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ON THE DENSITY OF EXTREMAL SELECTIONS FOR MEASURABLE MULTIFUNCTIONS

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§1. INTRODUCTION: In this paper we are concerned with a property of measurable multifunctions, related to the bang bang principle in the control theory. It is well known an infinite – dimensional version of the Lyapunov convexity theorem, stating if Γ is an integrable convex compact-valued function from a measure space $(\Omega, \mathcal{A}, \mu)$ with $\mu \geqslant 0$ non-atomić, σ-finite, into a separable Banach space E then the set S_Γ of measurable selections of Γ contains as a dense subset the one of its profil, i.e. the multifunction $\ddot{\Gamma}$ defined as : $\ddot{\Gamma}(\omega) =$ the set of extremal points of $\Gamma(\omega)$, for the topology $\sigma(L^1_E, L^\infty_E)$.

In the present paper the above density property will be established for measurable multifunctions which need not be integrable with emphasis on the topology in which the density is considered. It will be shown that for the case when Ω is an interval in $\mathbf{R}$, the density property in question still holds for some topologies in L^1_E finer than the weak one.

We shall deal with measurable multifunctions taking values in a locally convex Suslin spaces E and consider S_Γ as subset of a generalized Orlicz space M_E of measurable functions from Ω into E, including thus the spaces L^p_E with $1 \leqslant p < \infty$, with E separable Banach.

The main results will be stated in § 2 and proved in **§ 4. Auxiliar lemmas** are given § 3.

§ 2. NOTATIONS AND STATEMENT OF RESULTS.

1. Notations. We shall deal with a complete measure space $(\Omega, \mathcal{A}, \mu)$ with $\mu > 0$ need not be σ – finite, a locally **convex** Suslin space E not necessarily metrizable [4]. As always, E' denotes the topological dual of E, $<.,.>$ the canonical bilinear form between E and E', $\mathcal{B}(E)$ the Borel tribe in E, $\mathcal{A}\otimes\mathcal{B}(E)$ the smallest δ-field in $\Omega\times E$ containing all the sets of form $A\times B$ with $A\in\mathcal{A}$ and $B\in\mathcal{B}(E)$. By a *measurable multifunction* from Ω into E is meant any function Γ from Ω into the collection of all the subsets of E such that $\mathrm{Graph}\,\Gamma \underline{\underline{\Delta}}$ $\{(\omega, x)\in\Omega\times E | x\in\Gamma(\omega)\}\in\mathcal{A}\otimes\mathcal{B}(E)$. For a function $f:\Omega\to E$, we shall say that it is *$\mathcal{A}$—measurable in the usual sense* if $f^{-}(U)\in\mathcal{A}$ for every open set U in E, or equivalently, $f^{-}(B)\in\mathcal{A}$ for every Borel set B in E,. Let us cite below two facts which will be used frequently without reference.

A/ *Let $(\Omega, \mathcal{A}. \mu)$ be a complete measure space with $\mu\geqslant 0$, finite. For a function $f:\Omega\to E$, the following statements are equivalent:*

a) Graph $\in\mathcal{A}\otimes\mathcal{B}(E)$

b) *f is scalarly measurable,* i.e. *for each $x'\in E'$, the function $\omega\to\langle f(\omega), x'\rangle$ is measurable in the usual sense.*

c) *f is the pointwise limit of an ordinary sequence of measurable functions assuming a finite number of values.*

d) *f is measurable in the usual sense*

e) *the function $\omega\to\langle f(\omega), g(\omega)\rangle$ is measurable (in the usual sense) for every scalarly measurable function $g:\Omega\to E'$.*

Proof : The statements $a)\Leftrightarrow b)\Leftrightarrow c)\Leftrightarrow d)$ can be found in [7] (Theorem III.3.6) while the implication $b)\Rightarrow e)$ follows immediately from $)\, c\Rightarrow e)$ and the inverse one is trivial.

B/ *With $(\Omega, \mathcal{A}, \mu)$ given as above, $E=\prod_{s=1}^{k}. E^s$ is the Cartesian product of locally convex spaces $E^1, E^2, \ldots, E^k$. Suppose that for each s either E^s is Lusin or $(E^s)'$ as well as E^s is Suslin. Then for each scalarly measurable function $g=(g^s)_{s=1}^{k}:\Omega\longrightarrow E'$, the multifunction defined by : $V(\omega)=\{x=(x^s)_{s=1}^{k}\in E \mid \max_{s=1,k}|\langle x^s, g^s(\omega)_s\rangle| < 1\}$ is measurable, where $\langle . , . \rangle_s$ denotes the canonical bilinear form between E^s and $(E^s)'$.*

Proof : It suffices to show that for each s, the function $(\omega, x^s)\to\langle x^s, g^s(\omega)\rangle_s$ is $\mathcal{A}\otimes\mathcal{B}(E^s)$—measurable. Indeed, since, clearly, the function $(\omega, x)\to(\omega, x^s)$ is measurable from $\Omega\times E$ into $\Omega\times E^s$, the function $(\omega, x)\to\langle x^s, g^s(\omega)\rangle_s$ is $\mathcal{A}\otimes\mathcal{B}(E)$— measurable, hence so is the function $(\omega, x)\to\max_{s=1,k}|\langle x^s, g^s(\omega)\rangle_s|$.

Consequently, Graph $V\in\mathcal{A}\otimes\mathcal{B}(E)$.

If E^s is Lusin then the function $(\omega, x^s) \to \langle x^s, g^s(\omega)\rangle_s$ which is continuousin x^s on E^s for each fixed $\omega \in \Omega$ and $\mathcal{A}$ —measurable for each fixed $x^s \in E^s$, is $\mathcal{A} \otimes \mathcal{B}(E^s)$—measurable by a result of Castaing [6, Cor. 3 to Theorem 1], If $(E^s)'$ is Suslin now, then by c) above applied to $(E^s)'$, this function is the limit of an ordinary sequence of $\mathcal{A} \otimes \mathcal{B}(E^s)$—measurable, hence it is $\mathcal{A} \otimes \mathcal{B}(E^s)$ —measurable itslef, what in turn yields the measurability of V.

