index such that

(2.16) $\qquad r'_k \mid_{U'} \cong r'_{k_1} \mid_{U'}$ in V for every $k > k_1$

(2.17) $\qquad r_k \mid_{U'} \cong r_{k_1} \mid_{U'}$ in V for every $k > k_1$

$$W_{k_1 - 1} \subset V$$

By (2.12) and (2.14) $r'_{k_1} \mid X \cong r'_{k_1} \mid X$ in W_{k_1}, by (1 — 5) there exists a uniform neighbourhood $U \subset U'$ of X such that

(2.18) $\qquad r_{k_1} \mid U \cong r'_{k_1} \mid U$ in $W_{k_1-1} \subset V$.

From (2.16) —— (2.18) we obtain

$r_k \mid U \cong r'_k \mid U$ in V for every $k > k_1$.

The lemma is proved.

2.19. PROPOSITION. *Let X and Y be metric spaces lying in ANRU—spaces P and Q respectively and let h be a uniform homeomorphism of X onto Y and $A \subset X$, $B = h(A) \subset Y$. If there exists a fundamental retraction of X to A in P then there exists a fundamental retraction of Y to B in Q.*

Proof. Let $r = \{r_k, X, A\}_P$ be a fundamental retraction of X to A in P and let U_0 be a uniform neighbourhood of X in P such that r_k is defined on U_0 for every $k \in N$, Since $P, Q \in$ ANRU, there exist uniform neighbourhoods U, V of X and Y in P and Q respectively and uniformly continuous maps $\varphi: U \to Q$, $\psi: V \to U_0$ such that

$$\varphi(x) = h(x) \text{ for every } x \in X$$

$$\psi(y) = h^{-1}(y) \text{ for every } y \in Y$$

Clearly we may assume that $r_k: U_0 \to U$ for every $k \in N$. Whence the formula

$$r'_k(y) = \varphi r_k \psi(y) \quad \text{for every } y \in V$$

defines a fundamental retraction $r' = \{r'_k Y, B\}$ Q of Y to B in Q.

A subset A of a metric space X is called a *fundamental retract* of X if there exist an ANRU - space $P \supset X$ and a fundamental retraction $r = \{r, X, A\}_P$ of

X to A in P. By proposition 2–19, the choice of an ANRU-space P containing X as a subset, is immaterial.

A metric space Y is called an *FANRU-space* iff for every metric space $Z \supset Y$ there exists a uniform neighbourhood U of Y in Z such that Y is a fundamental retract U. A metric space Y is called an *FARU-space* if one can take $U = Z$ for every metric space $Z \supset Y$.

It is clear that

(2.20). A metric space Y is an FANRU-space if and only if there exists an ANRU — space $Q \supset Y$ such that Y is a fundamental neighbourhood retract of Q.

(2.21). Every fundamental retract of an FANRU-space (res. FARU-space) is an FANRU-space (res. FARU-space).

REMARK. By an example of Borsuk ([5]) a fundamental neighbourhood retract of an FARU-space is not necessarily an FANRU-space.

§ 3. EXTENSION OF FUNDAMENTAL SEQUENCES.

A fundamental sequence $f' = \{f'_k, X, Y\}_{P,Q}$ is called an *extension* of a fundamental sequence $f = \{f_k, A, Y\}_{P,Q}$ where $A \subset X \subset P$, iff $f'_k(x) = f_k(x)$ for every $x \in A$ and $k \in N$.

The following theorem was proved in [14].

3.1. THEOREM ([14]). *Let $f = \{f_k, A, Y\}_{P,Q}$ and $g = \{g_k, A, Y\}_{P,Q}$ where $A \subset X \subset P$, be homotopic fundamental sequences, If f has an extension $f' = \{f'_k, X, Y\}_{P,Q}$ then g has an extension $g' = \{g_k, X, Y\}_{P,Q}$ homotopic to f'.*

Let us prove the following.

3.2. LEMMA. *Let X, Y be metric spaces lying in ANRU-spaces P and Q respectively, $A \subset X$ and V be a uniform neighbourhood of Y in Q such that Y is a fundamental retract of V and $f = \{f_k, A, Y\}_{P,Q}$ be a fundamental sequence. It there exists a fundamental sequence $\widehat{f} = \{\widehat{f}_k, X, V\}_{P,Q}$ such that $\widehat{f}_k|A = f_k|A$ then f has an extension $f' = \{f'_k, X, Y\}_{P,Q}$.*

Proof. By (2.8) there exist a complete sequence of uniform neighbourhoods $\{V_k\}$ of Y and a fundamental retraction $r = \{r_k, V, Y\}_Q$ such that $r_k(y) = y$ for every $y \in V_k$. Since $f = \{f_k, A, Y\}_{P,Q}$ is a fundamental sequence, we may assume that $f_k(A) \subset V_k$ for every $k \in N$.

We put $f' = r\widehat{f}$ to complete the proof.

3.3. THEOREM. *Let X and Y be metric spaces lying in ANRU-spaces P and Q respectively and $A \subset X$. If $Y \in FANRU$ then every fundamental sequence $f = \{f_k, A, Y\}_{P,Q}$ can be extended to a fundamental sequence $f = \{f'_k, U, Y\}_{P,Q}$, where U is a uniform neighbourhood of A in X.*

Proof. Let V_0 be a uniform neighbourhood of Y in Q such that Y is a fundamental retract of V_0. Then there exist a uniform neighbourhood U_0 of A in P and $k_0 \in N$ such that

$f_k|U_0 \simeq f_{k_0}|U_0$ in V_0 for every $k \geqslant k_0$.

Clearly we may assume that $k_0 = 1$.

