ACTA MATHEMATICA VIETNAMICA

Tom 5 N° 2

CONTROLLABILITY OF NONLINEAR DISCRETE DELAY SYSTEMS

VŨ NGỌC PHÁT

Institute of Mathematics, Hanoi

As is well known, the controllability of continuous nonlinear systems is considered completely by a number of authors ([1], [2], [3] and others). But for the discrete systems, it is studied only in linear case (see [4], [5]). In this paper we present some results for controllability of nonlinear discrete delay systems.

1. CONTROLLABILITY

In this section we consider the controllability of nonlinear discrete delay systems represented by the form ;

$$x(t) = \Lambda(x, u) + g(t, x(t-h_1), \dots, x(t-h_p), u(t-k_1), \dots, u(t-k_q)), \tag{1}$$

$$x(t) = \Lambda(x, u) \tag{2}$$

where the linear operator Λ is defined as :

$$\Lambda(x, u) = \sum_{i=1}^{p} A_i(t)\, x(t-h_i) + \sum_{j=1}^{q} B_j(t)\, u(t-k_j),$$

$t = 1, 2, \dots, N$; h_i, k_j are nonnegative integers satisfying the following conditions:

$$l = h_1 \leqslant h_2 \leqslant \dots \leqslant h_p,$$

$$0 = k_1 \leqslant k_2 \leqslant \dots \leqslant k_q.$$

$x(t)$ is a $(n \times l)$ state vector, $u(t)$ is a $(m \times l)$ input vector; $A_i(t)$, $B_j(t)$ are $(n \times n)$ and $(n \times m)$ matrices $g(t,.)$ – vector-function in R^n.

We shall show that if the linear system (2) is globally controllable then the nonlinear system (1) is also globally controllable provided the function $g(t,.)$ satisfies appropriate conditions. The class of admissible control functions for the system (1) will be all the vectors $u(t) \in R^m$ ($t = -k_q, \dots, N$) with $u(t) = 0$ for $t = 0, -1, \dots, -k_q$. Let $\vec{a}$ be n-vector, then there exists a solution of the system (1) satisfying $x(t) = \vec{a}$ for $t = 0, -1, \dots, -h_p$. For every admissible control function $u(t)$, define a solution $x_\Lambda(t)$ of the system (2) :

$$x(t) = \left[G_1^t + \sum_{i=2}^{p} \sum_{s=1}^{h_i-1} G_{s+1}^t A_i(s)\right]\vec{a} + \sum_{s=1}^{t} \sum_{j=1}^{q} G_{s+1} B_j(s) u(s-k_j) \tag{3}$$

where G_s^t is a $(n \times n)$ matrix function satisfying the following conditions:

$$G_s^t = \sum_{i=1}^{p} A_i(t)\, G_s^{t-h_i}, \quad t, s = 1, 2, ..., N,$$

$$G_s^t = 0, \text{ for } t+1 < s,$$

$$G_{t+1}^t = E - \text{the identity matrix.}$$

Lemma 1.1. (3) is a solution of the system (2).

Proof. We shall prove the lemma by the mathematical induction. Letting $t = 1$, we have:

$$x_\Lambda(1) = \left[G_1^1 + \sum_{i=2}^{p}\sum_{s=1}^{h_i-1} G_{s+1}^1 A_i(s)\right]a + \sum_{s=1}^{1}\sum_{j=1}^{q} G_{s+1}^1 B_j(s)\, u(s-k_j).$$

Since $G^{1-k_i} = 0$ for $i \geqslant 2$ and $u(l-k_j) = 0$ for $j \geqslant 2$ then

$$x_\Lambda(1) = \sum_{i=1}^{p} A_i(1)a + B_1(1)u(1).$$

The last expression is a solution of the system (2) for $t = 1$. Now assume that (3) is a solution of the system (2) for $t = k$. Letting $t = k+1$, we have:

$$x_\Lambda(k+1) = \left\{\sum_{i=1}^{p} A_i(k+1)G_1^{k+1-h_i} + \sum_{i=2}^{p}\sum_{s=1}^{h_i-1}\left(A_i(k+1)G_{s+1}^{k+1-h_i}\right)A_i(s)\right\}a +$$

$$+\sum_{s=1}^{k}\left(\sum_{i=1}^{p} A_i(k+1)G_{s+1}^{k+1-h_i}\right)\sum_{j=1}^{q} B_j(s)\, u(s-k_j) + \sum_{j=1}^{q} B_j(k+1)\, u(k+1-k_j) =$$

$$+\sum_{i=1}^{p} A_i(k+1)\left\{\left[G_1^{k+1-h_i} + \sum_{i=2}^{p}\sum_{s=1}^{h_i-1} G_{s+1}^{k+1-h_i} A_i(s)\right]a +\right.$$

$$\left.+\sum_{s=1}^{k} G_{s+1}^{k+1-h_i}\sum_{j=1}^{p} B_j(s)\, u(s-k_j)\right\} + \sum_{j=1}^{q} B_j(k+1)\, u(k+1-k_j).$$

