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PRINCIPAL SYSTEMS OF IDEALS

NGÔ VIỆT TRUNG
Institute of Mathematics, Hanoi

1. INTRODUCTION

The notion of a principal radical system of ideals is very important by proving that ideals of certain classes are radical; it was introduced by Hochster in [3] and then used more than once in [4], [5], [11], [12]. In this note we shall slightly extend this notion and show among other things that some nontrivial facts about regular sequences may be easily interpreted by the new notion (which we shall call « principal system of ideals »). Furthermore, one can construct new classes of radical ideals by certain specializations of known « generic » principal systems of ideals in polynomial rings over the integers. As an appendix we shall also show that the property of being a prime ideal is transitive by such specializations.

2. DEFINITION AND MAIN PROPERTIES

Throughout this note R will be a commutative noetherian ring with identity.

DEFINITION. A principal system of ideals in R is a family K of ideals in R with the following property: for every nonmaximal ideal $a \in K$ (« nonmaximal » means that one can find an ideal $c \in K$ with $c \supset a$) there exists an element $x \in R$ such that $(a, x) \in K$ and one of following conditions is satisfied:

(1) $a : x = a$ and $\bigcap_{n=1}^{\infty} (a,x^n) = a$,

(2) There is $b \in K$ with $b \supset a$, $b : x = b$, and $xb \subseteq a$.

REMARK. The condition $\bigcap_{n=1}^{\infty} (a,x^n) = a$ is satisfied if x is contained in the

Jacobson radical of R or if we have a graded situation (i.e. R is a graded ring, $\mathfrak{a}$ a graded ideal, and x a homogeneous element of positive degree).

EXAMPLE. For every regular sequence $x_1,\dots, x_r$ in R such that $\bigcap_{n=1}^{\infty} (x_1,\dots,x_i, x_{i+1}^n) = (x_1,\dots, x_i)$, $i = 0,\dots, r-1$, the family of ideals O_R, $(x_1),\dots, (x_1,\dots, x_r)$ forms a principal system of ideals. See [3, § 9], [4, (3.9)], [11, § 3] 12, § 3 for other examples.

The above notion has the following remarkable property:

PROPOSITION 1. *Let K be a principal system of ideals in R. Let U be a map from K to the set of ideals in R such that for all $\mathfrak{a} \in K$:*

(i) $\mathfrak{a} \subseteq U(\mathfrak{a})$,

(ii) $U(\mathfrak{a}) \subseteq U(\mathfrak{b})$ if $\mathfrak{a} \subseteq \mathfrak{b}$, $\mathfrak{b} \in K$,

(iii) $U(\mathfrak{a}) : x = U(\mathfrak{a})$ if $\mathfrak{a} : x = \mathfrak{a}$, $x \in R$.

Suppose further that $U(\mathfrak{a}) = \mathfrak{a}$ for all maximal $\mathfrak{a} \in K$. Then $U(\mathfrak{a}) = \mathfrak{a}$ for all $\mathfrak{a} \in K$.

Proof. Suppose the contrary. Choose $\mathfrak{a}$ maximal from the ideals $\mathfrak{c} \in K$ with $U(\mathfrak{c}) \neq \mathfrak{c}$. Then $\mathfrak{a}$ is nonmaximal in K. By the above definition there exists $x \in R$ such that $(\mathfrak{a},x) \in K$ and either (1) or (2) is satisfied. Note that $U(\mathfrak{a}, x) = (\mathfrak{a},x)$ because $\mathfrak{a} \subset (\mathfrak{a},x)$. Then by (i) and (ii) we have $\mathfrak{a} \subseteq U(\mathfrak{a}) \subseteq (\mathfrak{a},x)$, hence

$$U(\mathfrak{a}) = \mathfrak{a} + x(U(\mathfrak{a}) : x).$$

From this it follows in case (1) by using (iii):

$$U(\mathfrak{a}) = \mathfrak{a} + xU(\mathfrak{a}) = \mathfrak{a} + x^2U(\mathfrak{a}) = \dots \subseteq \bigcap_{n=1}^{\infty} (\mathfrak{a},x^n) = \mathfrak{a},$$

and in case (2) by using the relation $U(\mathfrak{a}) \subseteq U(\mathfrak{b}) = \mathfrak{b}$:

$$U(\mathfrak{a}) = \mathfrak{a} + x(\mathfrak{b} : x) = \mathfrak{a} + x\mathfrak{b} = \mathfrak{a}.$$

Therefore we have $U(\mathfrak{a}) = \mathfrak{a}$ in any case, which contradicts our choice of $\mathfrak{a}$.