If W is a uniform neighbourhood of A in P such that $\rho(W, P \setminus U_0) > 0$ and $U = W \cap X$, then setting $\widehat{f}_k(x) = f_k(x)$ for every $x \in U_0$ and every $k \in N$, we get a fundamental sequence $\widehat{f}\left\{\widehat{f}_k\ U, V_0\right\}_{P,Q}$ such that $\widehat{f}_k \,|\, A = f_k|_A$. By Lemma 3.2 there exists an extension $f' = \left\{f'_k, U, Y\right\}_{P,Q}$ of $f = \left\{f_k, A, Y\right\}_{P,Q}$.

3.4. COROLLARY. *Let X, Y be uniformly homeomorphic metric spaces. If X is an FANRU — space then so is Y.*

Proof. Let P and Q be ANRU-spaces containing X and Y respectively and let $f: X \to Y$ be a uniform homeomorphism. Let $g = \left\{g_k, Y, X\right\}_{Q,P}$ be the fundamental sequence generated by f^{-1}. Since $X \in$ FANRU by 3-3 there exist a uniform neighbourhood V of Y in Q and a fundamental sequence $g' = \left\{g'_k V, X\right\}_{P,Q}$ extending g.

Let $f = \left\{f_k, X, Y\right\}_{PQ}$ be the fundamental sequence generated by f, and put $r = f' g' = \left\{f_k g'_k, V, Y\right\}_Q$. Since

$r_k(y) = f_k g'_k(y) = f f^{-1}(y) = y$ for every $y \in Y$

we infer that r is a fundamental retraction.

3.5. DEFINITION. *A metric space Y lying in an ANRU-space Q is called an FAEU-space iff for every subset A of a metric space X lying in ANRU space P, every fundamental sequence $f \,\|\, \left\{f_k, A, Y\right\}_{P,Q}$ has an extension $f' = \left\{f_k, X, Y\right\}_{P,Q}$.*

It is clear that every FAEU-space is an EARU-space. Theorem 3. 7 shows that the converse is not true.

Given $\varepsilon > 0$ and $x_0 \in X$, put

$$U^0(x_0, \varepsilon) = \left\{x \in X,\ \rho(x, x_0) \leqslant \varepsilon\right\}$$

and define $U^n(x_0, \varepsilon)$ by induction

$$U^n(x_0, \varepsilon) = \left\{ x \in X, \rho(x,y) \leqslant \varepsilon \text{ for some } y \in U^{n-1}(x_0, \varepsilon) \right\}.$$

3 . 6. DEFINITION ([1], [7]). A metric space X is called *absolutely bounded* iff for every $\varepsilon > 0$ there exist $n_0 \in N$ and $x_0 \in X$ such that $X = U_0^n(x_0, \varepsilon)$.

3.7. THEOREM. *Every EAFU-space is absolutely bounded.*

Proof. Assume on the contrary that Y is not absolutely bounded. Then there exists an $\varepsilon_0 > 0$ sugch that

$$Y \setminus U^n(y, \varepsilon_0) \neq \phi \text{ for every } y \in Y \text{ and } n \in N.$$

Select an arbitrary point $y_0 \in Y$ and for every $n \geqslant 1$ put

(3.8) $$y_n \in Y \setminus U^{n-1}(y_{n-1}, \varepsilon_0) \text{ for every natural number } n.$$

Let A denote the set of all natural numbers and $f: A \to Y$ be a map defined by

$$f(n) = y_n \text{ for every } n \in A.$$

It is clear f is uniformly continuous. Let P and Q be ANRU-spaces containing R^1 and Y respectively and $f = \{f_k, A, Y,\}_{P,Q}$ be the fundamental sequence generated by f. Since $Y \in FAEU$ there exists a fundamental sequence $f' = \left\{f'_k, R^1, Y\right\}_{P,Q}$ extending f. In particular we get a uniformly continuous map $f: R^1 \to Q$ such that

(3.9) $$|x - y| < \delta_0 \text{ implies } \rho(f'(x), f'(y)) < \varepsilon_0.$$

Let $n_0 > 1/\delta_0$ and $a_k = f'\left(n_0 + \frac{k}{n_0}\right)$, $k = 0,1,\dots, n_0$

By (3.9) we get

$$a_k \in U^{k-1}(y_{n_0}, \varepsilon_0) \text{ for every } k = 1,2,\dots, n_0.$$

In particular

$$y_{n_0+1} = a_{n_0} = f(n_0 + 1) \in U^{n_0-1}(y_{n_0}, \varepsilon_0) \subset U^{n_0}(y_0, \varepsilon_0)$$

This contradicts to (3–8) and thus the theorem is proved.

3.10 COROLLARY. *A metric space $Y \in FAEU$ if and only if Y is an FARU-space and diam $Y < \infty$.*

Proof. By (3.7) every FAEU-space is an bounded FARU-space. Assume that Y is an FARU-space and diam $Y < \infty$. Let

$$Q_1 = \{ x \in l^\infty(Y) : \|x\| \leqslant M \} \subset Q = l^\infty(Y)$$

where $M = \text{diam } Y$. By Katětov theorem [8], $Q_1 \in AEU$.

Let A be a subset of a metric space X lying in an ANRU-space P and $f = \{f_k, A, Y\}_{P, Q}$ be a fundamental sequence. We may assumex that $f_k(A) \subset Q_1$ for every $k \in N$. Let $f'_k : U_0 \to Q_1$, where U_0 is a uniform neighbourhood of X in P, be uniformly continuous maps such that $f'_k \big| A = f_k$ for every $k \in N$.

Since all uniformly continuous maps from P into Q_1 are homotopic, we infer that $f' = \left\{ f'_k, X, Q_1 \right\}_{P, Q}$ is a fundamental sequence. Moreover Y is a fundamental retract of Q_1, the result follows from 3.2.

