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NON-STATIONARY INVERSE SCATTERING PROBLEM FOR PERTURBATION WAVE EQUATION

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In this work the non-stationary inverse scattering problem for perturbation wave equation is studied

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} + c(x, t)u(x,t) = 0 \quad 0 \leqslant x < \infty, -\infty < t < \infty, \tag{1}$$

with the boundary condition

$$u(0, t) = \varphi(t) \tag{2}$$

and the irradiation conditions

$$|u(x, t)| \leqslant C, \left|\frac{\partial u(x, t)}{\partial x}\right| \leqslant C, \left|\frac{\partial u(x, t)}{\partial t}\right| \leqslant C, \ C - \text{const},$$

$$\frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t} \to 0 \text{ as } x \to \infty. \tag{3}$$

The non-stationary inverse scattering problem for equation (1) with condition (2) which equals zero was resolved in [1]. First we examine the non-homogeneous wave equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} = \rho(x, t) \tag{4}$$

We demonstrate the following lemma

Lemma 1. The solution of problem (4), (2)—(3) is unique.

Proof : We suppose that there are two solutions of this problem. Thus, their difference $\tilde{u}(x, t)$ satisfies the homogenous equation

$$\frac{\partial^2 \tilde{u}(x, t)}{\partial t^2} - \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} = 0 \tag{5}$$

the boundary condition

$$\tilde{u}(0, t) = 0 \tag{6}$$

and irradiation conditions (3).

The solution of Cauchy's problem for equation (5) with the boundary condi tion (6) and initial date by $t = t_0$ has following form by $x \leqslant t - t_0$:

$$\tilde{u}(x, t) = \frac{1}{2}[\tilde{u}(x + t - t_0, t_0) - \tilde{u}(t - t_0 - x, t_0)] +$$

$$+ \frac{1}{2}\int_{t-t_0-x}^{x+t-t_0} \frac{\partial \tilde{u}(y, t_0)}{\partial t} dy = \frac{1}{2}\int_{t-t_0-x}^{x+t-t_0}\left[\frac{\partial \tilde{u}(y, t_0)}{\partial t} + \frac{\partial \tilde{u}(y, t_0)}{\partial y}\right]dy, \tag{7}$$

From (7) it is easy to obtain the valuation

$$|\tilde{u}(x, t)| \leqslant x \max_{t-t_0-x \leqslant y \leqslant t-t_0+x} \left| \frac{\partial \tilde{u}(y, t_0)}{\partial y} + \frac{\partial \tilde{u}(y, t_0)}{\partial t} \right|. \qquad (8)$$

Taking into consideration that from the condition $y \in [t - t_0 - x, t - t_0 + x]$ and $t_0 \to -\infty$ it results $y \to \infty$, and using the irradiation conditions (3) we find that the right member of inequality (8), as $t_0 \to -\infty$, becomes zero. Thus $\tilde{u}(x, t) = 0$ for any point (x, t). Therefore the uniqueness of solution is proved.

Lemma 2. Let the right member of equation (4) satisfy the condition

$$|\rho(x, t)| \leqslant \frac{C}{(1+x)^{1+\varepsilon}(1+|t|)^{1+\varepsilon}}, \quad \varepsilon > 0 \qquad (9)$$

and the function $\varphi(s)$, $-\infty < s < \infty$, of boundary condition (2) be a continuously differentiable function and

$$|\varphi(s)| \leqslant C, \quad \left|\frac{d\varphi(s)}{ds}\right| \leqslant C \qquad (10)$$

Thus function

$$u(x, t) = \varphi(t - x) + \frac{1}{2}\int_{-\infty}^{t-x} d\tau \int_{t-x-\tau}^{t+x-\tau} \rho(y, \tau)dy + \frac{1}{2}\int_{t-x}^{t} d\tau \int_{x-t+\tau}^{t+x-\tau} \rho(y, \tau)dy, \qquad (11)$$

is solution of problem (4), (2)—(3).