Since

$$\sum_{s=1}^{k} G_{s+1}^{k+1-h_i}\sum_{j=1}^{q} B_j(s)\, u(s-k_j) = \sum_{s=1}^{k+1-h_i} G_{s+s}^{k+1-h_i}\sum_{j=1}^{q} B_j(s)\, u(s-k_j),$$

then

$$x_\Lambda(k+1) = \sum_{i=1}^{p} A_i(k+1)x(k+1-h_i) + \sum_{j=1}^{p} B_i(k+1)\, u(k+1-k_j).$$

The last expression is a solution of the system (2) for $t = k+1$ Q.E.D.

Definition 1.1. The system (1) is globally controllable if for every initial state $\vec{a} \in R^n$ and every $x_1 \in R^n$, there exists an admissible control $u = \{u(t)\}$ such that the trajectory of the system (1) satisfies the following conditions $x(t) = \vec{a}$, $t = -k_p, ..., 0$, $x(N) = x_1$.

We shall need the following notations:

$$\widetilde{S}^{\,t}_{s+1} = \sum_{j=2}^{q} S^{t,j}_{s+1}, \quad s, t = 1, 2, ..., N, \tag{4}$$

where

$$S^{t,j}_{s+1} = \begin{cases} G^t_{s+1} B_1(s) + G^t_{s+1+k_j} B_j(s-k), & 1 \leqslant s \leqslant t-k_j \\ G^t_{s+1} B_1(s), & t-k_j < s \leqslant t \end{cases}$$

We notice that

$$\sum_{s=1}^{t} \widetilde{S}^{\,t}_{s+1} u(s) = \sum_{s=1}^{t} \sum_{j=1}^{q} G^t_{s+1} B(s)\, u(s-k_j),$$

then we can rewrite (3) in the following form:

$$x_A(t) = X(t)\vec{a} + S(t)\, U(t), \tag{5}$$

where

$$X(t) = G^t_1 + \sum_{i=2}^{p} \sum_{s=1}^{h_i-1} G^t_{s+1} A_i(s)$$

$$U(t) = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(t) \end{bmatrix} = [u(1), ..., u(t)]^*,$$

$$S(t) = \left[S^t_2, S^t_3, ... S^t_{t+1}\right]$$

Let

$$T(N) = S(N)\, S^*(N)$$

Lemma 1.2. The system (2) is globally controllable if and only if rang $T(N) = n$.

Proof. Assume that the system (2) is globally controllable but rang $T(N) < n$. Then there is a nonzero vector $\vec{b} \in R^n$ such that $\vec{b}^* S(N) = 0$. Let $\vec{b}_1$ be chosen so that $\vec{b}\, X(N) \vec{b}_1 = 0$. Furthermore, there exists an admissible control $u = \{u(t)\}$ such that the trajectory of the system (2) satisfies:

$$\vec{b} = X(N)\vec{b}_1 + S(N)\, U(N). \tag{6}$$

Multiplying (5) by $\vec{b}^*$, we obtain:

$$\vec{b}^2 = \vec{b}^* X(N) \vec{b}_1 + \vec{b}\, S(N)\, U(N) = 0$$

The last identity is impossible. To prove the converse part, assume that rang $T(N) = n$. Then for every pair $(\overrightarrow{a}, x_1) \in R^n \times R^n$ we define the admissible control function $u = \{u(t)\}$ of the system (2) by $u(t) = 0$, $t = 1 - k_q, \ldots, 0$,

$$u(t) = (\widetilde{S}^N_{t+1})^* T^{-1}(N)\,(x_1 - X(t)\overrightarrow{a}) \qquad t = 1, 2, \ldots, N$$

Inserting the last expression into (5) we obtain that $x_A(N) = x_1$. It implies that the system (2) is globally controllable Q.E.D.

Theorem I.1. Let the continuous function $g(t,.)$ be bounded on $\underbrace{R^n \times \ldots \times R^n}_{p} \times \underbrace{R^m \times \ldots \times R^m}_{q}$ for every $t = 1,2,\ldots, N$. Assume that the linear system (2) is globally controllable. Then the system (1) is globally controllable.