3.11. COROLLARY. *Every bounded FARU-space is absolutely bounded.* By the same argument as in the proof of **3. 4** one shows the following.

3.12. COROLLARY. *Let X and Y be uniformly homeomorphic metric spaces. If X is an FAEU-space then so is Y.*

3.13. REMARK. Let us note that the corresponding statement of 3.4 and 3.12 for FARU-spaces is not true. Indeed let R^1 denote the real line and define on R^1 a new metric ρ by

$$\rho(x, y) = min\{|x-y|, 1\},$$

Then $(R^1, |.|)$ and (R^1, ρ) are uniformly homeomorphic. Obviously $(R^1, |.|) \in$FARU but by 3.11 (R^1, ρ) is not.

§ 4. CHARACTERIZING FAEU-SPACES

Two metric spaces X and Y lying in ANRU-spaces P and Q respectively are said to be of the *same shape* (denote Sh $X =$ Sh Y) iff there exist two fundamental sequences $f = \{f_k, X, Y\}_{P,Q}$ and $g = \{g_k, Y, X\}_{P,Q}$ such that

$$f g \cong id_Y \text{ and } gf \cong id_X. \tag{4.1}$$

Let us prove the following.

4.2. THEOREM. *Sh X does not depend on ANRU-Space $P \supset X$.*

Proof. Assume that X is contained in both ANRU-Spaces P and P', Y is contained in both ANRU-spaces Q and Q'. Then there exist uniform neighbourhoods U, U' of X in P, P' and V, V' of Y in Q, Q' respectively and uniformly continuous maps.

$$\varphi : U \to P' \qquad \varphi' : U' \to P$$
$$\psi : V \to Q' \qquad \psi' : V' \to Q$$

such that

$$\varphi(x) = \varphi'(x) = x \quad \text{for every point } x \in X$$
$$\psi(y) = \psi'(y) = y \quad \text{for every point } y \in Y$$

If there exist two fundamental sequences

$$f = \{f_k, X, Y\}_{P,Q} \qquad \text{and } g = \{g_k, Y, X\}_{P,Q}$$

such that

$$g\,f \cong id_X \qquad \text{and } f\ g \cong id_Y$$

then the formulas

$$f'_k = \psi f_k \varphi' \qquad \text{and } g'_k = \varphi\, g_k \psi'$$

define fundamental sequences $f' = \{f'_k, X, Y\}_{P',Q'}$ and $g' = \{g'_k, Y, X\}_{P',Q'}$ satisfying the conditions

$$g' f' \cong id_X \qquad \text{and} \qquad f' g' \cong id_Y.$$

The theorem is proved.

If $ShX = Sh(a)$, where (a) is a metric space consisting of only one point a then we say that the shape of X is *trivial.*

It is clear that

(4—3) If $Sh\,X \geqslant Sh\,Y$ and $Sh\,X$ is trivial then $Sh\,Y$ is trivial.

A metric space $X \subset Y$ is called *uniformly contractible* in Y iff the identity map id X is homotopic in Y to a constant map in X.

Obviously every AEU-space is uniformly contractible in itself.

Let us prove the following

4.4. THEOREM. *Let Y be a metric space lying in an ANRU-space Q. Then the following conditions are equivalent:*

a) *Y is an FAEU-space.*

b) *For every uniform neighbourhood U of Y in Q there exists a uniform neighbourhood U_0 of Y in Q that is uniformly contractible in U to a point $a \in Y$.*

c) *For every uniform neighbourhood U of Y in Q, Y is uniformly contractible in U.*

d) *$Sh\,Y$ is trivial.*

c) *There exists an AEU-space $Q_0 \supset Y$ such that for every uniform neighbourhood U of Y in Q_0 there exists a uniformly continuous map $f: Q_0 \to U$ such that $f(y) = y$ for every $y \in Y$.*

Proof. Clearly we may assume that $Q = I^\infty(Y)$.

a) $\Rightarrow$ b). Let $\{V_k\}$ be a complete sequence of uniform neighbourhoods of Y in Q and $r = \{r_k, Q, Y\}_Q$ be a fundamental retraction satisfying the condition (2-9).

Consider now a uniform neighbourhood U of Y in Q. Let k_0 be an index such that $r_{k_0}(Q) \subset U$. Since $r_{k_0}(V_{k_0}) = V_{k_0}$, we infer that $V_{k_0} \subset U$. Setting

$U_0 = V_{k_0}$ and $\varphi(x,t) = r_{k_0}(tx)$ for every $x \in U_0$, $t \in I$ we get a homotopy $\varphi: U_0 \times I \to U$ such that $\varphi(x, 1) = r_{k_0}(x) = x$, $\varphi(x,o) = r_{k_0}(o) = o \in Y$ for every $x \in U_0$.

Since $Y \in$ FAEU, by 3-7 we may assume that diam $U_0 < \infty$. Thus φ is uniformly continuous, that means U_0 is uniformly contractible in U.

b) $\Rightarrow$ c) is trivial.

c) $\Rightarrow$ b). Let U be a uniform neighbourhood of Y in Q. By hypothesis there exists a point $a \in Y$ such that

$$id_Q|_Y \cong C_a|_Y \text{ in } U_0$$

where $C_a: Q \to Y$ is the constant map to a and U_0 is a uniform neighbourhood of Y in Q such that $\rho(U_0, Q \setminus U) > 0$. By 1-5 there exists a uniform neighbourhood V_0 of Y in Q such that

$$id_Q|_{V_0} \cong C_a|_{V_0} \text{ in } U.$$

That means V_0 is uniformly contractible in U.