In what follows, if $(\Omega, \mathcal{A}, \mu)$ is an arbitrary measurable space with $\mu \geqslant 0$, $\mathcal{M}_E(\Omega)$, or simply $\mathcal{M}_E$ ($\mathcal{N}_{E'}(\Omega)$, or simply $\mathcal{N}_{E'}$) will denote a vector space of measurable (scalarly measurable functions) from Ω into E(E', resp.) We write M_E ($N_{E'}$) for the quotient of $\mathcal{M}_E$ ($\mathcal{N}_{E'}$, resp.) by the almost everywhere equality equivalence relation. It will be always supposed that *$\mathcal{M}_E$ and $\mathcal{N}_{E'}$ form a duality pair*, i. e. for all $f \in \mathcal{M}_E$, $g \in \mathcal{N}_{E'}$ the function $\omega \to \langle f(\omega), g(\omega)\rangle$ is μ — integrable on Ω. For a set $A \subset E$ we say that $\mathcal{M}_E$ is A—*decomposable* if for every measurable set $\Omega_0 \subset \Omega$ of finite measure and for every measurable function $f_0 : \Omega_0 \to E$ such that $f_0(\omega) \in A$ a.e. on Ω_0 the function f_1 defined by: $f_1(\omega) = f_0(\omega)$ if $\omega \in \Omega_0$, $f_1(\omega) = f(\omega)$ if $\omega \in \Omega \setminus \Omega_0$, belongs to $\mathcal{M}_E(\Omega)$ for any $f \in \mathcal{M}_E(\Omega)$. We say simply that $\mathcal{M}_E$ is *decomposable* if it K—decomposable for every compact set K [7, VIII.3].

Finally, for a multifunction Γ from Ω into E, $\Gamma|\Omega_0$ denotes the restriction of Γ on Ω_0, $\mathcal{S}_\Gamma$ (S_Γ) denotes the set of all measurable selections (classes of measurable selections, resp.) of Γ, and ClCo Γ is the multifunction defined by, (ClCoΓ) (ω) = ClCoΓ (ω), where Co stands for the convex hull and Cl the closure of sets in E.

2. Statements of results.

In what follows, J is a closed subset of R, $\mu \geqslant 0$ is a Radon measure on J, E is a locally convex Suslin space. Let us introduce in M_E the topology defined by the system of neighborhoods in $\mathcal{M}_E$, given by:

$$U_{\varepsilon; g_1, g_2, \dots, g_m} = \{f \in \mathcal{M}_E \mid \sup_{\Delta \in \mathcal{Y}} |\int_{\Delta \cap J} \langle f(t), g_j(t)\rangle \mu(dt)| < \varepsilon \ \forall_{j=1,2,\dots,n}\}$$

where $\{g_1, g_2, \dots, g_m\} \subset \mathcal{N}_E$ and $\mathcal{Y}$ denotes the family of all intervals in R. We shall call it $s(M_E, N_{E'})$ — *topology*.

THEOREM 1: *Let μ be a non atomic measure on* J, Γ *be a measurable multifunction admitting at least one selection in $\mathcal{M}_E$. Suppose:*

(i) *either E is Lusin, or E' as well as E is Suslin*

(ii) *there is a balanced convex Borel set A in E such that $\mathcal{M}_E$ is A—decomposable and the range of Γ is contained in the linear hull of A.*

(ii') (*replacing* (ii)) $\mathcal{M}_E$ *is decomposable.*

Then $S_\Gamma \cap M_E$ *is dense in* $S_{ClCo\Gamma} \cap M_E$ *in the* $s(M_E, N_{E'})$ — *topology.*

Let (J, μ) be given as in Theorem 1 and E a separable Banach space. We introduce in $L_E(J)$ the norm defined by:

$$\|f\|_s = \sup_{\Delta \in \mathcal{Y}} \left\| \int_{\Delta \cap J} f(t)\mu(dt) \right\| \quad \text{for all } f \in \mathcal{L}^1_E(J) \text{ where } \|.\| \text{ is the norm in } E.$$

THEOREM 2: Under the above assumptions on (J, μ), *E, let* Γ *be a measurable multifunction from J into E, admitting at least one integrable selection. Then* $S_\Gamma \cap L^1_E$ *is dense in* $S_{ClCo\Gamma} \cap L^1_E$ *in the s-norm.*

THEOREM 3: Let $(\Omega, \mathcal{A}, \mu)$ *be an arbitrary complete measure space with* $\mu \geqslant 0$, *non atomic,* $\Gamma : \Omega \to 2^E$ *be a measurable multifunction, admitting at least one selection in* $\mathcal{M}_E$. *Suppose*

(i) *either E is Lusin, or E' (as well as E) is Suslin.*

(ii) *there is a balanced convex Borel set A in E such that* $\mathcal{M}_E$ *is A-decomposable and the range of* Γ *in contained in the linear hull of A.*

(ii') (*replacing* (ii)) $\mathcal{M}_E$ *is decomposable.*

Then $S_\Gamma \cap M_E$ *is dense in* $S_{ClCo\Gamma} \cap M_E$ *in the* $\sigma-(M_E, N_{E'})$ — *topology.*

THEOREM 4: Let $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $\mu \geqslant 0$, *nonatomic, E be a separable Banach space,* Γ *be a measurable multifunction from* Ω *into E admitting at least one integrable selection.*

Then the set $\int_\Omega (S_\Gamma \cap L^1_E)$ *is dense in the set* $\int_\Omega (S_{ClCo\Gamma} \cap L^1_E)$ *in the norm of E.*