Proof. It is easy to see that a solution of the system (1) is defined as:

$$x(t) = x_A(t) + \sum_{s=1}^{t} G^t_{s+1}\, g(s, x(s-h_1), \ldots, x(s-h_p)u(s-k_1), \ldots, u(s-k_q))$$

Let us set

$$x = \{x(1-h_p), \ldots, x(0), \ldots, x(N)\} \in R^{n(N+h_p)},$$

$$u = \{u(1-k_q), \ldots, u(0), \ldots, u(N)\} \in R^{m(N+k_q)},$$

$$\mathbf{R} = R^{n(N+h_p)} \times R^{m(N+k_q)},$$

$$\| z \|_{\mathbf{R}} = \| x \| + \| u \| = \max_t \| x(t) \|_{R^n} + \max_t \| u(t) \|_{R^m}.$$

For every pair $(\overrightarrow{a}, x_1) \in R^n \times R^n$ and every element $z = (x, u) \in \mathbf{R}$ we define the function $\Phi(z) = \Phi_1(z) \times \Phi_2(z)$ where:

$$\Phi_1(z) = \left\{ \hat{x} \in R^{n(N+h_p)} \;\middle|\; \begin{array}{l} \hat{x}(t) = \overrightarrow{a}, \quad t = 1 - h_p, \ldots, 0 \\ \hat{x}(t) = x_A(t) + \sum_{s=1}^{t} G^t_{s+1}\, g(s, x, u), \\ t = 1, 2, \ldots, N. \end{array} \right\},$$

$$\Phi_2(z) = \left\{ \hat{u} \in R^{m(N+k_q)} \;\middle|\; \begin{array}{l} \hat{u}(t) = 0, \quad t = 1 - k_q, \ldots, 0 \\ \hat{u}(t) = (\widetilde{S}^N_{t+1})^* T^{-1}(N)\,(x_1 - X(t)\overrightarrow{a} - \\ \quad - \sum_{s=1}^{t} G^t_{s+1}\, g(s, x, u), \quad t = 1, 2, \ldots, N. \end{array} \right\}.$$

From the boundedness and continuity of the function $g(t,.)$, there exists a positive number k, such that the continuous function $\Phi(z)$: $D_k \to D_k$, where $D_k = \{z \in \mathbf{R} \mid \| z \| \leqslant k\}$. Since D_k is a convex and compact subset of $\mathbf{R}$, by the Schauder's fixed-point theorem, the function $\Phi(z)$ has at least one fixed point in D_k. It implies that the system:

$$x(t) = x_A(t) + \sum_{s=1}^{t} G_{s+1}\, g(s, x, u),$$

$$u(t) = (\tilde{S}^{N}_{t+1})^{*} T^{-1}(N)(x_1 - X(t)a - \sum_{s=1}^{t} G^{t}_{s+1} g(s, x, u),$$

with the initial conditions $u(t) = 0$ $(t = 1-k_q, ..., 0)$, $x(t) = 0$ $(t = 1-h_p, ..., 0)$ has a solution $\{\overset{\circ}{x}(t), \overset{\circ}{u}(t)\}$ satisfying $\overset{\circ}{x}(N) = x_1$ Therefore, system (1) is globally controllable. Q. E. D.

By the similar way as in the proof of theorem 1.1, we can now extend the previous results even in the general case, when matrices $A_i(t)$ and $B_j(t)$ depend on control and state i.e:

$$A_i(t,...) = A_i(t, x(t-h_1), ..., u(t-k_q)), \quad (7)_1$$

$$B_j(t,...) = B_j(t, x(t-h_1), ..., u(t-k_q)). \quad (7)_2$$

Let us assume in this case that $\inf_{z \in R} \det T(N, z) > 0.$ (8)

Theorem 1.2. Given the system (1), where matrices $A_i(t,.)$, $B_j(t,.)$ are defined by $(7)_1$, $(7)_2$. Assume that the functions $g(t,.)$, $A_i(t,.)$, $B_j(t,.)$ are continuous and bounded for every t, i, j. If the condition (8) is satisfied, then the system (1) is globally controllable.