b) $\Rightarrow$ d). Assume that for every uniform neighbourhood U of Y in Q there exists a uniform neighbourhood U_0 of Y in Q that is uniformly contractible in U to a point $a \in Y$. Putling $X = (a) = P$ and $f_k(a) = a$, $g_k(x) = a$ for every $x \in Q$, we get fundamental sequences $f = \{f_k, (a), Y\}_{(a), Q}$ and $g = \{g_k, Y, (a)\}_{Q, (a)}$ such that $f\ g \cong id_Y$ and $g\ f = id_{(a)}$

Thus $Sh\ Y$ is trivial.

d) $\Rightarrow$ a). Let (a) be a metric space consisting of only one point a and $f = \{f_k, Y, (a)\}_{Q, (a)}$ and $g = \{g_k, (a), Y\}_{(a), Q}$ be fundamental sequences such that

$$(4.5) \qquad g\, f \cong id_Y$$

Let A be a subset of a metric space X lying in an ANRU — space P and $h = \{h_k, A, Y\}_{P, Q}$ be fundamental sequence. From (4 – 5) we obtain

$$g\ f\ h \cong h.$$

Since (a) consisting of only one point, the map $g\ f\ h$ has an extension over X. By 3 . 1, h has an extension $h' = \{h'_k, X, Y\}_{P, Q}$. Thus $Y \in FAEU$.

a) $\Rightarrow$ e). By 3 . 7, diam $Y < \infty$, thus by 1 . 3 there exists an AEU-space Q_0 containing Y.

Given a uniform neighbourhood U of Y in Q_0 and a fundamental retraction $r = \{r_k, Q_0, Y\}_{Q_0}$, there exists an index k_0 such that $r_{k_0}(Q_0) \subset U$. Since $r_{k_0}(y) = y$ for every $y \in Y$ we have e).

e) $\Rightarrow$ a). Let Q_0 be an AEU-space containing Y and $\{U_k\}$ be a complete sequence of uniform neighbourhoods of Y in Q_0. For every k, let $r_k : Q_0 \to U_k$ be a uniformly continuous map such that $r_k(y) = y$ for every $y \in Y$. Let us show that $r = \{r_k, Q_0, Y\}_{Q_0}$ is a fundamental sequence.

Since $Q_0 \in AEU$ there exists a uniformly continuous map $\varphi : Q_0 \times I \to Q_0$ such that $\varphi(y, o) = y$ and $\varphi(y, 1) = y_0 \in Y$ for every point $y \in Q_0$.

Given a uniform neighbourhood U of Y in Q_0 there exists an index k_0 such that $U_k \subset U$ for every k_0. Since

$$r_k\varphi : r_k \cong C_{y_0} \text{ in } U_k \text{ for every } k \in N$$

where $C_{y_0} : Q_0 \to Y$ is constant map to y_0, we infer that

$$r_k \cong r_{k_0} \text{ in } U \text{ for every } k \geqslant k_0.$$

The theorem is proved.

4 . 6 COROLLARY. *If $Sh\ X \geqslant Sh\ Y$ then $X \in FAEU$ implies $Y \in FAEU$.*

4 . 6 contains 3 . 12 *as a special case.*

Let us note that for the compacta, 4 . 4 and 4 . 6 has been proved by Bosruk [3], [5].

By 3 . 7 and 4 . 4 we get

4 . 7. COROLLARY. *If $Sh\ X$ is trivial then X is absolutely bounded.*

§ 5. A CONDITION CHARACTERIZING THE HOMOTOPY OF TWO FUNDAMENTAL SEQUENCES.

If Z is a metric space then by $\widehat{Z}$ we denote the cartesian product $Z \times I$. For compacta the following theorem has been established by Borsuk [5].

5 . 1. THEOREM. Let X and Y be metric spaces lying in ANRU-spaces P and Q *respectively. Two fundamental sequences $f = \{f_k, X, Y\}_{P,Q}$ and $g = \{g_k, X, Y\}_{P,Q}$ are homotopic if and only if there exists a fundamental sequence $\varphi = \{_k, \widehat{X}, Y\}_{\widehat{P},Q}$ such that for every $k = 1, 2, \ldots$*

(5 . 2) $\varphi_k(x, o) = f_k(x)$ and $\varphi_k(x, 1) = g_k(x)$ for every $x \in X$.

Proof. Assume that there exists a fundamental sequence $\varphi = \{\varphi_k, X, Y\}_{\widehat{P},Q}$ satisfying the condition (5 . 2). Let U_0 be a uniform neighbourhood of X in P

such that φ_k is defined on $U_0 \times I$ for every k. Set $f'_k(x) = \varphi_k(x,o)$ and $g'_k(x) = \varphi_k(x,1)$ for every $x \in U_0$.

. It is clear that $f \cong f' = \{f'_k, X, Y\}_{P,Q}$ and $g \cong g' = \{g'_k, X, Y\}_{P,Q}$. Therefore it suffices to prove that $f' \cong g'$.

For every uniform neighbourhood V of Y in Q. there exists a uniform neighbourhood $U \subset U_0$ of X in P such that

$\varphi_k | U \times I \cong \varphi_{k+1} | U \times I$ in V for almost all k. In particular

$\varphi_k (U \times I) \subset V$. Whence we get

$f_k | U \cong g'_k | U$ in V for almost all k.

Conversly, assume that $f \cong g$. Let $P = V_1 \supset V_2 \supset \dots$ be a complete sequence of uniform neighbourhoods of Y in Q. Let $\{U_n\}$ be a decreasing sequence of uniform neighbourhoods of X in P and $\{m_n\}$ be a sequence of indexes $1 = m_1 < m_2 < \dots$ such that $f_k | U_n \cong f_{k+1} | U_n \cong g_k | U_n$ in V_n for every $k \geqslant m_n$.