Proof: We verify directly that (11) is solution of equation (4) with $u(0, t) = \varphi(t)$. Furthermore by using (9)—(10), from (11) we obtain the regular limitation of function $u(x, t)$ and its first derivatives, except that

$$\left|\frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t}\right| = \left|\int_{-\infty}^{t} \rho(x + t - \tau, \tau)d\tau\right| \leqslant$$

$$\leqslant \frac{C}{(1+x)^{1+\varepsilon}} \int_{-\infty}^{\infty} \frac{d\tau}{(1+|\tau|)^{1+\varepsilon}} \to 0 \qquad \text{as } x \to \infty \qquad (12)$$

In (12), when $x \to \infty$ the right member tends to zero regularly in t, i.e. function $u(x, t)$ satisfies the irradiation conditions (3) So this lemma is proved.

We go on to the solution of scattering problem (1)—(3).

Theorem 1. Let $\varphi(s)$ be a continuously diffierentiable function and $|\varphi(s)| \leqslant C$, $\left|\frac{d\varphi(s)}{ds}\right| \leqslant C$, C — const and potential $c(x, t)$ satisfies the condition

$$|c(x, t)| \leqslant \frac{C}{(1+x)^{1+\varepsilon}(1+|t|)^{1+\varepsilon}}, \quad \varepsilon > 0 \qquad (13)$$

Thus the solution $u(x, t)$ of scattering problem (1)—(3) exist and is unique. This solution as $x \to \infty$ presents in form

$$u(x, t) = f(t - x) + 0(1) \qquad (14)$$

here $f(t-x)$ — scattering wave which is given by function $\varphi(t-x)$ and connec-
ed with this by equality

$$f(s) = \varphi(s) - \frac{1}{2}\int\limits_0^{\infty} dy \int\limits_{s-y}^{s+y} c(y, \tau)u(y, \tau)d\tau \qquad (15)$$

Proof: Using lemmas 1 and 2 we found that problem (1) – (3) is
quivalent to the following integral equation in the space $C(E^2)$ of conti-
uous functions in Euclidean 2-dimensional space E^2:

$$u(x, t) = \varphi(t-x) - \frac{1}{2}\int\limits_{-\infty}^{t-x} d\tau \int\limits_{t-x-\tau}^{t+x-\tau} c(y, \tau)\,u(y, \tau)dy - \frac{1}{2}\int\limits_{t-x}^{t} d\tau \int\limits_{x-t+\tau}^{x+t-\tau} c(y,\tau)\,u(y, \tau)dy, \quad (16)$$

Using the valuation (13) and the properties of function $\varphi(s)$ we can easily
emonstrate the uiqueness of solution of equation (16) and the convergence
consecutive approximating method for (16), i.e. equation (16) has a unique
lution.

If we examine the scattering wave $f(t-x)$ after (15), equation (16) has
llowing form

$$u(x, t) = f(t-x) + \frac{1}{2}\int\limits_{x}^{\infty} dy \int\limits_{t+x-y}^{t-x+y} c(y, \tau)\,u(y, \tau)d\tau \qquad (17)$$

om (17) in consideration of (13) we obtain easily (14). So the theorem is
monstrated.

According to the theorem 1 each function $\varphi(s)$ giving wave $\varphi(t-x)$ is
rrespondent to the function $f(s)$ which determines the scattering wave $f(t-x)$.
us the operator S which transfers $\varphi(s)$ in $f(s)$ is determined. This operator
is called the operator of scattering

$$S\varphi(s) = f(s) \qquad (18)$$

The operator S will be examined in space $L_2(-\infty, \infty)$.

We remark that in case the potential $c(x, t) = 0$, the scattering wave
$-x) = \varphi(t-x)$ and in this case the operattor S is equal to the unit operator.

In order to resolve the inverse scattering problem (1) — (3) we prove
following lemma.