Remarks: In the case where the multifunction Γ is integrable, the results of Theorem 1 and Theorem 2 were obtained in [12] for the finite dimensional E and in [13] for E being locally convex Lusin space. It should be noted that the s-topology in M_E is, obviously, finer than the $\sigma-(M_E, N_{E'})$-topology, wheaeras such an assertion fails when replacing the latter topology by the topology defined by the s-norm. Indeed, by virtue of Theorem 2, any element in $S_{ClCo\Gamma} \cap L^1_E$ can be approximated in the s-norm by an ordinary sequence in $S_\Gamma \cap L_E$ whenever the latter set is noempty. However such an assertion fails to hold if the s-norm is replaced by the $\sigma-(M_E, N_E)$-topology and, a fortiory, by s-topology. Namely, Valadier gave recently an example showing that even in the two—dimensional case one can not in general, approximate in the weak topology an element of $S_{ClCo\Gamma} \cap L^1_{R^2}$ by ordinary sequences in $S_\Gamma \cap L^1_{R^2}$ [19]. Thus Theorem 2 together with the mentioned example shows in particular that the s-norm topology is not stronger than the weak one in L^1_E. Theorem 3 is not quite new. It was obtained in [18] [19] in other ways for the case $\mathcal{M}_E(\Omega)=\mathcal{L}^1_E(\Omega), \mathcal{W}_E(\Omega) = \mathcal{L}^\infty_{E'}(\Omega)$ with separable Banach E under the σ-finiteness hypothesis on μ.

The first results on density property in question were apparently obtained in [5] [9] [10] [11] [16] [17] [3]. In context of control theory this question was considered in [20] [15] [9] [10] [1], [2], [8], [7], [14], where one can find more complete references on the subject as well as its applications.

§ 3 LEMMAS: In this section we shall prove some auxiliary lemmas which are essentially intermediate steps in proving the main results, stated in § 2. Some of these facts could be of interest in itself. At first, recall that a measurable space $(\Omega, \mathcal{A})$ is said to be *complete* if $\mathcal{A} = \bigcap_{\mu \in M^1_+} \widehat{\mathcal{A}}_\mu$, where $\widehat{\mathcal{A}}_\mu$ is the tribe of all the subsets in Ω of form: $A = A^1 \cup M$ with $A^1 \in \mathcal{A}$, $M \subset N \in \mathcal{A}$ and $\mu(N) = 0$, M^1_+ is the space of all probability measures on $(\Omega, \mathcal{A})$. It is easy to see that if $(\Omega, \mathcal{A}, \mu)$ is a complete measure space with $\mu > 0$, σ-finite, then $(\Omega, \mathcal{A})$ is a complete measurable space (for this notion and related facts see [7, Ch. III]).

LEMMA 1: *Let $(\Omega, \mathcal{A})$ be a complete measurable space, E be a locally-convex Suslin space, $\mathcal{C} = \{a_i\}_{i=1}^\infty$ be a sequence of measurable functions from Ω into E, and $a(.)$ be a measurable function such that $a(\omega) \in ClCo\{a_i(\omega)\}_{i=1}^\infty$ for all $\omega \in \Omega$. Then for any measurable multifunction V from Ω into E taking values in a system of neighbourhoods of the origin in E, there exist a nondecreasing sequence $\{\Omega^i\}_{i=1}^\infty$ of measurable sets in Ω and for each $i \in \mathbb{N}$, a set $\{\lambda^i_k\}_{k=1}^i$ of i nonnegative measurable functions on Ω^i such that $\sum_{k=1}^i \lambda^i_k(\omega) = 1$,*

$$\sum_{k=1}^i \lambda^i_k(\omega)\, a_k(\omega) \in a(\omega) + V(\omega) \text{ for all } \omega \in \Omega^i \ (i = 1, 2, \ldots) \text{ and } \bigcup_{i=1}^\infty \Omega^i = \Omega$$

Proof. Let S^i denotes the (countable) set of all rational vectors of the stardard simplex in $\mathbb{R}^i_+$ i. e.

$$S^i = \left\{ (\lambda_k)_{k=1}^{i} \;\middle|\; \lambda_k \geqslant 0, \text{ rational}, \sum_{k=1}^{i} \lambda_k = 1 \right\}. \text{ Let}$$

$$S^i = \left\{ \lambda^{i,n} \right\}_{n=1}^{\infty}, \quad \lambda^{i,n} = \left(\lambda_k^{i,n} \right)_{k=1}^{i}. \text{ Set } \lambda_e^{i,n}(\omega) =$$

$$\sum_{k=1}^{n} \lambda_k^{i,n} \cdot a_k(\omega), \quad \Omega^{i,n} = \left\{ \omega \in \Omega \mid \lambda_e^{i,n}(\omega) \in a(\omega) + V(\omega) \right\} \quad (\forall n \in \mathbb{N})$$

and $\Omega^i = \bigcup_{n=1}^{\infty} \Omega^{i,n}$ $(\forall\, i \in \mathbb{N})$.

It is easily seen that $\Omega^{i,n} \in \mathcal{A}$. Indeed, by hypotheses on $a(.)$, $a_i(.), \lambda_i(.)$ $(i = 1, 2, \ldots)$ and in virtue of A $(a) \Leftrightarrow (b)$ in § 1, the function $-a(.) + .\lambda_e^{i,n}(.)$ is measurable and our assertion follows then from the Yankov-Neumann's projection theorem for Suslin space [7. Theorem. III. 23]. Clearly, $\Omega^i \subset \Omega^{i+1} (\forall i \in \mathbb{N})$ and $\bigcup_{i=1}^{\infty} \Omega^i = \Omega$ by hypotheses on $a(.)$. For each $i \in \mathbb{N}$ set $\left\{ \lambda_k^i(\omega) \right\}_{k=1}^{i} = \left\{ \lambda_k^{i,n} \right\}_{k=1}^{i}$ for all $\omega \in \Omega^{i,n} \setminus \bigcup_{m<n} \Omega^{i,n}$. It is easy to check that the sets Ω^i and the functions $\left\{ \lambda_k^{i}(\omega) \right\}_{k=1}^{i}$ $(\forall i \in \mathbb{N})$ possess all desired properties.