2. THE MAXIMUM PRINCIPLE AND CONTROLLABILITY.

Now, consider the following nonlinear discrete delay-time system:

$$\begin{cases} x(t) = f(t, x(t-1), x(t-h), u(t)) & t = 1, 2, ..., N, \\ x(t) = 0, \quad t = 1-h, ..., 0, \quad (h \geq 2). \end{cases} \quad (9)$$

Let $U_t(x(t-1), x(t-h))$ be a nonempty subset of R^m depending on $(x(t-1), x(t-h))$ for every $t = 1, 2, ... N$. The class of admissible control functions for the system (9) will be all the $u(t)$, $t = 1, 2, ..., N$ satisfying $u(t) \in U_t(x(t-1), x(t-h))$. Let $\mathcal{K}$ be a controllable set of the system (9), i. e.

$$\mathcal{K} = \left\{ \begin{array}{l} a \in R^n \mid \exists u(t) \in U_t(x(t-1), x(t-h)), x(t) = f(t, x(t-1), x(t-h), u(t)) \\ x(N) = a, \ x(t) = 0 \text{ for } t = 1-h, ..., 0. \end{array} \right\}.$$

Let us set

$$H(t, p(t), x, y, u) = \langle p(t), f(t, x, y, u) \rangle,$$

$$\delta_u H(t, p(t), x, y, u) = \left(\frac{\partial H(t, p(t), x, y, u)}{\partial u}, \delta u \right) = \langle p(t), \langle \frac{\partial f(t, x, y)}{\partial u} \delta u \rangle,$$

where the vectors p(t) are defined by

$$\left. \begin{array}{l} p(t-1) = \left[\dfrac{\partial f(t, x, y, u)}{\partial x(t-1)} \right]^{*} p(t) + \left[\dfrac{\partial f(t, x, y, u)}{\partial x(t-1)} \right]^{*} p(t+h-1), \\ p(N) \neq 0, \ p(N+1) = ... = p(N+h-1) = 0, \ t = 1, 2, ..., N. \end{array} \right\} \quad (10)_1$$

Definition 2.1. We shall say that the system (9) is globally controllable if $\mathcal{K} = R^n$.

Now as in [6] we introduce a notion of the tangential cones (TC(.)) and cones of admissible variations (AC(.)) which are defined by the following formulas:

$$AC(M, x) = \{\delta x \mid \exists \varepsilon_1 > 0,\ \forall \varepsilon \in (0, \varepsilon_1),\ x + \varepsilon\delta x \in M\},$$

$$TC(M, x) = \left\{\delta x \mid \exists \varepsilon_2 > 0,\ \exists o(\varepsilon): \| o(\varepsilon) \| / \varepsilon \underset{\varepsilon \to 0}{\to} 0,\ \forall \varepsilon \in (0,\varepsilon_2)\ x + \varepsilon\delta x + o(\varepsilon) \in M\right\}.$$

With the system $(10)_1$ we consider the auxiliary matrix-system of the form:

$$\left.\begin{array}{l} \Phi(s,t-1) = \Phi(s,t)\dfrac{\partial f(t, x, y, u)}{\partial x(t-1)} + \Phi(s,t-1+h)\dfrac{\partial f(t, x, y, u)}{\partial x(t-1)}, \\ \Phi(t,t) = E, \quad \Phi(s,t) = 0 \qquad (t > s), \qquad s,t = 1, 2, \dots, N. \end{array}\right\} \quad (10)_2$$

Lemma 2.1. If $p(t)$ and $\Phi(s,t)$ satisfy the conditions $(10)_1$ and $(10)_2$ then for every $t = 1, 2, \dots, N$ we have

$$p^*(N)\ \Phi(N, t) = p^*(t).$$

Lemma 2.2. Let $\Phi(s,t)$ be the matrices satisfying the conditions $(10)_2$. Then:

i) $\Phi(t, t-1) = \dfrac{\partial f[t, x(t-1), x(t-h), u(t)]}{\partial x(t-1)}$.

ii) For $h > s$:

$$\Phi(t+s,t) = \frac{\partial f[t+s, x(t+s-1), x(t+s-h), u(t+s)]}{\partial x(t+s-1)}\ \phi(t+s-1, t),$$

For $h \leqslant s$:

$$\Phi(t+s,t) = \frac{\partial f(t+s, x(t+s-1), x(t+s-h)\ u(t+s)]}{\partial x(t+s-1)}\ \Phi(t+s-1,t) +$$

$$+ \frac{\partial f(t+s, x(t+s-1), x(t+s-h), u(t+s))}{\partial x(t+s-h)}\ \Phi(t+s-h,t).$$

The proof of these lemmas follows from the definition of $p(t)$ and $\Phi(s,t)$ by $(10)_1$ and $(10)_2$

