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ON DOÉBLIN THEOREM FOR RANDOM MEASURES

by HỒ ĐĂNG PHÚC and NGUYỄN VĂN THU
Institute of Mathematics, Hanoi

A famous theorem of Doéblin [2] asserts that there exists a distribution belonging to the domain of partial attraction of every one-dimensional infinitely divisible distribution. The multi-dimensional version of this theorem is made by J. Baranska [1] in a Hilbert space and recently, by Hồ Đăng Phúc [5] in a Banach space. The aim of this note is to prove Doéblin theorem for random measures on a locally compact second countable Hausdorff topological space.

Throughout the paper we shall denote by σ a locally compact second countable Hausdorff topological space. Such a space is known to be Polish. Let $\mathcal{M}$ denote the class of all Radon measures on σ endowed with the vague topology. Then $\mathcal{M}$ is also a Polish space. Further, by $\mathcal{M}_b$ we shall denote the subclass of $\mathcal{M}$ consisting of totally finite measures. It hints at, $\mathcal{M}_b$ is a dense subset of $\mathcal{M}$ in the vague topology. By a random measure on σ we mean any probability measure on Borel subsets of $\mathcal{M}$. In what follows the convergence of random measures will be considered in the weak sense. Let $\mathcal{F}$ denote the class of all positive Borel functions on σ. Then the Laplace transform L_ξ of a random measure ξ is defined on $\mathcal{F}$ by the formula:

$$L_\xi(f) = \int_{\mathcal{M}} exp(-\mu f)\ \xi(d\mu) \qquad (f \in \mathcal{F})$$

Since $\mathcal{M}$ is an additive topological semigroup then the concept of convolution and hence the concept of infinite divisibility of random measures on σ are well-defined.

Recall ([3], Theorem 6.1) that a random measure ξ on σ is infinitely divisible if and only if L_ξ has the following canonical representation

$$-\log L_\xi(f) = \alpha f + \int [1 - exp(-\mu f)]\ \lambda(d\mu)$$

$(f \in \mathcal{F})$, where $\alpha \in \mathcal{M}$ while λ is a measure on $\mathcal{M}\setminus\{0\}$ satisfying the condition:

$$\int [1 - exp(-\mu f)]\ \lambda(d\mu) < \infty$$

for any $f \in \mathcal{F}$ with compact support. Since the canonical measures α and λ determine ξ uniquely it will be convenient to write $I(\alpha, \lambda)$ instead of ξ.

Let ξ be a random measure. For every $c > 0$ define a random measure $T_c\xi$ by the formula:

$$(T_c \xi)(E) = \xi(\{\mu : c\mu \in E\}) \qquad (E \subset \mathcal{M}).$$

Following Doéblin [2] we say that a random measure η is universal for the class of all infinitely divisible random measures on σ if for every infinitely divisible random measure ξ on σ there exist a sequence $\{\alpha_k\} \subset \mathcal{M}$, a subsequence $\{n_k\}$ of natural numbers and a sequence $\{a_k\}$ of positive numbers such that the sequence $\{T_{a_k}\eta^{*n_k} * \delta\alpha_k\}$ $k = 1, 2, \ldots$, converges to ξ. Here the asterisk $*$ denotes the convolution and $\delta\alpha$ denotes the unit mass at the point $\alpha \in \mathcal{M}$.

Modifying the technique developed by Doéblin [2] one can prove the following theorem:

Theorem. There exists a universal random measure for the class of all infinitely divisible random measures on σ.

Proof. By vitue of Lemma 6.6 [3] it follows that the class of infinitely divisible random measures $I(\alpha, \lambda)$ with $\lambda(\mathcal{M}) < \infty$ is dense in the class of all infinitely divisible random measures. Further, by Theorem 6.3 [4] such a measure λ can be approximated by measures whose supports are finite subsets of $\mathcal{M}_b$. Consequently, one can choose a countable dense subset ξ_n, $n = 1, 2, \ldots$, of the set of all infinitely divisible random measures on σ such that $\xi_n = I(\alpha_n, \lambda_n)$ and λ_n is supported by $\mathcal{M}_b$ $(n = 1, 2, \ldots)$. Put, for $j = 1, 2, \ldots$,

$$\mathcal{M}_j = \left\{\mu \in \mathcal{M} : \frac{1}{j} \leqslant \mu\sigma < j\right\}$$

It is clear that $\mathcal{M}_b = \bigcup_{j=1}^{\infty} \mathcal{M}_j$. Hence we may assume that λ_n is concentrated on $\mathcal{M}_n$ and that $\lambda_n(\mathcal{M}_b) \leqslant n$ for $n = 1, 2, \ldots$. Putting $k_n = 2^{n^3}$ and taking into account the inequality

$$\sum_{n=1}^{\infty} 2^{-n^2} T_{k_n}\lambda_n(\mathcal{M}_b) \leqslant \sum_{n=1}^{\infty} 2^{-n^2} n < \infty$$

we get a totally finite measure Υ defined by the formula

$$\Upsilon = \sum_{n=1}^{\infty} 2^{-n^2} T_{k_n}\lambda_n$$

We shall show that the random measure $\pi = I(o, \Upsilon)$ is universal in the class of all infinitely divisible random measures on σ.