For every k, there exist n and a homotopy $\varphi_k : \hat{U}_n \to V_n$ such that $m_n \leqslant k < m_{n+1}$ and that

$\varphi_k(x,o) = f_k(x)$ and $\varphi_k(x,1) = g_k(x)$ for every $x \in U_n$.

Put

$\psi_k((x,t),s) = \varphi_k(x,(1-s)t)$ for $(x,t) \in \widehat{U}_n$ and $s \in I$.

Then we get

$\psi_k((x,t),o) = \varphi_k(x,t)$ and $\psi_k((x,t),1) = f_k(x)$ for every point $(x,t) \in \widehat{U}_n$.

Since $f_k | U_n \cong f_{k+1} | U_n$ in v_n we obtain

$\psi_{k+1} | \widehat{U}_n \cong \widehat{f}_{k+1} | \widehat{U}_n \cong \widehat{f}_k | \widehat{U}_n \cong \varphi_k | \widehat{U}_n$ in V_n for every $k \geqslant m_n$ where $\widehat{f}_i(x,t) = f_i(x)$ for $(x,t) \in \widehat{U}_i$.

Setting

$U_k = U_n$, $V_k = V_n$ for $k = m_n, m_{n+1}, \dots, m_{n+1} - 1$, we get sequences $\varphi_k : \widehat{U}_k \to V_k$ satisfying the condition 2.7 of Theorem 2.5. Therefore there exists a fundamental sequence $\varphi = \{\widehat{\varphi_k}, \widehat{X}, Y\}_{\widehat{P},Q}$ such that $\widehat{\varphi_k} | \widehat{X} = \varphi_k$. It is elear that the condition (5.2) is satisfied.

§ 6. STRONG MOVABILITY.

Let X be *a* metric space lying in an ANRU-space. We say that X is *strongly movable* in P iff for every uniform neighbourhood U of X in P there exists a uniform neighbourhood U_o of X in P such that for every uniform neighbourhood W of X in P there exists a homotopy $\varphi: U_0 \times I \to U$ satisfying the following conditions

(6 . 1) $\varphi(x,0) = x$, $\varphi(x,1) \in W$ for every $x \in U_0$

(6 . 2) $\varphi(x, 1) = x$ for every $x \in X$.

It is easy to see that choice of an ANRU-space $P \supset X$ is immaterial.

Let us prove the following.

6.3. THEOREM. *A metric space X is an FANRU-space if and only if X is strongly movable.*

Proof. Assume that $X \in$ FANRU. Let $r = \{r_k, V, X\}_P$ be a fundamental retraction of a uniform neighbourhood V of X in P to X. Given a uniform neighbourhood U of X in P, there exists an index k_0 such that

(6 . 4) $\quad r_k|V \simeq r_{k_o}|V$ in U forevery $k \geqslant k_o$.

Since $r_{k_0}|_X = id_P|_X$, by 1 . 5 there exists a uniform neighbourhood $U_0 \subset V$ of X in P such that

(6.5) $\quad r_{k_0}|U_o \simeq id_P|U_o$ in U.

Let W be an arbitrary uniform neighbourhood of X in P and $k_1 \geqslant k_0$ be an index such that $r_{k_1}(V) \subset W$. From (6 . 4) and (6 . 5) we obtain

$$r_{k_1}|U_o \simeq id_P|U_o \text{ in } U.$$

Thus there exists a homotopy $\varphi: U_0 \times I \to U$ such that $\varphi(x, o) = x$, $\varphi(x,1) = r_{k_1}(x) \in W$ for every $x \in U_0$ and $\varphi(x,1) = r_{k_1}(x) = x$ for every $x \in X$. Thus both conditions (6 . 1) and (6 . 2) are satisfied.

Now let us assume that X is strongly movable. Then there exists a complete sequence of uniform neighbourhoods $\{A_k\}$ of X in P such that for every $k = 1,2,\ldots$ there exists a homotopy $\varphi_k: V_k \times I \to V_{k-1}$ (we let $V_0 = P$) such that

(6 . 6) $\quad \varphi_k(x,o) = x$, $\varphi_k(x,1) \in V_{k+1}$ for every $x \in V_k$.

(6 . 7) $\quad \varphi_k(x,1) = x$ for every $x \in X$.

Let $r_1(x) = x$ for every $x \in V_1$ and assume that for every $i = 1, 2,..., k$, $r_i : V_1 \longrightarrow V_i$ has been defined satisfying the following conditions

$$r_i(x) = x \text{ for every } x \in X \text{ and } i = 1, 2,..., k.$$

$$r_i \cong r_{i+1} \text{ in } V_{i-1}.$$

Putting $r_{k+1}(x) = \varphi_k(r_k(x), 1)$ for every $x \in V_1$ we get a uniformly continuous map $r_{k+1}: V_1 \rightarrow V_{k+1}$ such that

$$r_{k+1}(x) = x \text{ for every } x \in X$$

$$r_{k+1} \cong r_k \text{ in } V_{k-1}.$$

Let V be a uniform neighbourhood of X in P such that $\rho(V, P\setminus V_1) > 0$. Then $r = \{r_k V, X\}_P$ is a fundamental retraction of V to X.

Let us note that for compacta (6.3) has been established by Borsuk [3].

§ 7. SHAPE INVARIANCE FOR FANRU-SPACES.

3.4 is a special case of the following.

7.1. THEOREM. *If $X \in FANRU$ and $ShX \geqslant ShY$ then $Y \in FANRU$.*

Proof. Assume that $X \in FANRU$. Consider X, Y as subsets of ANRU-spaces P and Q respectively. Let $r = \{r_k, G, X\}_P$ be a fundamental retraction of a uniform neighbourhood G of X in P to X. By hypothesis there exist two fundamental sequences $f = \{f_k, X, Y\}_{P,Q}$ and $g = \{g_k, Y, X\}_{Q,P}$ such that $f\,g \cong id_Y$.