Theorem 2.1. Let the function $f(t,.)$ be continuously differentiable $(1 \leqslant t \leqslant N)$ and $U_t\ (x(t-1), x(t-h)) = U_t$. Assume that $\operatorname{int} U_t \neq \phi$, $TC(\mathcal{K}, x^o(N)) \neq R^n$; Cone $AC\{U_t, u^o(t)\}$ is convex, where $x^o = \{x^o(t)\}$ is a trajectory of the system (9) corresponding to the admissible control function $u^o = \{u^o(t)\}$. Then there is a nonzero vector $\vec{n} \in R^n$, such that:

$$\delta_u H(t, p^o(t), x^o(t-1), x^o(t-h), u^o(t)) \leqslant 0, \quad t = 1, 2, \dots, N,$$

for all $\delta u^o(t) \in \overline{AC}(U_t, u^o(t))$, where the vectors $p^o(t)$ are defined by $(10)_1$ provided $x(t) = x^o(t)$, $u(t) = u^o(t)$, $p^o(N) = \vec{n}$ Furthermore, if $u^o(t) \in \operatorname{int} U_t$ then for every $t = 1,2,\dots, N$, we have:

$$\frac{\partial H(t, p^o(t), x^o(t-1), x^o(t-h), u^o(t))}{\partial u^o(t)} = 0.$$

Proof. Fixing the process $\{x^o(t), u^o(t)\}$ satisfying the conditions of the theorem, consider the variation equation of the form:

$$\delta x^o(t) = \frac{\partial f(t, x^o, y^o, u^o)}{\partial x^o(t-1)}\,\delta x^o(t-1) + \frac{\partial f(t, x^o, y^o, u^o)}{\partial x^o(t-h)}\,\delta x^o(t-h) + + \frac{\partial f(t, x^o, y^o, u^o)}{\partial u^o(t)}\,\delta u^o(t),\ t = 1, 2, \ldots, N.$$

where $\partial u^o(t) \in AC(U_t, u^o(t))$, $\delta x^o(t-h) = \ldots = \delta x^o(o) = 0$.

From the lemma 2.2 it is easy to verify that

$$\delta x^o(t) = \sum_{S=1}^{t} \Phi(t, s)\,\frac{\partial f(t, x^o, y^o, u^o)}{\partial u^o(s)}\,\delta u^o(s), \tag{11}$$

Now we set

$$\Omega(N, x^o, v^o) =$$

$$= \left[\Phi(t, 1)\,\frac{\partial f(1, x^o(o), x^o(1-h), u^o(1))}{\partial u^o(1)}, \ldots, \Phi(t, t)\,\frac{\partial f(t, x^o(t-1), x^o(t-h), u^o(t))}{\partial u^o(t)}\right].$$

$$\delta U^O_t = [\delta u^o(1), \delta u^o(2), \ldots, \delta u^o(t))^*,$$

then we can rewrite (11) as:

$$\partial x^o(t) = \Omega(N, x^o, u^o)\,\delta U^O_t.$$

Let

$$\widetilde{\mathcal{K}}(x^o, u^o) = \{a \in R^n,\ a = \delta x^o(N) = \Omega(N, x^o, {}^o u)\delta U^O_N \,|\, \delta u^o(t) \in AC(U_t, u^o(t)\}.$$

From convexity of $AC(U_t, u^o(t))$ it follows that $\widetilde{\mathcal{K}}(x^o, u^o)$ is a convex set. Now we shall prove that

$$\widetilde{\mathcal{K}}(x^o, u^o) \subseteq TC(\mathcal{K}, x^o(N)).$$

Indeed, consider an arbitrary element $\vec{a} \in \widetilde{\mathcal{K}}(x^o, u^o)$. There exists variations $\delta u^o(t)$, $t = 1, 2, \ldots, N$ such that

$$\delta u^o(t) \in AC(U_t, u^o(t)) \text{ and } \vec{a} = \Omega(N, x^o, u^o)\delta U^O_t.$$

Furthermore, there is a positive number $\varepsilon_1 > 0$, such that for every $\varepsilon \in (0, \varepsilon_1)$: $u^o(t) + \varepsilon\delta u^o(t) \in U_t$. Let $\tilde{u}(t) = u^o(t) + \varepsilon\delta u^o(t)$ using lemma 2.2 we can prove that the trajectory $\{\tilde{x}(t)\}$ corresponding to the admissible control $\{\tilde{u}(t)\}$ of the system (9) is given by the form:

$$\tilde{x}(t) = x^o(t) + \varepsilon \sum_{s=1}^{t} \Phi(t, s)\,\frac{\partial f(s, x^o, y^o, u^o)}{\partial u^o(s)}\,\delta u^o(s) + o_t(\varepsilon).$$

Putting $t = N$, we obtain:

$$\tilde{x}(N) = x^o(N) + \varepsilon\vec{a} + O_N(\varepsilon) \in \mathcal{K}$$