COROLLARY: *Let $(\Omega, \mathcal{A})$, E given as in Lemma 1, let Γ be a measurable multifunction from Ω into E and $a(.) \in \mathcal{S}_{ClCo\Gamma}$. Then for every measurable multifunction V from Ω into E taking values in a system of neighborhoods of the origin in E, there exists a nondecreasing sequence $\left\{ \Omega^i \right\}_{i=1}^{\infty}$ of measurable subsets of Ω such that $\bigcup_{i=1}^{\infty} \Omega^i = \Omega$ and for each $i \in \mathbb{N}$, there exists a set $\left\{ \lambda_k^i(.) \right\}_{k=1}^{i}$ of i non-negative measurable functions on Ω^i such that $\sum_{k=1}^{i} \lambda_k^i(\omega) = 1$ and $\sum_{k=1}^{i} \lambda_k^i(\omega)\, a_k(\omega) \in a(\omega) + V(\omega)$ $(\forall \omega \in \Omega^i)$ where $\left\{ a_k(.) \right\}_{k=1}^{\infty}$ is a Castaing representation of Γ.*

Proof: The existence of such a sequence $\left\{ a_k(.) \right\}_{k=1}^{\infty}$ is well known [7, Theorem III. Since $a(\omega) \in Cl \left\{ a_i(\omega) \right\}_{i=1}^{\infty}$ $(\forall \omega \in \Omega)$ the conclusion follows readily from Lemma 1.

LEMMA 2 : *Let* $(\Omega, \mathcal{A})$ *be a measurable space, E be a locally convex space, A be a balanced convex Borel set in E and* $a_1, a_2, \ldots, a_n$ *be measurable functions with ranges contained in the linear hull of A.*

Then there exists a nondecreasing sequence $\{\Omega_p\}_{p=1}^{\infty}$ *of measurable sets in* Ω *such that* $\bigcup_{p=1}^{\infty} \Omega_p = \Omega$ *and*

(3.1) $\bigcup_{k=1}^{n} a_k(\Omega_p) \subset p.A \; (\forall_p \in \mathbb{N})$

Proof. Set $\Omega_p = \bigcap_{k=1}^{n} a_k^{-1}(p.A)$. Clearly, $\Omega_p \in \mathcal{A}$ and $\Omega_p \subset \Omega_{p+1}$ $(\forall_p \in \mathbb{N})$

Furthermore, since the range of $a_k (= k1,2,\ldots, n)$ is contained in the set $\bigcup_{p=1}^{\infty} p.A$ by hypothesis, we have $\Omega = \bigcup_{p=1}^{\infty} \Omega_p$. The inclusion (3.1) follows immediately from the definition of Ω_p $(p \in \mathbb{N})$

LEMMA 3 : *Let* $(\Omega, \mathcal{A}, \mu)$ *be a complete measure space with* $\mu \geqslant 0$*, finite, E be a locally convex Suslin space,* $u(.)$ $(i = 1,2,\ldots,n)$ *be measurable functions from* Ω *into* E. *Then there exists a nondecreasing sequence* $\{\Omega_p\}_{p=1}^{\infty}$ *of measurable sets in* Ω *such that* $\mu(\Omega \setminus \bigcup_{p=1}^{\infty} \Omega_p) = 0$ *and for each* $p \in \mathbf{N}$ *and* $i = 1, 2, \ldots, n$ $\mathrm{Cl} u_i (\Omega_p)$ *is compact.*

Proof : If $n = 1$ the result of Lemma 3 is well-known [7, Proposition VII-4]. The general case is reduced to the previous one by setting $u = (u_1, u_2, \ldots, u_n)$: $\Omega \rightarrow E^n$ and observing that the projection on E of a compact set in E^n is compact.

LEMMA 4 : *Let I be a compact set* $\mathbf{R}$, μ *be a nonatomic nonnegative Radon measure on I,* $(E, \|\,.\,\|)$ *be a separable Banach space,* $b_i \in \mathcal{L}_E^1(I)$, $\lambda_i \in \mathcal{L}_{R+}^{\infty}(I)$ $(i = 1,2,\ldots, n)$ $\sum_{i=1}^{n} \lambda_i(t) = 1$ *for all* $t \in I$. *Then, for every* $\varepsilon > 0$ *there exists a measurable partition of I into n sets* M_i $(i = 1,2,\ldots,n)$ *such that* $\sup_{\Delta \in Y} \left\| \int_{\Delta \cap I} \left[\Sigma_{i=1}^{n} \lambda_i(t) b_i(t) - \Sigma_{i=1}^{n} \lambda_{M_i}(t) b_i(t) \right] \mu(dt) \right\| < \varepsilon$ *where* $\mathcal{Y}$ *denotes the family of all intervals in* $\mathbf{R}$

Proof : In the case I is a compact interval the proof is given in [13]. The general case amounts to the above by taking a compact interval I_1 containing I and setting $\mu(G) = 0$ $(\forall G \subset I_1 \setminus I)$ $b_i(t) = o$, $\lambda_i(t) = 1/n$ $(i = 1, 2, \ldots, n)$ for all t in $I_1 \setminus I$.

LEMMA 5; *Let* $(\Omega, \mathcal{A}, \mu)$ *be a complete measure space with* $\mu > 0$, *finite, nonatomic,* $(E, \|\cdot\|)$ *be a separable Banach space,* $a_i(\cdot) \in \mathcal{L}^1_E$ $(i = 1, 2, \ldots, n)$ *and* $\lambda_i(\cdot)$ $(i = 1, 2, \ldots, n)$ *be nonnegative measurable functions on* Ω *such that* $\sum_{i=1}^{n} \lambda_i(\omega) = 1$ $(\forall \omega \in \Omega)$. *Then for every* $\varepsilon > 0$ *there exists a measurable partition* $\{M_i\}_{i=1}^{n}$ *of* Ω *such that*

$$\left\| \int_{\Omega} \left[\sum_{i=1}^{n} \lambda_i(\omega)\, a_i(\omega) - \sum_{i=1}^{n} \chi_{M_i}(\omega)\, a_i(\omega) \right] \mu(d\omega) \right\| < \varepsilon$$