REMARK. Many maps from an arbitrary family K of ideals in R to the set of ideals in R have the properties (i), (ii), (iii); for example, the maps:

(a) $\mathfrak{a} \to \sqrt{\mathfrak{a}}$ (the radical of $\mathfrak{a}$),

(b) $\mathfrak{a} \to$ the intersection of the primary components of $\mathfrak{a}$ whose associated primes are minimal over $\mathfrak{a}$,

(c) $\mathfrak{a} \to \varphi^{-1}(\mathfrak{a}R_M)$, where R_M is the localizalion of R at a multiplicatively closed set M and φ is the natural homomorphism from R to R_M.

Especially, if M is the set of the elements in R which are nonzerodivisors on all maximal ideals of K, i. e. $M = R \setminus$ the union of all associated primes of $\mathfrak{a}$

contained in some maximal ideals of K, then $\varphi^{-1}(aR_M) = a$ for all maximal $a \in K$ and therefore for all $a \in K$ by Proposition 1.

Now we obtain the following nontrivial fact about regular sequences:

COROLLARY 2. *Let $x_1,\ldots, x_r$ be a regular sequence in R such that $\bigcap_{n=1}^{\infty} (x_1,\ldots, x_i, x_{i+1}^n) = (x_1,\ldots, x_i)$, $i = 0,\ldots, r-1$. If $(x_1,\ldots,x_r)$ is an unmixed or prime ideal, then $(x_1,\ldots, x_i)$ is also an unmixed or prime ideal, respectively, for all $i = 0,\ldots, r$.*

Proof. It is easy to see from the above remark that we need only consider the case of a semilocal ring R whose maximal ideals are associated to $(x_1,\ldots,x_r)$. In this case R is a Cohen-Macaulay or a regular local ring, respectively; and for such a ring the statement is already known (see [8, Theorem 137] or [9, (17.F)]).

To check whether a family of ideals in R is a principal system of ideals we have the following criterion, which was used in [3, §5] to define the notion of a principal radical system of ideals (where U is the map $a \to \sqrt{a}$):

PROPOSITION 3. *Let K be a family of ideals in R and U a map from K to the set of ideals in R which have the following property: for every non maximal ideal a in K there exists an element $x \in R$ such that $(a,x) \in K$ and one of the following conditions is satisfied:*

(1) $U(a): x = U(a)$ and $\bigcap_{n=1}^{\infty} (a, x^n) = a$,

(2) There is $b \in K$ with $b \supset a$, $U(b): x = U(b)$, and $xb \subseteq a$.
Suppose further that $U(a) = a$ for all maximal $a \in K$. Then $U(a) = a$ for all $a \in K$, and therefore K is a principal system of ideals.

Proof. The proof is similar to the one of Proposition 1.

For regular sequences we have:

COROLLARY 4. (cf. [1, Theorem 1 and Corollary 1]) *Let $x_1,\ldots,x_r$ be elements in R such that they form a regular sequence in R_M, where M denotes the set of nonzerodivisors on $(x_1,\ldots,x_r)$ in R, and $\bigcap_{n=1}^{\infty} (x_1,\ldots,x_i, x_{i+1}^n) = (x_1,\ldots,x_i)$, $i = 0,\ldots, r-1$. Then $x_1,\ldots,x_r$ also form a regular sequence in R.*

Proof. Let U be the map $a \to \varphi^{-1}(aR_M)$, where φ denotes the natural homomorphism from R to R_M. Note that $\varphi^{-1}((x_1,\ldots,x_i) R_M) : x_{i+1} = \varphi^{-1}((x_1,\ldots,x_i) R_M)$, $i = 0,\ldots, r-1$. Then it is easy to see that 0_R, $\varphi^{-1}(x_1 R_M),\ldots,$

$\varphi^{-1}((x_1,\ldots,x_i)R_M)$ satisfies the hypothesis of Proposition 3. Therefore we have $\varphi^{-1}((x_1,\ldots,x_i)R_M)=(x_1,\ldots,x_i)$, and hence $(x_1,\ldots,x_i):x_{i+1}=(x_1,\ldots,x_i)$ for $i=0,\ldots,r-1$.

3. TRANSITIVITY

In this section we shall exhibit some conditions for the transitivity of principal systems of ideals by a homomorphism $R\to S$ of rings. Given an element $x\in R$, we shall denote by $\overline{x}$ the image of x in S.

PROPOSITION 5. *Let $R\to S$ be a flat homomorphism. Let K be a principal system of ideals in R. Then $\overline{K}=\{aS;\ a\in K\}$ is also a principal system of ideals in S.*

Proof. Let a be an ideal and x an element in R. Then, if $a:x=a$, $aS:\overline{x}=$

$=aS$ [9, (3. H) (1)], and, if $\bigcap_{n=1}^{\infty}(a,x^n)=a$, $\bigcap_{n=1}^{\infty}(aS,\overline{x}^n)=$ aS (see the proof of [9, (3.H) (2)]). With these facts we can easily check that K is a principal system of ideals in S.