From this last inclusion it follows that $\vec{a} \in TC(\mathcal{K}, x^o(N)$. On the other hand, since $TC(\mathcal{K}, x(N)) \neq R^n$ there is a nonzero vector $\vec{n} \in R^n$, such that $< n, \vec{a} > \leqslant 0$ for all $\vec{a} \in \tilde{\mathcal{K}}(x^o, u^o)$. Let $p^o(t)$ be a solution of the system (10) provided $p^o(N) = 0$. We have

$$\langle p^o(N), \delta x^o(N) \rangle = \sum_{t=1}^{N} \langle p^o(t), \frac{\partial f(t, x^o, y^o\ u^o)}{\partial u^o(t)} \delta u^o(t) \rangle \leqslant 0.$$

In the last identity, define

$$\delta u^o(j) = \begin{cases} o \text{ for } j \neq t, \\ \delta u^o(t) \text{ for } j = t. \end{cases}$$

Then we have that for *all* $\delta u^o(t) \in \overline{AC}(U_t, u^o(t))$:

$$\delta_u H(t, p^o(t), x^o(t-1), x^o(t-h), u^o(t)) \leqslant 0 \ (t = 1, 2,.., N).$$

If $u^o(t) \in \text{int } U_t$, then $AC(U_t, u^o(t)) = R^m$. Hence $\pm \delta u^o(t) \in AC(U_t, u^o(t))$. It follows that:

$$\frac{\partial H(t, p^o(t), x^o, y^o, u^o)}{\partial u(t)} = 0, \ (t = 1, 2,..., N) \quad \text{Q.E.D}$$

To obtain the necessary conditions in the form of the maximum principle, we shall consider the following discrete delay conclusion:

$$\left.\begin{aligned} &x(t) \in F(t, x(t-1), x(t-h)), \qquad t = 1, 2,..., N, \\ &x(t) = \vec{a} = 0, \quad t = l-h, ..., 0. \end{aligned}\right\} \qquad (12)$$

where $F(t,.)$ —multivalued function.

Definition 2.2. We shall say that the trajectory $x = \{x(t)\}$ of the conclusion (12) possesses a local section if for every t, there is a smooth function $\sigma(t, x, y)$ defined in the neighbourhood of points $(x(t-1), x(t-h))$ such that

$$x(t) = \sigma(t, x(t-1), x(t-h)) \text{ and } \sigma(t, x, y) \in F(t, x, y).$$

Lemma 2.3. Let $F(t, x, y)$ be a convex subset of R^n $(t = 1, 2,..., N)$, Let $x^o = \{x^o(t)\}$ be a trajectory of the system (12) possessing the local section $\sigma(t, x, y)$ such that $\text{conv } TC(\mathcal{K}_F, x^o(N) \neq R^n$ where $\mathcal{K}_F$ is the controllable-set of the system (12). Then there exists a nonzero vector $\vec{n} \in R^n$: such that:

$$\max \langle p^o(t), x \rangle = \langle p^o(t), x^o(t) \rangle, \ t = 1, 2,..., N. \ x \in F(t, x^o(t-1), x^o(t-h))$$

where the vectors $p^o(t)$ are defined by the system:

$$\left.\begin{aligned} p^o(t-1) = &\left[\frac{\partial\sigma(t, x^o(t-1), x^o(t-h))}{\partial x^o(t-1)}\right]^* p^o(t) + \\ &+ \left[\frac{\partial\sigma(t, x^o(t-h+2), x^o(t-1))}{\partial x^o(t-1)}\right]^* p^o(t+h-1) \\ p^o(N) = \vec{n}, \ &p^o(N+1) = ... = p^o(N+h-1) = 0, \ t = 1, 2,..., N. \end{aligned}\right\} \qquad (13)$$

Proof. Consider the following system:

$$\left.\begin{aligned}\Phi(s,t-1) &= \Phi(s,t)\frac{\partial\sigma(t,x^o(t-1),x^o(t-h))}{\partial x^o(t-1)} + \\ &+ \Phi(s,t+h-1)\frac{\partial\sigma(t,x^o(t-h+2),x^o(t-1))}{\partial x^o(t-1)} \\ \Phi(t,t) &= E,\quad \Phi(s,t)=0 \quad \text{for } t>s,\ t,s=1,2,\dots,N.\end{aligned}\right\} \tag{13}$$

Let

$$\widetilde{\mathcal{K}}_F^t = \{a\in R^n,\ a=\Phi(N,t)(x-x^o(t)) \mid x\in F(t,x^o(t-1),x^o(t-h))\}.$$

It is no hard to prove that $\bigcup_1^N \widetilde{\mathcal{K}}_F^t \leqslant TC(\mathcal{K}_F, x^o(N))$