Proof. Note first that if $a(.) \in \mathcal{L}^1_E(\Omega)$ and $\varepsilon_1 > o$ then there exists an (integrable) function $\overline{a}: \Omega \to E$ assuming at most a countable number of values such that $\|\overline{a}(\omega) - a(\omega)\| < \varepsilon_1$ $(\forall \omega \in \Omega)$. Indeed, if $\{x_k\}_{k=1}^{\infty}$ is a sequence dense in E, it suffices to set, for each $\omega \in \Omega$ $\overline{a}(\omega) = x_k$, if k is the smallest integer such that $a(\omega) \in B_{\varepsilon_1}(x_k)$, where $B_{\varepsilon_1}(x_k)$ denotes the ball around x_k with radius ε_1. Lef now $\overline{a}_i(.)$ be a measurable function with at most countable range such that $\|\overline{a}_i(\omega) - a_i(\omega)\| < \varepsilon/2\mu(\Omega)$ $(i = 1, 2, \ldots, n)$ and $\{\Omega^k\}_{k=1}^{\infty}$ be a measurable partition of Ω such that on each Ω^k the functions $a_i(.)$ take constant values, say, $\overline{a}_i(\omega) = c_i^k$ $(i = 1, 2, \ldots, n)$. By virtue of Lyapunov's theorem [7, Theorem IV, 17] for each $k \in \mathbf{N}$ there exists a measurable partition $\{\Omega_i^k\}_{i=1}^{n}$ of Ω^k such that $\mu(\Omega_i^k) = \int_{\Omega^k} \lambda_i(\omega)\, \mu(d\omega)$. Set $\Omega_i = \bigcup_{k=1}^{\infty} \Omega_i^k$. We have:

$$\int_{\Omega^k} \sum_{i=1}^{n} \lambda_i(\omega)\, \overline{a}_i(\omega)\, \mu(d\omega) = \sum_{i=1}^{n} c_i^k \int_{\Omega^k} \lambda_i(\omega)\, \mu(d\omega) = \sum_{i=1}^{n} c_i^k\, \mu(\Omega_i^k) =$$

$$= \sum_{i=1}^{n} \int_{\Omega_i^k} \overline{a}_i(\omega)\, \mu(d\omega). \text{ Hence, } \int_{\Omega} \sum_{i=1}^{n} \lambda_i(\omega)\, \overline{a}_i(\omega)\, \mu(d\omega) =$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{n} \int_{\Omega_i^k} \overline{a}_i(\omega)\, \mu(d\omega) = \sum_{i=1}^{n} \int_{\Omega} \chi_{\Omega_i}(\omega)\, \overline{a}_i(\omega)\, \mu(d\omega)$$

Finally, it follows from the choice of a(.) that

$$\left\|\int_\Omega \left[\sum_{i=1}^n \lambda_i(\omega)\, a_i(\omega) - \sum_{i=1}^n \chi_{\Omega_i}(\omega)\, a_i(\omega)\right]\mu(d\omega)\right\| =$$

$$\left\|\int_\Omega \left\{\sum_{i=1}^n \lambda_i(\omega)[a_i(\omega)-\overline{a}_i(\omega)] - \sum_{i=1}^n \chi_{\Omega_i}(\omega)[a_i(\omega)-\overline{a}_i(\omega)]\right\}\mu(d\omega)\right\| < \varepsilon$$

Thus the lemma is proved.

§ 4. PROOF OF THEOREMS:

1. Proof Theorem 1: Since the case $\mu(J) = 0$ is trivial, we can suppose that $\mu(J) > 0$. Let $b(.) \in \mathcal{S}_\Gamma \cap \mathcal{M}_E$ which exists by hypothesis, $a(.) \in \mathcal{S}_{ClCo\Gamma} \cap \mathcal{M}_E$, $\varepsilon > 0$ and $\{g_1, g_2, \dots, g_m\} \subset \mathcal{N}_E$, be given. We have to show the existence of an element $\overline{b}(.) \in \mathcal{S}_\Gamma \cap \mathcal{M}_E$ such that

(4.1) $\sup\limits_{\Delta\in\mathcal{Y}} \left| \int_{\Delta\cap J} \langle a(t)-\overline{b}(t), g_j(t)\rangle \mu(dt) \right| < \varepsilon \ (\forall_j = 1,2,\dots,m)$

To do it take first a bounded interval $I \subset \mathbb{R}$ such that

(4.2) $\int_{J\setminus I} |\langle a(t) - b(t), g_j(t)\rangle| \mu(dt) < \varepsilon/4 \ (\forall_j = 1, 2,\dots,n)$

Since as noted above in §2 the multifunction $t \to V(t) =$ $= \{x\in E \mid |\langle x, g_j(t)\rangle| < \varepsilon/4\mu(I\cap J)\}$ is measurable, according to Corollary to Lemma 1 there exist a measurable set $J_\varepsilon \subset J\cap I$ a finite set $\{b_i\}_{i=1}^n$ in $\mathcal{S}_\Gamma$ along with n nonnegative measurable functions $\lambda_i(.)$ $(i = 1,2,\dots,n)$ such that

(4.3) $\max\limits_{j=1,2,\dots m} \int_{(J\cap I)\setminus J_\varepsilon} |\langle a(t) - b(t), g_j(t)\rangle| \mu(dt) < \varepsilon/8$

$$\sum_{i=1}^n \lambda_i(t) = 1 \ (\forall t \in J_3) \text{ and}$$

$$a(t) - \sum_{i=1}^n \lambda_i(t)\, b_i(t) \in V(t) \ (\forall t \in J_\varepsilon), \text{ i.e.,}$$

(4.4) $\max\limits_{j=1,2,\dots,m} \left|\langle a(t) - \sum_{i=1}^n \lambda_i(t) b_i(t), g_i(t)\rangle\right| < \varepsilon/4\mu(I\cap J) \ (\forall t\in J_\varepsilon)$

Applying Lemma 2, or Lemma 3 under Assumption (ii)', to the functions $t \to \left(b_i(t), \langle b_i(t), g_1(t)\rangle, \langle b_i(t), g_2(t)\rangle, \dots \langle b_i(t), g_m(t)\rangle\right)$ $(i = 1,2,\dots, n)$ from J_ε into $E\times\mathbb{R}^m$; there exist a measurablese $J^\circ \subset J_\varepsilon$ and a number $p > 0$ such that

$$(4.5)\int_{J_\varepsilon / J^0} \max_{j=1,2,\ldots,m} \left|\left\langle a(t) - b(t), g_j(t)\right\rangle\right| \mu(dt) < \varepsilon/8,$$

$$(4.6') \max_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m}} \left|\left\langle b_i(t), g_j(t)\right\rangle\right| < p \ (\forall t \in J^0)$$

(4.6'') $b_i(J^0) \subset pA$, or, under Assumption (ii)', $b_i(J^0) \subset K$, where K is a compact set ($i = 1.2,\ldots,n$).