The flatness of $R\to S$ in Proposition 5 may be simplified in certain situations :

PROPOSITION 6. *Let K be a principal system of ideals in R with only a maximal ideal m such that*

(i) $\bigcap_{n=1}^{\infty}(aS,m^nS)=aS$ for all $a\in K$,

(ii) $\mathrm{Tor}_1^R(R/m,S_M)=0$, *where M denotes the set of the nonzerodivisors on mS in S.*

Add to the condition (2) of the definition of a principal system of ideals the demand that $(b,x)\in K$. Then $\overline{K}=\{aS;\ a\in K\}$ is also a principal system of ideals in S.

Proof. Let U be the map $A\to\varphi^{-1}(AS_M)$, where A denotes an arbitrary ideal in S and φ the natural homomorphism from S to S_M. Then it is not hard to see from Proposition 3 that it suffices to show the following facts:

(iii) $\bigcap_{n=1}^{\infty}(aS,\overline{x}^n)=aS$ for all $a\in K$, $x\in m$,

(iv) $aS_M:\overline{x}=aS_M$ for $a\in K$, $x\in R$ with $(a,x)\in K$, $a:x=a$.

For (iii), using (i) we have

$$aS \subseteq \bigcap_{n=1}^{\infty} (aS, \bar{x}^{n}) \subseteq \bigcap_{n=1}^{\infty} (aS, m^n S) = aS.$$

For (iv) we shall first assume that $\mathrm{Tor}_1^R(R/(a,x), S_M) = 0$. Then, from the exact sequence

$$(*) \qquad 0 \to R/a \xrightarrow{x} R/a \to R/(a,x) \to 0$$

we get the exact sequence

$$0 \to S_M/aS_M \xrightarrow{\bar{x}} S_M/aS_M \to S_M/(a,x)S_M \to 0,$$

which implies that $aS_M : \bar{x} = aS_M$. Therefore, the proof will be complete if we can show below that $\mathrm{Tor}_1^R(R/a, S_M) = 0$ for all $a \in K$. Suppose the contrary. Choose a maximal from the ideals $c \in K$ with $\mathrm{Tor}_1^R(R/c, S_M) \neq 0$. Then a is nonmaximal in K because (ii). Therefore there exists $x \in R$ such that $(a, x) \in K$ and either (1) or (2) of the definition of a principal system of ideals is satisfied. In case (1) we have the exact sequence(*). Note that $\mathrm{Tor}_1^R(R/(a, x), S_M) = 0$ because $(a, x) \supset a$. Then, from this sequence we get the exact sequence

$$\mathrm{Tor}_1^R(R/a, S_M) \xrightarrow{\bar{x}} \mathrm{Tor}_1^R(R/a, S_M) \to 0,$$

or, equivalently, the relation $\bar{x}\,\mathrm{Tor}_1^R(R/a, S_M) = \mathrm{Tor}_1^R(R/a, S_M)$. But $\bar{x}$ is contained in the Jacobson radical of S_M; hence, by Nakayama's lemma [9, (1. M)], $\mathrm{Tor}_1^R(R/a, S_M) = 0$, and we get a contradiction. In Case (2) we have $a : x = b$ and therefore the exact sequence

$$0 \to R/b \xrightarrow{x} R/a \to R/(a,x) \to 0.$$

Note that $\mathrm{Tor}_1^R(R/(a,x), S_M) = 0$. and $\mathrm{Tor}_1^R(R/b, S_M) = 0$. So from the above sequence we easily get $\mathrm{Tor}_1^R(R/a, S_M) = 0$, a contradiction.

REMARK. Proposition 5 can be considered as a generalization of [10, Corollary 1. 2 and Corollary 1. 4].

In Proposition 6, if m is perfect, i. e. the homological dimension of R/m as R-module is equal to the grade of m (the lenght of a maximal regular sequence of R in m), and grade $mS_M) =$ grade m, then the condition (ii) is satisfied. Especially, if m is generated by a regular sequence $x_1, \ldots, x_r$, then m is perfect; and grade mS_M = grade $m = r$ means that $x_1, \ldots, x_r$ is also a regular sequence of S_M [8, Theorem 29] or, equivalently, of S by Corollary 4.