Indeed, consider an arbitrary element $\vec{a}\in\bigcup_1^N\widetilde{\mathcal{K}}_F^t$. There is a number $\bar t\in\{1,2,\dots,N\}$, $\bar x\in F(\bar t, x^o(\bar t-1), x^o(\bar t-h))$ such that $\vec a=\Phi(N,\bar t)(\bar x-x^o(\bar t))$.
We define:

$$\tilde x(t)=\begin{cases}x^o(t) & 1-h\leqslant t<\bar t,\\ x^o(\bar t)+\varepsilon(\bar x-x^o(\bar t)) & t=\bar t,\\ \sigma(t,\tilde x(t-1),\tilde x(t-h)) & N\geqslant t>\bar t\end{cases}$$

where ε is an arbitrary positive enough small number. It is easy to see that $\{\tilde x(t)\}$ is the trajectory of the system (12) Furthermore, from lemma 2.2. there is a function $0(\varepsilon)$ such that $\|0(\varepsilon)\|/\varepsilon \underset{\varepsilon\to 0}{\to} 0$ and

$$\tilde x(\bar t+k)=x^o(\bar t+k)\,\varepsilon\,\Phi(\bar t+k,\bar t)(\bar x-x^o(\bar t))+0(\varepsilon).$$

In particular, we have for $k=N-t$

$$\tilde x(N)=x^o(N)+\varepsilon\vec a+0(\varepsilon)\in\mathcal{K}_F.$$

The last conclusion implies that $\vec a\in TC(\mathcal{K}_F, x^o(N))$ On the other hand, since $\operatorname{conv} TC(\mathcal{K}_F, x^o(N))\neq R^n$ then we again define a nonzero vector $\vec n\in R^n$ such that

$$\langle\vec n,\vec a\rangle\leqslant 0 \text{ for all } \vec a\in\bigcup_{t=1}^N\widetilde{\mathcal{K}}_F^t.$$

In the system (13), putting $p(N)=n$, we can obtain that

$$\langle p^o(N),\Phi(N,t)(x-x^o(t))\rangle=\langle p^o(t),x-x^o(t)\rangle\leqslant 0.$$

for all $x\in F(t,x^o(t-1),x^o(t-h))$, $t=1,2,\dots,N$. Q.E.D.

Remark 2.1. The condition $\operatorname{conv} TC(\mathcal{K}_F, x^o(N))\neq R^n$ can be replaced by the following condition:

$$F(t,\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2)\geqslant\lambda F(t,x_1,y_2)+(1-\lambda)F(t,x_2,y_2),$$

where $(x_i,y_i)\in R^n\times R^n$, $\lambda\in[0,1]$, $i=1,2$. $t=1,2,\dots,N$.

Remark 2.2. Let $U_t(x(t-1), x(t-h))$ be given by the system of identities and inequalities of the form:

$$G_\alpha(t, x(t-1), u(t)) = 0, \quad \alpha = 1, 2, \dots, k,$$
$$g_\beta(t, x(t-1), u(t)) \leqslant 0, \quad \beta = 1, 2, \dots, k.$$

Then problem of existence of local sections in this case can be found in [7].

Definition 2.3. We shall say that the trajectory $x^o = \{x^o(t)\}$ corresponding to the admissible control $u^o = \{u^o(t)\}$ of the system (9) possesses a local section if there exists a smooth function $\beta(t, x, y)$ defined in the neighbourhood of points $(x^o(t-1), x^o(t-h))$ such that:

i) the function $\varphi(t, x, y) = f(t, x, y, \beta(t, x, y))$ is continuously differentialabe.

ii) $\beta(t, x^o(t-1), x^o(t-h)) = u^o(t)$,

iii) $\beta(t, x, y) \in U_t(x,y)$ for all (x, y) in the domain of the function $\beta(t, x, y)$.

Theorem 2.2. Assume that for every (t, x, y), $f(t, x, y, U_t)$ is a convex subset of R^n. Let $\beta(t, x, y)$ be a local section of $\{x^o(t), u^o(t)\}$. Furthermore, assume that conv TC $(\mathcal{K}, x^o(N)) \neq R^n$. Then there exists a nonzero vector $\vec{n} \in R^n$ such that:

$$\max_{u \in U_t(x^o(t-1), (t-h))} H(t, p^o(t), x^o, u) = H(t, p^o(t), x^o, u^o), \quad t = 1, 2, \dots, N$$

where the vectors $p^o(t)$ are defined by the following relation

$$p^o(t-1) = \left[\frac{\partial\varphi(t, x^o(t-1), x^o(t-h))}{\partial x^o(t-1)}\right]^* p^o(t) + \left[\frac{\partial\varphi(t, x^o(t-h+2), x^o(t-1))}{\partial x^o(t-1)}\right]^* p^o(t-h-1)$$

$$p^o(N) = \vec{n}, \; p^o(N+1) = \dots = p^o(N+h-1) = 0.$$

Theorem 2.2 immediately follows from the lemma 2.3.