Clearly, we can suppose that J^0 is compact.

Taking into account (4.6'), we are enable to apply Lemma 3 to the functions $t \to \left(\left\langle b_i(t), g_1(t)\right\rangle, \left\langle b_i(t), g_2(t)\right\rangle, \ldots, \left\langle b_i(t), g_m(t)\right\rangle\right)$ $(i = 1.2,\ldots,n)$ from J^0 into $\mathbb{R}^m$ (endowed with the norm $|\mathbf{x}| = \max_{j=1,2,\ldots,m} |x_j|$ for all $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$). According to this lemma there exists a measurable partition $\{M_i\}_{i=1}^n$ of J^0 such that

$$(4.7) \max_{j=1,2,\ldots,m} \sup_{\Delta \in \mathcal{Y}} \left|\int_{\Delta \cap J^0}\left[\sum_{i=1}^{n} \lambda_i(t) \left\langle b_i(t), g_j(t)\right\rangle - \sum_{i=1}^{n} \chi_{M_i}(t) \left\langle b_i(t), g_j(t)\right\rangle\right]\right.$$

$$\mu(dt) < \frac{3}{4}$$

Set

$$\overline{b}(t) = \begin{cases} b_i(t) & \text{if } t \in M_j \\ b(t) & \text{if } t \in J \setminus J^0 \end{cases}$$

By decomposability hypothesis on $\mathcal{M}_E$, it follows from (4.6'') that $\overline{b}(.) \in \mathcal{S}_{\Gamma} \cap \mathcal{M}_E$. Moreover, by (4.2), (4.3), (4.5) and the definition of $\overline{b}(.)$ we have

$$\left|\int_{\Delta \cap J}\left\langle a(t) - \overline{b}(t), g_i(t)\right\rangle \mu(dt)\right| < \left|\int_{\Delta \cap J^0}\left\langle a(t) - \sum_{i=1}^{n} \chi_{M_i}(t)\, b_i(t), g_j(t)\right\rangle \mu(dt)\right| + \varepsilon/^{\bullet}$$

and by (4.4), (4.7).

$$\left|\int_{\Delta \cap J^0} < a(t) - \sum_{i=1}^{n} \chi_{M_i}(t) b_i(t), g_j(t) > \mu(dt)\right| \leqslant \int_{J^0} \left| < a(t) - \right.$$

$$\left. - \sum_{i=1}^{n} \lambda_i(t) b_i(t), g_i(t) \right\rangle \Big| \mu(dt) + \left|\int_{\Delta \cap J^0}\left\langle \sum_{i=1}^{n} \lambda_i(t) b_i(t)\right.\right.$$

$$\left.\left. - \sum_{i=1}^{n} \chi_{Mi}(t) b_i(t), g_i(t) \right\rangle \mu(dt)\right| < \varepsilon/2$$

which shows (4.1).

Proof of Theorem 2: Let $b(.) \in \mathcal{S}_{\Gamma} \cap \mathcal{L}^1_E$. Let there be given $a(.) \in \mathcal{S}_{ClCo\Gamma} \cap \mathcal{L}^1_E$ and $\varepsilon > 0$. As above, we can suppose that $\mu(J) > 0$. Likewise to what have been done in the poof of Theorem 1, by using Corollary to Lemma 1 and Lemma 2 and the integrability of a,b we find a compact set $J^0 \subset J$ of nonzero measure, a finite set $\{b_i(.)\}_{i=1}^n$ in $\mathcal{S}_{\Gamma|J^0}$ and n nonnegative measurable functions $\lambda_i(.)$ $(i = 1,2,\dots,n)$ on J^0 such that

(4.8) $$\int_{J \setminus J^0} \| a(t) - b(t) \| \mu(dt) < \varepsilon/2$$

$$\sum_{i=1}^{n} \lambda_i(t) = 1 \qquad \text{for all } t \text{ in } J^0$$

(4.9) $$\| a(t) - \sum_{i=1}^{n} \lambda_i(t) b_i(t) \| < \varepsilon/4\mu(J^0)$$

(4.10) $\| b_i(t) \| <$ p for all t in J^0, for a suitable number $p \in \mathbf{N}$

By (4.10), it follows Lemma 3 the existence of a measurable partition $\{M_i\}_{i=1}^n$ of J^0 such that

(4.11) $$\sup_{\Delta \in \mathcal{Y}} \| \int_{\Delta \cap J^0} \left[\sum_{i=1}^{n} \lambda_i(t) b_i(t) - \sum_{i=1}^{n} \chi_{M_i}(t) b_i(t) \right] \mu(bt) \| < \varepsilon/4$$

Set

$$\bar{b}(t) = \begin{cases} \sum_{i=1}^{n} \chi_{M_i}(t) b_i t & \text{if } t \in J^0 \\ b(t) & \text{if } t \in J \setminus J^0 \end{cases}$$

Clearly, $\mathcal{S}_{\Gamma} \cap \mathcal{L}^1_E \ni \bar{b}(.)$. Moreover, it follows from (4.8), (4.9), (4.11) that

$$\| \int_{\Delta \cap J} [a(t) - \bar{b}(t)] \mu(dt) \| \leqslant (\varepsilon/2) + \| \int_{\Delta \cap J^0} [a(t) - \sum_{i=1}^{n} \chi_{M_i}(t) b_i(t)] \mu(dt) \| \leqslant$$

$$(\varepsilon/2) + \left\| \int_{\Delta \cap J^0} \left[a(t) - \sum_{i=1}^{n} \lambda_i(t) b_i(t) \right] \mu(dt) \right\| + \left\| \int_{\Delta \cap J^0} \left[\sum_{i=1}^{n} \lambda_i(t) b_i(t) - \sum_{i=1}^{n} \chi_{Mi}(t) b_i(t) \right] = \mu(dt) \| < \varepsilon$$

This concludes the proof.