Until now a lot of principal systems K of (radical) ideals in polynomial rings $Z[X] = Z[X_1, \ldots, X_r]$ over the ring Z of integers, where r is large enough are known in [3, §9], [4, (3.9)], [5, (2.2)], [11, §3], [12, §3], where K may be always chosen to have only a maximal ideal which is just $(X_1, \ldots, X_r)$. From these «generic» cases one can easily construct new classes of radical ideals. For, one only need to send $X_1, \ldots, X_r$ to a regular sequence $x_1, \ldots, x_r$ of a given ring S (that induces a homomorphism from $Z[X]$ to S) such that $(x_1, \ldots, x_r)$ is a radical ideal and $\bigcap_{n=1}^{\infty} (aS + (x_1, \ldots, x_r)^n) = aS$ for all $a \in K$; and $K = \{aS; a \in K\}$ is then a principal system of (radical) ideals in S.

Notice that the prime members of the known principal systems of ideals in the above mentioned papers are always homogeneous and that they generate prime ideals in $k[X]$ for every field k. Then we shall show below as an appendix that the image of a such prime ideal by a such homomorphism $Z[X] \to S$ is also a prime ideal if $(x_1, \ldots, x_r)$ is prime.

APPENDIX. (cf. [4, Lemme (3.14)] and Corollary 2) *Let $Z[X] = Z[X_1, \ldots, X_r]$ be a polynomial ring over the integers and I a homogeneous prime ideal which generates a prime ideal in $k[X]$ for every field k. Let $Z[X] \to S$ be a homomorphism of rings such that*

(i) *The images $x_1 \ldots, x_r$ of $X_1, \ldots, X_r$ in S form a regular sequence in S,*

(ii) *$P = (x_1, \ldots, x_r)$ is a prime ideal and $\bigcap_{n=1}^{\infty} (IS, P^n) = IS$. Then IS is also a prime ideal.*

Proof. By [9, (17.D)] it suffices to show that $\mathrm{gr}_{P/IS}(S/IS)$ is a domain. First, we have: $\mathrm{gr}_P(S) \cong R[X]$, where R denotes the factor ring S/P (see [9, (16.6) Theorem 32 (2) ⇒ (3)]). Let I^* denote the ideals of the leading forms of the elements of IS in $\mathrm{gr}_P(S)$. Then, by [9, (17.C)], $\mathrm{gr}_{P/IS}(S/IS) \cong R[X]/I^*$. Therefore, we must show that I^* is a prime ideal in $R[X]$. Since I is homogeneous, $IR[X] \subseteq I^*$. Further, $IR[X]$ is a prime ideal by [4, Lemma (3.14) (1)]. Thus, we only need to show that $ht\ I^* \leqslant ht\ IR[X]$ because this will imply $I^* = IR[X]$. Let K denote the quotient field of R. Then $ht\ I^* \leqslant ht\ I^*K[X]$ because $K[X]$ is a localization of $R[X]$. But, since P^n is primary for all n by [5, (2.1)], $I^*K[X]$ is the ideal of the leading forms of the elements of IS_P

in $K[X] \cong \mathrm{gr}_{PS_P}(S_P)$. Hence, $ht\ I^{*}K[X] = ht\ IS_P$ by [2, Kap. II § 3]. On the other hand, by the superheight theorem [7, (7. 1)], $ht\ IS_P \leqslant ht\ IR[X]$. Thus, summing up all these facts we get $ht\ I^{*} \leqslant htIR\ X$, as required.

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REFERENCES

1. Eisenbud, D., Hermann, M., Vogel, W., *Remarks on regular sequences*, Nagoya Math. J. 67 (1977), 177 — 180.

2. Gröbner, W., *Algebraische Geometrie II*, Bibliographisches Institut Mannheim — Wien — Zurich 1970.

3. Hochster, M., Eagon, I.A., *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. M., Math. 93 (1971), 1020—1058.

4. Hochster, M., *Grassmannians and their Schubert subvarieties are arithmetically Cohen—Macaulay*, J. Algebra 25 (1973), 40 — 57.

5. Hochster, M., *Criteria for equality of ordinary and symbolic powers of primes*, Math. Z. 133 (1973), 53 — 65.

6. Hochster, M., *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. 29 (1974), 55 — 76.

7. Hochster, M., *Big Cohen-Macaulay modules and algebras and embeddaility in rings of Witt vectors*, Proc. conference Comm. Algebra Queen's univ., Kingston, Ontario 1975.

8. Kaplansky, I., *Commutative rings*, Allyn and Bacon 1970.

9. Matsumura, H., *Commutative Algebra*, Benjamin, New York 1970.

10. Ramras, m., *Constructing new R-sequences*, Pacific J. Math. 71 (1977), 133 — 138.

11. Trung, N.V., *On the symbolic powers of determinantal ideals*, J. Algebra 58 (1979), 361 — 369.

12. Trung, N.V., *A class of imperfect prime ideals having the equality of ordinary and symbolic powers*, to appear.