Since G is a uniform neighbourhood of X in P there exist a uniform neighbourhood V' of Y in Q and an index k_1 such that

(7.2) $\quad g_k \mid V' \cong g_{k_1} \mid V'$ in G for every $k \geqslant k_1$.

Let V be a uniform neighbourhood of Y in Q such that $\rho(V, Q\setminus V') > 0$. Let us show that $\{f_k\, r_k\, g_k, V, Y\}_Q$ is a fundamental sequence.

Given a uniform neighbourhood W of Y in Q, let U be a uniform neighbourhood of X in P and $k_2 \geqslant k_1$ be an index such that

(7.3) $\quad f_k \mid U \cong f_{k_2} \mid U$ in W for every $k \geqslant k_2$.

Since $k_2 \geqslant k_1$, by (7.2) we infer that

(7.4) $\quad g_k \mid V' \cong g_{k_2} \mid V'$ in G for every $k \geqslant k_2$.

Moreover, we can assume that k_2 is so large that

(7.5) $\quad r_k \mid G \cong r_{k_2} \mid G$ in U for every $k \geqslant k_2$.

From (7.3) — (7.5) it follows that

$f_k r_k g_k \mid V' \cong f_{k_2} r_{k_2} g_{k_2} \mid V'$ in W for every $k > k_2$.

Since V' is a uniform neighbourhood of V in Q we infer that $\{f_k r_k g_k, V, Y\}_Q$ is a fundamental sequence.

Let $\widehat{r}$ denote the fundamental sequence $r \big| X = \{r_k, X, X\}_P$. Let us show that

(7.6) $f \widehat{r} g \cong id_Y$.

Given a uniform neighbourhood $\widehat{W}$ of Y in Q, let $\widehat{U}$ be a uniform neighbourhood of X in P such that

(7.7) $f_k(\widehat{U}) \subset \widehat{W}$ for almost all k.

By 1.5 one can assign to this uniform neighbourhood $\widehat{U}$ a uniform neighbourhood $\widehat{U_0}$ of X such that

(7.8) $r_k \big| \widehat{U_0} \cong id_P \big| \widehat{U_0}$ in U for almost all k.

We now can assign to the uniform neighbourhood $\widehat{U_0}$ of X a uniform neighbourhood $\widehat{V}$ of Y in Q such that

(7.9) $g_k(\widehat{V}) \subset \widehat{U_0}$ for almost all k, and that

(7.10) $f_k g_k \big| \widehat{V} \cong id_Q \big| \widehat{V}$ in $\widehat{W}$ for almost all k.

From (7.7) — (7.10) we obtain

$f_k r_k g_k \big| \widehat{V} \cong f_k g_k \big| \widehat{V} \cong id_Q \big| \widehat{V}$ in $\widehat{W}$ for almost all k.

Thus (7.6) is proved.

Since $\{f_k r_k g_k, V, Y\}_Q$ is an extension of fundamental sequence $f \widehat{r} g$, by 3.1 and (7.6), we infer that the fundamental sequence id_Y is extendable to a fundamental retraction r' of V to Y. Thus $Y \in FANRU$ and the theorem is proved.

REMARK. 1) As we have seen in 3.13 the corresponding theorem of 7.1 FARU — spaces is not true.

2) For compacta, Theorem 7.1 has been proved by Borsuk [5]: The proof of 7.1 given here is a slight modification of Boruk's argument.

§ 8. THE UNION OF TWO FAEU-SETS.

In this section we establish a theorem on the union of two FAEU — sets similar to that for ANRU — spaces [11] and L — spaces [12].

Let X_1 and X_2 be subsets of a metric space (X, ρ) such that $X_0 = X_1 \cap X_2 \neq \Phi$. We say that $X_1 \cup X_2$ is the uniform union of X_1 and X_2 iff the metric d on $X_1 \cup X_2$ defined by

8.1 $$d(x,y) = \begin{cases} \rho(x,y) \text{ if } x,y \in X_i, \text{ for } i = 1,2. \\ \inf \{\rho(x,z) + \rho(y,z) : z \in X_0\} \text{ otherwise.} \end{cases}$$

is uniformly equivalent to the metric ρ on $X_1 \cup X_2$.

Let us prove the following.

(8.2). THEOREM. *Let X_0, X_1, X_2 be FAEU – sets of a metric space X such that $X_0 = X_1 \cap X_2$ and $X = X_1 \cup X_2$ is the uniform union of X_1 and X_2. Then $X \in EAEU$.*

The proof of 8.2 is based on the following

8.3. LEMMA. Let $X = X_1 \cup X_2$ be a metric space lying in a metric space P. If $X = X_1 \cup X_2$ is uniform then for every uniform neighbourhood (U_1, U_2) of (X_1, X_2) in (P, P) there exists a uniform neighbourhood (V_1, V_2) of (X_1, X_2) in (P, P) such that $v_i \subset U_i$ for $i = 1,2$, and $V_1 \cup V_2$ is uniform.

Proof. Put

$$\delta = \frac{1}{2} \min \left\{ \rho(X_i, P \setminus U_i), i = 1,2 \right\}.$$

$$m = \inf \{\rho(x_1, x_2) : x_i \in A_i, i = 1,2\}.$$

where $A_i = \{x \in X_i, \rho(x, X_1 \cap X^2) \geqslant \sigma\}$.