3. Proof of Theorem 3: Let $b(.) \in \mathcal{S}_\Gamma \cap \mathcal{M}_E$. Given $a(.) \in \mathcal{S}_{clco\Gamma} \cap \mathcal{M}_E$, $\{g_j\}_{j=1}^m \subset \mathcal{N}_E$, and $\varepsilon > 0$, we have to show the existence of an element $\bar{b}(.) \in \mathcal{S}_\Gamma \cap \mathcal{M}_E$ such that

$$(4.12) \quad \left| \int_\Omega \langle a(\omega) - \bar{b}(\omega), g_j(\omega) \rangle \mu(d\omega) \right| < \varepsilon \ (j = 1, 2, \ldots, m)$$

Set $\Omega_k = \left\{ \omega \in \Omega \,\middle|\, \left(1/k+1\right) < \max_{j=1,2,\ldots,m} \left| \langle a(\omega) - b(\omega), g_j(\omega) \rangle \right| < k \right\}$

By A) e) $\Omega_k \in \mathcal{A}$ $(k = 0,1,2,\ldots,)$. Clearly $\{\Omega_k\}_{k=1}^\infty$

is nondecreasing and $< a(\omega) - b(\omega), g_j(\omega) > = 0$ for all $\omega \in \Omega \setminus \bigcup_{k=1}^\infty \Omega_k$

and $j = 1,2,\ldots, m$. Hence,

$$\sum_{k=1}^\infty \int_{\Omega_{k+1} \setminus \Omega_k} \max_{j=1,2,\ldots,m} \left| \langle a(\omega) - b(\omega), g_j(\omega) \rangle \right| \mu(d\omega) = \int_\Omega \max_{j=1,2,\ldots,m} \left| \langle a(\omega) - b(\omega), g_j(\omega) \rangle \right| \mu(d\omega) < +\infty,$$

which implies $\mu(\Omega_k) < \infty$ by definition of Ω_k and

$$(4.13) \quad \int_{\Omega \setminus \Omega_{k_0}} \max_{j=1,2,\ldots,m} \left| \langle a(\omega) - b(\omega), g_j(\omega) \rangle \right| \mu(d\omega) < \varepsilon/4$$

for some k_0 large enough. We can suppose that $\mu(\Omega_{k_0}) > 0$. By using Corollary to Lemma 1, applied to Ω_{k_0}, Γ, $a|\Omega_{k_0}$ and the multifunction $V: \omega \mapsto V(\omega) \triangleq \{x \in E \,\|\, \langle x, g_j(\omega) \rangle | < \varepsilon/4\mu(\Omega_{k_0})\}$ which is measurable by *B*) in §2, there exist a measurable set $\Omega_\varepsilon \subset \Omega_{k_0}$ together withi n elements $b_1, b_2, \ldots, b_n$ in $\mathcal{S}_{\Gamma|\Omega_\varepsilon}$ and n nonnegative measurable functions $\lambda_1(.), \lambda_2(.), \ldots, \lambda_n(.)$ on Ω_ε such that

$$(4.\ 14) \quad \int_{\Omega_{k_0} \setminus \Omega_\varepsilon} \max_{j=1,2,\ldots,n} |\langle a(\omega) - b(\omega), g_j(\omega) \rangle| \mu(d\omega) \qquad < \varepsilon/8$$

$\sum_{i=1}^n \lambda_i(\omega) = 1 \ (\forall \omega \in \Omega_\varepsilon)$ and $a(\omega) - \sum_{i=1}^n \lambda_i(\omega) b_i(\omega) \in V(\omega)$, i, e., (4.15) $$\max_{j=1,2,\dots,m} |\langle a(\omega), g_j(\omega)\rangle - \sum_{i=1}^{n} \lambda_i(\omega) \langle b_i(\omega), g_j(\omega)\rangle| < \varepsilon/2\mu(\Omega_{k_0}) \ (\forall \omega \in \Omega_\varepsilon)$$

By virtue of Lemma 2 (or Lemma 3, under Assumption 2 (ii)'), applied to the space $E \times R^m$ and the functions

$$\omega \mapsto (b_i(\omega), \langle b_i(\omega), g_1(\omega)\rangle, \langle b_i(\omega), g_2(\omega)\rangle, \dots, \langle b_i(\omega), g_m(\omega)\rangle) \quad (i = 1, 2, \dots, n)$$

there exist a measurable set Ω^0 in Ω_ε and an integer p such that

(4.16) $$\int_{\Omega_\varepsilon \setminus \Omega^0} \max_{j=1,2,\dots,m} |\langle a(\omega) - b(\omega), g_j(\omega)\rangle| \, \mu(d\omega) < \varepsilon/8$$

$|\langle b_i(\omega), g_j(\omega)\rangle| < p \ (\forall \omega \in \Omega^0, i = 1, 2, \dots, n; \ j = 1, 2, \dots, m)$ and $b_i(\Omega^0) \subset p \cdot A$, or, under Assumption 2) (ii'), $b_i(\Omega^0) \subset K$, where K is a compact set in E. By Ljapunov Theorem (see [1], [7]) there exists a measurable partition $\{M_i\}_{i=1}^{n}$ of Ω^0 such that

(4.17) $$\int_{\Omega^0} \left[\sum_{i=1}^{n} \lambda_i(\omega) \langle b_i(\omega), g_j(\omega)\rangle - \sum_{i=1}^{n} \chi_{M_i}(\omega) \langle b_i(\omega), g_j(\omega)\rangle \right] \mu(d\omega) = 0$$