Lemma 3. The solution $u(x, t)$ of equation (17) with any right member
$) \in C(E)$ exists and is unique in space $C(E^2)$. The solution of this equation
be written in following form

$$u(x, t) = f(t-x) + \int\limits_{-\infty}^{t} H_+(x, t, \xi)\, f(\xi - x)d\xi = (I + H_+(x))T_{-x}f(t). \qquad (19)$$

ere the operator $H_+(x)$ by fixed x is operator of Hilbert-Schmidt and T_{-x} is
operator of translation $T_{-x}f(t) = f(t-x)$. By $x \to \infty$ $\| H_+(x) \| L_2 \to 0$ (20)

The proof of existence and uniqueness of solution in $C(E^2)$ of
iation (17) with any right member $f(t-x) \in C(E)$ is analogous with the

demonstration of theorem 1. We get the solution of this equation in form (19). With this aim by replacing $u(x, t)$ from (19) into (17) with any f any $\xi \leqslant t$ we obtain

$$H_+(x, t, \xi) = \frac{1}{2} \int\limits_{x+\frac{t-\xi}{2}}^{\infty} c(y, \xi - x + y)dy + \frac{1}{2} \int\limits_{x}^{\infty} dy \int\limits_{t+x-y}^{t+y-x} c(y+\tau) H_+(y, \tau, \xi + y - x)d\tau -$$

$$- \frac{1}{2} \int\limits_{x}^{\infty} dy \int\limits_{t+x-y}^{t+y-x} c(y, \tau) H_+(y, \tau, \xi + y - x)d\tau \tag{21}$$

By using the valuation (13) we demonstrate easily that the consecutive approximating method is convergent for equation (21), where

$$| H_+(x, t, \xi) | \leqslant \frac{1}{2} \int\limits_{x+\frac{t-\xi}{2}}^{\infty} | c(y, \xi - x + y) | \cdot \exp \left\{ \int\limits_{x}^{\infty} \frac{C dz}{(1 + | z |^{1+\varepsilon})} \right\}, \tag{22}$$

We get easily the uniqueness of solution of equation (21) from the fact, that this equation is that of Voltera type with the variable x, where $c(x, t)$ satisfies inequality (13). Furthermore, the valuations (13) and (22) permit to demonstrate the statement (20) and to estimate

$$\iint\limits_{\xi \leqslant t} H_+^2(x, t, \xi) dt d\xi < \infty$$

i.e. $H_+(x, t, \xi)$ with variables t and ξ is a kernel of Hilbert—Schmidt. So the lemma is demonstrated.

Here $H_+(x, t, \xi)$ is the analogy of kernel transformation operator of stationary scattering problem for equation of Sturm—Liouville.

From equation (21), by supposing $\xi = t$ we get

$$H_+(x, t, t) = \frac{1}{2} \int\limits_{x}^{\infty} c(y, t - x + y)dy \tag{23}$$

By applying operator $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$ for (23) we get

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) H_+(x, t, t) = - \frac{1}{2} c(x, t) \tag{24}$$

The formulas (23), (24) express the potential $c(x, t)$ through the kernel of the transformation operator.

We designate the operator which transfens the given wave $\varphi(t)$ in solution $u(x, t)$ of non-stationary scattering problem by $U(x)$, thus

$$U(x) \varphi(t) = u(x, t) .$$

In consideration of (18) equality (19) can be written in form

$$U(x)\varphi = [I + H_+(x)]T_{-x}S\varphi \tag{25}$$

From the boundary condition $u(0, t) = \varphi(t)$ we get

$$U(0) = I \tag{26}$$

Because $H_+(x)$ is an operator of Hilbert—Schmidt, therofore the operator $(I + H_+(x))^{-1}$ exists also. And from (25) in consideration of (26) we obtain the expression of scattering operator in $L_2(-\infty, \infty)$:

$$S = (I + H_+(0))^{-1} \tag{27}$$

From the equality (27) we examine operator function

$$S(x_o) = (I + H_+(x_o))^{-1}, \ x_o \geqslant 0.$$

By using equation (21) we can prove the following lemma.

Lemma 4. The operator $S(x_o)$ is operator of scattering of non-stationary problem on semi-axis $x \geqslant x_o$ with boundary condition $u(x_o, t) = \varphi(t)$.

We examine the properties of solution of problem (1)—(3) and properties of scattering operator of this problem.

Lemma 5. If the given wave $\varphi(t-x)$ is equal to zero in area $t - x < \lambda$ the solution $u(x, t)$ and also the scattering wave $f(t-x)$ of problem (1)—(3) are equal zero in this area.