Since $X_1 \cup X_2$ is uniform we infer that $m > o$. Let

$$\varepsilon = \min \left\{ \sigma, \frac{1}{4} m \right\} \text{ and put}$$

$$V'_i = \{x \in P : \rho(x, X_i) \leqslant \varepsilon\}, \ i = 1,2$$

$$V_0 = \{x \in P : \rho(x, X_1 \cap X^2) \leqslant \sigma\}$$

$$V_i = V_i' \cup V_0, \ i = 1,2$$

It is easy to see that $V_i \subset U_i$ for $i = 1,2$ and $V_1 \cup V_2$ is uniform

8.4. LEMMA. If $X = X_1 \cup X_2$ *is the uniform union of* X_1 *and* X_2 *then the map* $f : X \to Y$ *is uniformly continuous whenever* $f|_{X_1}$ *and* f_{X_2} *are.*

Proof. If $(x_n) \subset X_1$ and $(y_n) \subset X_2$ are sequences such that $\rho(x_n, y_n) \to 0)$ then there exists a sequence of points $(z_n) \subset X_0$ such that $\rho(x_n, z_n) + \rho(y_n, z_n) \to 0$.

Whence we get

$$\rho(f(x_n), f(y_n)) \leqslant \rho(f(x_n), f(z_n)) + \rho(f(z_n), f(y_n)) \to 0)$$

8.5. LEMMA. *A metric space Y lying in an AEU — space Q is an EAEU-space if and only if for every uniform neighbourhood U of Y in Q there exists a uniform neighbourhood U_0 of Y in Q such that for every metric space X, every map uniformly continuous map of a subset $A \subset X$ into U_0 can be extended to a uniformly continuous map from X into U.*

Proof. The sufficiency of the condition follows from 4.4 e). To prove the necessity, let given a uniform neighbourhood U of Y in Q, select a uniform neighbourhood U_1 of Y in Q such that $\rho(U_1, Q \setminus U) > 0$. By 4 — 4 e) there exists a uniformly continuous map $\varphi: Q \to U_1$ such that $\varphi(x) =$ for every $x \in Y$.

By **1.5** there exists a uniform neighbourhood U_0 of Y in Q such that, $\varphi \mid U_0 \cong \mathrm{id}_Q \mid U_0$ in U_2, where U_2 is a uniform neighbourhood of U_1 in Q such that $\rho(U_2, X \setminus U) > 0$. By Homotopy Extension Lemma there exits a uniformly continuous map $r: Q \to U$ such that $r \mid U_0 = id_Q \mid U_0$ and $r \cong \varphi$ in U.

Now let us assume that f is a uniformly continuous map of a subset A of foa metric space X into U_0. Since $Q \in AEU$ there exists a uniformly continuous map $\tilde{f}: X \to Q$ such that $\tilde{f} \mid A = f$. We put $f' = r\tilde{f}$ to complete the proof.

Proof of Theorem 8.2. By 3.7 and 1.3 there exists an AEU — space $P \supset X$. By 4.4 b) it is sufficient to show that for every uniform neighbourhood U of X in P there exists a uniform neighbourhood V of X in P such that V is uniformly contractible in U.

By 8.5 for every $i = 1,2$ there exists a uniform neighbourhood U_i of X_i in P such that every uniformly continuous map with the range in U_i can be extended to a uniformly continuous map with the range in U.

Since $X_0 \in FAEU$, by 4.4 b) there exist a uniform neighbourhood U_0' of X_0 in P and a homotopy $\varphi: U_0 \times I \to U_1 \cap U_2$ such that

(8.6) $\varphi(x,o) =$ x and $\varphi(x,1) = x_0 \in X_0$ for every $x \in U_0$

By 8.3 there exists uniform neighbourhoods V_1, V_2 of X_1, X_2 such that $V_i \subset U_i$ for every i $= 1,2$: $V_1 \cap V_2 \subset U_0$ and $V_1 \cup V_2$ is uniform. Put $V_0 = V_1 \cap V_2$.

For every $i = 1, 2$, let $f_i: V_i \times \{0,1\} \cup V_0 \times I \to U_i$ be a map defined by

(8.7) $\qquad f_i | V_i \times \{0\} = id_{V_i}$

(8.8) $\qquad f_i | V_0 \times I = \varphi$

(8.9) $\qquad f_i | V_i \times \{1\} = x_0$.

It is easy to see that f_i is uniformly continuous. Let $\widetilde{f}_i : V_i \times I \to U$ be an extension of f_i. Put $V = V_1 \cup V_2$ and ler $f : V \times I \to U$ be a map defined by

$$f(x,t) = f_i(x,t) \text{ if } (x,t) \in V_i \times I.$$

By 8.4, f is uniformly continuous. From (8.6) – (8.9) we obtain.

$$f(x, o) = x \text{ and } f(x, 1) = x_0 \text{ for every } x \in V.$$

The proof of Theorem 8.2 is finished.

8.10. **Example.** Let A, B, C be three points in the plane R^2 and put

$$X_1 = [A, B] \cup [B, C)$$
$$X_2 = [A, B] \cup [A, C)$$
$$X_0 = X_1 \cap X_2 \text{ and } X = X_1 \cup X_2.$$

It is easy to see that X_0, X_1, $X_2 \in FAEU$ but $X \notin FAEU$. Thus the assumption of the uniform union of X_1 and X_2 is essential.

Let us note that the corresponding theorem of 8.2 for FARU-spaces is not true. We see this by the following example.

8.11. EXAMPLE. In the plane R^2, consider the set

$$X_1 = [0, \infty) \times \{0\} \cup \{0\} \times [0,1]$$
$$X_2 = \{0\} \times [0,1] \cup [0,\infty) \times \{1\}$$
$$X_0 = X_1 \cap X_2 \text{ and } X = X_1 \cup X_2.$$

It is easy to see that X_0, X_1, X_2 are FARU-spaces (even *ARU*-spaces) and $X = X_1 \cup X_2$ is uniform. Let us show that X is not an FARU-space.

Put $U = \left\{x \in R^2 : \| x - X \| \leqslant \frac{1}{4}\right\}$.