$(j = 1, 2, \dots, m)$

Set $$\overline{b}(\omega) = \begin{cases} \sum_{i=1}^{n} \chi_{M_i}(\omega)\, b_i(\omega) & \text{if } \omega \in \Omega^0 \\ b(\omega) & \text{if } \omega \in \Omega \setminus \Omega^0 \end{cases}$$

By the decomposability hypothesis on $\mathcal{M}_E$ and the choice of Ω^0, one has $\overline{b}(\cdot) \in \mathcal{S}_\Gamma \cap \mathcal{M}_E$. From (4.13), (4 14), (4.16) it follows

(4.18) $$\left|\int_\Omega \langle a(\omega) - \overline{b}(\omega), g_j(\omega)\rangle \mu(d\omega)\right| < (\varepsilon/2) + \left|\int_{\Omega^0} \langle a(\omega) - \overline{b}(\omega), g_j(\omega)\rangle \mu(d\omega)\right|$$

The second term on the right is less then or equal to

$$\int_{\Omega^0} \left| a(\omega) - \sum_{i=1}^{n} \lambda_i(\omega)\, b_i(\omega), g_j(\omega)\rangle \right| \mu(d\omega) + \left| \int_{\Omega^0} \left\langle \sum_{i=1}^{n} \lambda_i(\omega)\, b_i(\omega) - \right.\right.$$

$$\left.\left. - \sum_{i=1}^{n} \chi_{M_i}(\omega)\, b_i(\omega), g_j(\omega)\right\rangle \mu(\omega) \right|$$

Thus, by virtue of (4.15), (4.17), the inequality (4.18) yields (4.12). This completes the proof.

4. Proof of Theorem 4: As above, it suffices to examine the case $\mu(\Omega) > 0$. Let $b(.) \in \mathcal{S}_\Gamma \cap \mathcal{L}^1_E$ existing by hypothesis. Given an $a(.) \in \mathcal{S}_{ClCo\Gamma} \cap \mathcal{L}^1_E$ and a number $\varepsilon > 0$, we have to find an element $\bar{b}(.) \in \mathcal{S}_\Gamma \cap \mathcal{L}^1_E$ such that

$$(4.19) \quad \left\| \int_\Omega [a(\omega) - \bar{b}(\omega)] \, \mu(d\omega) \right\| < \varepsilon$$

Set $\Omega_k = \{ \omega \in \Omega \mid (1/k+1) < \| a(\omega) \| + \| b(\omega) \| < k \}$. It is clear that

$\Omega_k \in \mathcal{A}$, $\Omega_k \subset \Omega_{k+1}$ $(k = 0, 1, 2, \ldots)$ and

$a(\omega) = b(\omega) = 0$

for all $\omega \in \Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k$. Hence,

$$\sum_{k=0}^{\infty} \int_{\Omega_{k+1} \setminus \Omega_k} (\| a(\omega) \| + \| b(\omega) \|) \, \mu(d\omega) = \int_\Omega [\| a(\omega) \| + \| b(\omega) \|] \, \mu(d\omega) < +\infty$$

It follows that $\mu(\Omega_k) < +\infty$ $(k = 0, 1, 2, \ldots)$ and

$$(4.20) \qquad \int_{\Omega/\Omega_{k_0}} [\|a(\omega)\| + \| b(\omega) \|] \, \mu(d\omega) < \varepsilon/4$$

for some integer k_0 large enough. We can suppose that $\mu(\Omega_{k_0}) > 0$. By virtue of Corollary to Lemma 1 there exists a measurable subset $\Omega_\varepsilon \subset \Omega_{k_0}$, n elements b_i $(i = 1, 2, \ldots, n)$ in $\mathcal{S}_{\Gamma|\Omega_\varepsilon}$ and n nonnegative measurable functions on Ω_ε, λ_i $(i = 1, 2, \ldots, n)$ such that

$$(4.21) \qquad \int_{\Omega_{k_0} \setminus \Omega_\varepsilon} [\| a(\omega) \| + \| b(\omega) \|] \, \mu(d\omega) < \varepsilon/8$$

$\sum_{i=1}^{n} \lambda_i(\omega) = 1$ $(\forall \omega \in \Omega_\varepsilon)$ and

$$(4.22) \qquad \left\| a(\omega) - \sum_{i=1}^{n} \lambda_i(\omega) \, b_i(\omega) \right\| < \varepsilon / 4\mu(\Omega_0) \quad (\forall \omega \in \Omega_\varepsilon)$$

Let Ω^0 be a measurable set in Ω_ε such that

$$(4.23) \int_{\Omega\varepsilon \setminus \Omega^0} [\| a(\omega)\| + \| b(\omega) \|] \, \mu(d\omega) < \varepsilon/8$$

and on Ω^0 the functions b_i $(i = 1, 2, \ldots, n)$ are bounded. The exis tence of such a set is ensured by Lemma 2. We can now apply Lemma 5 to the functions b_i $(i = 1, 2, \ldots, n)$. According to this lemma there exists a measurable partition

$\{M_i\}_{i=1}^n$ of Ω^0 such that

$$(4.24) \qquad \int_{\Omega^0} \left[\sum_{i=1}^n \lambda_i(\omega)\, b_i(\omega) - \sum_{i=1}^n \chi_{M_i}(\omega)\, b_i(\omega) \right] \mu(d\omega)\| < \varepsilon/4$$

It is easily seen that the element $\overline{b}$ defined as

$$\overline{b}(\omega) = \begin{cases} \sum_{i=1}^n \chi_{M_i}(\omega)\, b_i(\omega) & if\ \omega \in \Omega^0 \\ b(\omega) & if\ \omega \in \Omega \setminus \Omega^0 \end{cases}$$

is one to be found. Indeed, the inclusion $\overline{b} \in \mathcal{S}_\Gamma \cap \mathcal{L}_E^1$ is evident while the inequality (4.19) follows fom (4.20) — (4.24).

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