If $\varphi(s)$ is equal to zero by $s \geqslant t - x$ the transformation operator is connected with the scattering operator through following equality

$$H_+(x, t, \xi) + F(t-x, \xi-x) + \int_{-\infty}^{t} H_+(x, t, \eta)\, F(\eta-x, \xi-x)\, d\eta = 0, \ \xi \leqslant t, \tag{28}$$

where $F(t, \xi)$ is the kernel of the operator $S-I$.

Proof: The first conclusion of lemma results from (16) — (17). Indeed' if $\varphi(t-x) = 0$ by $t - x < \lambda$, i.e. the free term of equation (16) is equal to zero by $t-x<\lambda$. Because of the Voltera-property of equation with variable t, we get easily $u(x, t) = 0$ by $t-x<\lambda$. But in equation (17) from condition $\tau \in [t+x-y, t-x+y]$ we get $\tau - y \leqslant t - x < \lambda$ and so $u(y, \tau) = 0$, therefore the scattering wave $f(t-x)$ is equal to zero in area $t-x<\lambda$, Now let $\varphi(s) = 0$ by $s \geqslant t-x$, and by using (16) and (18) from (19) we get

$$F\varphi(t-x) + \int_{-\infty}^{t} H_+(x, t, \xi)\, \varphi(\xi - x)\, d\xi + \int_{-\infty}^{t} H_+(x, t, \xi)\, F\varphi(\xi - x)\, d\xi =$$

$$= \int_{-\infty}^{t} F(t-x, \xi-x)\, \varphi(\xi - x)\, d\xi + \int_{-\infty}^{t} H_+(x, t, \xi)\, \varphi(\xi - x)\, d\xi +$$

$$+ \int_{-\infty}^{t} \varphi(\xi - x)\, d\xi \int_{-\infty}^{t} H_+(x, t, \eta)\, F(\eta-x, \xi - x)\, d\eta = 0$$

From the last equality we get equality (28) by $\xi \leqslant t$. So the lemma is demonstrated.

Furthermore, by supposing $F(t-x, \xi-x)$ be the kernel of operator F_x, we write (28) in operator form

$$-[I + H_+(x)] F_x = H_+(x),$$

therefore

$$-F_x = [I + H_+(x)]^{-1} H_+(x) \tag{29}$$

We suppose

$$F(x) = [I + H_+(x)]^{-1} - I \tag{30}$$

and adding (30) with (29) we get

$$F(x) - F_x = [I + H_+(x)]^{-1} [H_+(x) + I] - I = 0, \tag{31}$$

Supposing $F(x, t, \xi)$ be the kernel of operator $F(x)$ we write equality (31) for the kernel of corresponding operators in form

$$F(x, t, \xi) - F(t-x, \xi-x) = 0 \text{ by } \xi \leqslant t,\ x \geqslant 0; \tag{32}$$

Thus, if the operator S is known, the operator $F = S - I$ is known also, and the operator $F(x)$ is determined by formulas (31) – (32). Hereby and from (30) we find operator

$$H_+(x) = [I + F(x)]^{-1} - I,$$

on which kernel the potential $c(x, t)$ is recovered after formula (24), furthermore the retation between the operator S and potential $c(x, t)$ is single-valued. We formulate this result.

Theorem 2. The non-stationary potential $c(x, t)$ of scattering problem which is given by perturbation wave equation (1) and the boundary conditions (2)—(3) is recovered after the scattering operator S, furthermore the relation between S and $c(x, t)$ is single-valued.

The potential $c(x, t)$ is determined after $H_+(x, t, \xi)$ by equality

$$c(x, t) = -2\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) H_+(x, t, t),$$

where $H_+(x, t, \xi)$, $\xi \leqslant t$ is kernel of operator $[I + F(x)]^{-1} - I$, and $F(x)$ is connected with the given operator S by relation:

$$F(x, t, \xi) = F(t-x, \xi-x),\ \xi \leqslant t,\ x \geqslant 0,$$

where $F(x, t, \xi)$, $F(t, \xi)$ are kernel of the operators $F(x)$ and $S-I$.

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