It suffices to show that there is no uniformly continuous map f from R^2 into U fixing all points of X.

Indeed, if $f : R^2 \to U$ is any continuous map with $f(x) = x$ for $x \in X$ then we have

$$\operatorname{diam} f(\{n\} \times [0,1]) > 2\left(n - \frac{1}{4}\right) + \frac{1}{2} = 2n.$$

Therefore for every $n \in N$ there exist points $x_n, y_n \in \{n\} \times [0,1]$ such that

$$\| x_n - y_n \| \leqslant \frac{1}{n} \quad \text{and} \quad \| f(x_n) - f(y_n) \| > 1.$$

Thus f fails to be uniformly continuous.

§9. IMAGES OF FAEU-SPACES UNDER R'MAPS.

Adapting the terminology of Toruńczyk [8] let us say that uniformly continuous map $f : X \to Y$ is an *rf*map* if for every $\varepsilon > 0$ there exists a uniformly continuous map $g : Y \to X$ such that $\rho(f\,g(y), y) < \varepsilon$ for every point $y \in Y$.

9.1. THEOREM. *If Y is the image of an FAEU-space X under an r^* — map then $Y \in FAEU$.*

Proof. Let $\varphi : X \to Y$ be an r^*-map and $X \in FAEU$. By 3.7 diam $X < \infty$, hence diam $Y < \infty$. Since the choice of ANRU-spaces $P \supset X$ and $Q \supset Y$ are immaterial we may assume that.

$$P = \{x \in l^\infty(X) : \| x \| \leqslant \text{diam } X + 1\}$$
$$Q = \{y \in l^\infty(Y) : \| y \| \leqslant \text{diam } Y + 1\}$$

By 8.5 it suffices to show that given a uniform neighbourhood V of Y in Q there exists a uniform neighbourhood V_0 of Y in Q such that every uniformly continuous map with the range in V_0 can be extended to a uniformly continuous map with the range in V.

Let U be a uniform neighbourhood of X in P and $\tilde{\varphi} : U \to V$ be a uniformly continuous map such that $\tilde{\varphi}|_X = \varphi$. By 8.5 there exists a uniform neighbourhood U_0 of X in P such that every uniformly continuous map with the range in U_0 can be extended to a uniformly continuous map with the range in U. Put

$$\varepsilon = \frac{1}{3}\rho(Y, Q \setminus V)$$

and let $\psi : Y \to X$ be a uniformly continuous map such that

$$\| y - \varphi\psi(y) \| < \varepsilon/2 \quad \text{for every } y \in Y.$$

It is easy to see that there exists a positive number $\delta \leqslant \varepsilon$ and a uniformly continuous map $\tilde{\psi}$ from $V_0 = \{y \in Q : \| y - Y \| \leqslant \delta\}$ into U_0 such that $\tilde{\psi}|_Y = \psi$ and

(9.2) $\quad \| y - \tilde{\varphi}\tilde{\psi}(y) \| < \varepsilon$ for every point $y \in V_0$.

Put

$$V' = \{y \in Q : \| y - Y \| \leqslant 2\varepsilon\}$$

and let f be a uniformly continuous map from a subset A of a metric space Z into V_0. Then there exists a uniformly continuous map $g : Z \to U$ such that $g|A = \tilde{\psi} f$.

Let us put

$$\tilde{g}(x) = \tilde{\varphi}\, g(x) \quad \text{for every } x \in Z$$

Then $\tilde{g}(Z) \subset V$ and by (9.2)

$$\tilde{g} | A \cong f \text{ in } V'.$$

Thus by Homotopy Extension Lemma there exists a uniformly continuous map $\tilde{f} : X \to V$ such that $\tilde{f}|_A = f$. The theorem is proved.

REMARK. By 3.13, the corresponding statement of 9.1 for FARU-spaces in not true.

§ 10. CARTESIAN PRODUCT OF FAEU-SPACES.

Let (X_n, ρ_n) be a sequence of metric spaces. Put $X = \prod_{n=1}^{\infty} X_n$ and equip X with the metric ρ defined by

(10.1) $$\rho((x_n), (y_n)) = \sum_{n=1}^{\infty} \min \{\rho_n (x_n, y_n), 2^{-n}\} \text{ for } (x_n), (y_n) \in X.$$

10.2. THEOREM. *The Cartesian product $X = \prod_{n=1}^{\infty} X_n$ is an FAEU-space if and only if $X_n \in FAEU$ forevery n.*

Proof. Since X_n is uniformly homeomorphic to a uniform retract of X, by (2.21) and (3.12), $X \in FAEU$ implies that X_n is an FAUE-space for every n.

Now let us assume that $X_n \in$ FAEU for every n. By 3-7 and 1-3 for every n there exist an AEU-space $P_n \supset X_n$ and a fundamental retraction $r^n = \{r^n_k, P_n, X_n\}_{P_n}$ for every n = 1,2,... . Since $P_n \in$ AEU for every $n \in N$, $P = \prod_{n=1}^{\infty} P_n \in$ AEU. Setting $r_k(x) = (r^1_k(x_1); r^2_k(x_2),...)$ for $x = (x_1, x_2,...) \in P$, we get a sequence of uniformly continuous maps $r_k ; P \to P$ such that

$$r_k(x) = x \text{ for every } x \in X.$$

A straight forward verification shows that r is a fundamental sequence. The theorem is proved.

Since diam $X \leqslant 1$, by 3.10 and 10.2 we get

10-3. COROLLARY. *The Cartesian product $X = \prod_{n=1}^{\infty} X_n$ of FARU-spaces is not an AFRU-space if there exists an index n_0 such that diam $X_{n_0} = \infty$.*

Received June 5, 1981.

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