Let $\xi = I(\alpha, \lambda)$ be an arbitrary infinitely divisible random measure on σ. Without loss of generality we may assume that $\alpha = 0$. Then there is a subsequence $\{n_p\}$ of natural numbers such that $\{\xi_{n_p}\}$ tends to ξ. Let $a_p = 2^{n_p^2}$. Our further aim is to prove that the sequence

$$\pi_p = T_{k_{n_p}^{-1}} \pi^{*a_p} \qquad (p = 1, 2, \ldots)$$

converges to ξ. In fact, let us put

$$N_p^1 = \sum_{m > n_p} a_p 2^{-m^2} T_{k_m k_{n_p}^{-1}} \lambda_m.$$

and

$$N_p^2 = \sum_{m < n_p} a_p 2^{-m^2} T_{k_m k_{n_p}^{-1}} \lambda_m.$$

It is evident that

$$\pi_p = I(o, \lambda_{n_p} + N_p^1 + N_p^2)$$
$$= I(o, \lambda_{n_p}) * I(o, N_p^1 + N_p^2).$$

Consequently, to prove that $\pi_p \to \xi$ it suffices to prove that

$$\lim_p N_p^1 (\mathcal{M}_b) = 0 \text{ and } \lim_p N_p^2 (\mathcal{M}_b) = 0.$$

The first limit is clear, because

$$N_p^1(\mathcal{M}_b) \leqslant \sum_{m > n_p} a_p 2^{-m^2} m \leqslant \sum_{m=1}^{\infty} (n_p + 1) 2^{-(2n_p+m)m} \to 0 \text{ as } p \to \infty.$$

To prove the second limit let us denote $S_\varepsilon = \{\mu \in \mathcal{M}_b : \mu\sigma \geqslant \varepsilon\}$ for every $\varepsilon > 0$. Since λ_m is supported by $S_{1/m}$ then $T_{k_m k_{n_p}^{-1}} \lambda_m$ is supported by $S_{k_{n_p}/mk_m}$. Therefore all $T_{k_m k_{n_p}^{-1}}$, $m = 1, 2, \ldots, n_p - 1$, are concentrated at

$$C_p := S_{k_{n_p}/(n_p-1)k(n_p-1)} = S_{(n_p-1)^{-1} \cdot 2^{(3n_p^2 - 3n_p + 1)}}$$

which implies that N_p^2 is concentrated at C_p. If $N_p^2(\mathcal{M}_b)$ is not convergent to zero then there exists a positive number δ such that

$$N_{p_q}^2(C_{p_q}) = N_{p_q}^2(\mathcal{M}_b) > \delta$$

for some subsequence p_q, $q = 1, 2, \ldots$, of natural numbers. In this case we have

$$\int_{\mathcal{M}_b} \mu\sigma \; N_{p_q}^2(d\mu) = \int_{S_{(n_{p_q}-1)^{-1} 2^{(3n_{p_q}^2 - 3n_{p_q} + 1)}}} \mu\sigma \; \cdot \; N_{p_q}^2(d\mu)$$

$$\geqslant (n_{p_q} - 1)^{-1} 2^{(3n_{p_q}^2 - 3n_{p_q} + 1)} \delta \to \infty \text{ as } p_q \to \infty.$$

On the other hand,

$$\int_{\mathcal{M}_b} \mu\sigma \; N_p^2(d\mu) = \sum_{m < n_p} \cdot \int_{\mathcal{M}_b} k_m/k_{n_p} \, \mu\sigma \cdot a_p \cdot 2^{-m^2} \lambda_m(d\mu)$$

$$< \sum_{m < n_p} 2^{-m^2} m^2 \; 2^{(n_p^2 + (n_p - 1)^3 - n_p^3)}$$

$$\leqslant 2^{-2n_p^2 + 3n_p - 1} \sum_{m=1}^{\infty} 2^{-m^2} m^2 \leqslant \sum_{m=1}^{\infty} 2^{-m^2} m^2 < \infty$$

for all p, which contradicts the above proved relation that

$$\int_{\mathcal{M}_b} \mu\sigma\, N^2_{p_q}(d\mu) \to \infty .$$

Consequently, the sequence $\{N_p^2(\mathcal{M}_b)\}$ must converge to zero, which completes the proof.

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REFERENCES

1. J.Baranska, *Domain of partial attraction for infinitely divisible distributions in a Hilbert space.* Colloq. Math. Vol. XXVIII (1973), 317 – 322.

2. W. Doéblin. *Sur l'ensemble de puissences d'une loi de probabilite.* Bull. Sci. Math 6 (1939) 71 – 96.

3. O.Kalenberg, *Random measures*, New York – London 1976.

4. K.R.Parthasarathy, *Probability measures on metric spaces*, New York – London, 1967.

5. Hồ Đăng Phúc, *Universal distribution for infinitely divisible distribution in a Banach space*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astrom. et Phys. (to appear).