Theorem 2.3. Let us consider the system (9), where $U_t(.) = R^m$. Assume that the function $f(t,.)$ is continuously differentiable for $t = 1, 2, \dots, N$. Assume that rang $\Omega(N, x, u) = n$ for all admissible control functions $u = \{u(t)\}$ and trajectories $x = \{x(t)\}$ of the system (9) If the controllable set of the system (9) is closed, then the system (9) is globally controlable.

Proof. Propose that $\mathcal{K} \neq R^n$. We shall prove that in this case there is a point $x^o \in \mathcal{K}$ such that TC $(\mathcal{K}, x^o) \neq R^n$. Indeed, since $\mathcal{K}$ is nonempty and closed, there exists a point $x \in R^n \setminus \mathcal{K}$ and $x^o \in \mathcal{K}$ such that:

$$\rho(x, \mathcal{K}) = \inf_{a \in \mathcal{K}} \rho(x, a) = \rho(x, x^o) = \delta > 0.$$

Let $S_\delta(x)$ be a open sphere, center and radius of which accordingly are x and σ, we have

$$S_\delta(x) \cap \mathcal{K} = \phi.$$

Therefore, there is a positive number $\beta > 0$, such that for every $\lambda \in (o, \beta)$ we have $\widetilde{x} = x^o + \lambda(x - x^o) + 0(\lambda) \in \mathcal{K}$, where $\| 0(\lambda) \| / \lambda \underset{\lambda \to 0}{\to} 0$ On the other hand:

$$\rho(x, \widetilde{x}) = \| (1-\lambda)(x - x^o) - 0(\lambda) \| \leqslant (1-\lambda)\,\delta + \lambda \frac{\| 0(\lambda) \|}{\lambda}.$$

Hence, we can find a suitable positive λ, such that $\rho(x, \widetilde{x}) < \delta$, i. e., $\widetilde{x} \in S_\delta(x)$ It is impossible. Thus, there exists a trajectory $x^o = \{x^o(t)\}$ corresponding to the admissible control $u^o = \{u^o(t)\}$ of the system (9) such that $x^o(N) = x^o$ and TC $(\mathcal{K}, x^o(N)) \neq R^n$. Now, using the theorem 2.1 there is a nonzero vector $n \in R^n$ such that

$$\frac{\partial H(t, p^o(t), x^o(t-1), x^o(t-h), u^o(t))}{\partial u^o(t)} = 0,\ t = 1,2,\dots, N,$$

where $p^o(N) = n$.

Notice that:

$$n^*\Omega(N, x^o, u) = p^o(N)\,\Omega(N, x^o, u^o)$$

$$= \left[\frac{\partial H(1, p^o(1), x^o(0), x^o(1-h), u^o(1))}{\partial u^o(1)}, \dots, \frac{\partial H(N, p^o(N), x^o(N-1), x^o(N-h), u^o(N))}{\partial u^o(N)}\right]$$

$$= 0. \tag{13}$$

From this, we see that there exists the pair $\{x^o, u^o\}$, such that rang $\Omega(N, x^o, u^o) < n$. This contradicts the assumption of the theorem 2.3. Q.E.D.

Received April 15th, 1979

REFERENCES

1. L, Markus. *Controllability of nonlinear processes.* SIAM J. on control 3 (1965) p.p 78–90.

2. Pham Huu Sach. *Controllability and invariance in certain linear processes.* Automatica i Telemekhanika N7 (1976), pp.26–35 (in Russian)

3. J. Yorke, *The maximum principle and controllability of nonlinear equation.* SIAM J. on control Vol.10. N.2, (1972), pp 336–340,

4. Gabasov R., Kirillova F.M., Krakhotko V., Minkius s. *Controllability of linear discrete system.* Differential equation. T VIII, N. 5, 6, 7 (1972) (in Russian).

5. L. Weiss. *Controllability, realization and stability of discrete–time system.* SIAM J. on control Vol. 10, N.2, (1972), pp. 230 – 262.

6. A. Propoi. *Optimal theory of discrete processes.* Moscow – Nauka 1973.

7. V.G. Boltrianskii. *Optimal control of discrete systems.* Moscow–Nauka. 1973.