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ON THE KERNEL REPRESENTAION OF SOME CLASSES OF LINEAR OPERATORS

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§ 1 — Preliminaries.

Let T and S be two separated locally compact spaces with σ – finite positive Radon measures μ and λ, respectively. Let E be an arbitrary B - space over real (or complex) number field Φ. For a real number $q (1 \leqslant q \leqslant \infty)$ by $L^q_E(S, \lambda)$ or $L^q_E(S)$ we shall denote the space of λ – measurable functions f on S to E such that $\| f \|_q < \infty$, where $\| f \|_q$ is the norm of an element $f \in L^q_E(S, \lambda)$ and is defined as:

$$\| f \|_q \equiv \| f \|_{L^q_E(S)} = \begin{cases} [\int\limits_S \| f(s) \|^q_E \lambda(ds)]^{1/q} & \text{if} \quad 1 \leqslant q < \infty \\ \underset{\lambda}{\text{vrai}} \sup\limits_{s \in S} \{ \| f(s) \|_E \} & \text{if} \quad q = \infty \end{cases}$$

In particular, if $E = \Phi$, the space $L^q_\Phi(S, \lambda)$ is denoted by $L^q(S, \lambda)$ (or $L^q(S)$).

Let $X(T)$ be a linear manifold dense in the space $L^1(T, \mu)$ and let $Y^q_E(S)$ be a B - space imbedded in the space $L^q_E(S, \lambda)$. By $[X(T) \underset{1}{\to} Y^q_E(S)]$ we shall denote the class of all continuous linear operators $F : X(T) \to Y^q_E(S)$, where the continuity of each operator F belonging to $[X(T) \underset{1}{\to} Y^q_E(S)]$ is defined by the norms of $L^1(T, \mu)$ and $Y^q_E(S)$, i. e.

$$\| Ff \|_{Y^q_E(S)} \leqslant \| F \|_{[X(T) \underset{1}{\to} Y^q_E(S)]} \cdot \| f \|_{L^1(T)} \quad (\forall f \in X(T)). \tag{1.1}$$

$$\| F \|_{[X(T) \underset{1}{\to} Y^q_E(S)]} = \sup \{ \| Ff \|_{Y^q_E(S)} : f \in X(T), \| f \|_{L^1(T)} \leqslant 1 \} \tag{1.2}$$

In the special case that $X(T) \equiv L^1(T, \mu)$, by $[L^1(T) \to Y^q_E(S)]$ we denote the class $[L^1(T) \underset{1}{\to} Y^q_E(S)]$. In this paper, we shall consider the kernel representation of an operator $F \in [X(T) \underset{1}{\to} Y^q_E(S)]$, $(1 \leqslant q \leqslant \infty)$ – namely, consider the existence of a function $k: S \times T \to E$, such that

$$[Ff](s) = \int_T k(s, t) f(t) \mu(dt) \; (\forall s \in S (\text{mod } \lambda), \forall f \in X(T)) \tag{1.3}$$

In particular, if $X(T) = L^1(T, \mu)$ and $Y^q_\Phi(S) = L^q(S, \lambda)$, the above problem has been investigated under any more strict assumptions for the operator F. Concretely, it supposes (see [4; p. 379]) that F is a compact operator from $L^1(T, \mu)$ to $L^q(S, \lambda)$ $(1 \leqslant q \leqslant \infty)$ or F is a separable operator from $L^1(T,\mu)$ to $L^q(S, \lambda)$ $(1 < q < \infty)$. If $q = 1$ suppose (see[5; p. 547]) that F is a weakly compact operator from $L^1(T, \mu)$ to a separable subspace of $L^1(S, \lambda)$.

It is well known that the kernel representation of **F** belonging to some other classes of linear operators has been also considered in Refs [8], [9], [10], [1], [2], [15]. As applied examples, in this paper we shall also consider the kernel representation (1.3) of an operator $F \in [X(T) \underset{1}{\to} Y^q_\Phi(S)]$, where $X(T)$ and $Y^q_\Phi(S)$ are a number of functional spaces playing an important role in the theory of partial differential equations.

§ 2 — The kernel representation of some operators belonging to $[X(T) \underset{1}{\to} Y^q_E(S)]$.

First we study the special case when $X(T) = L^1(T, \mu)$.

Theorem 2.1 Let F is a weakly compact linear operator from $L^1(T, \mu)$ to $Y^q_E(S)$ $(1 \leqslant q \leqslant \infty)$, where T, S, E, $Y^q_E(S)$ satisfy the conditions quoted in the preliminaries. Then,

1) there is a $\lambda \times \mu$-essentially unique, $\lambda \times \mu$-measurable function k on $S \times T$ to E such that

$$[Ff](s) = \int_T k(s, t) f(t) \mu(dt) \; (\forall s \in S (\text{mod } \lambda), \; \forall f \in L^1(T, \mu)), \tag{2.1}$$

2) for a fixed $t \in T$: $k(., t) \in Y^q_E(S)$ and

(2.2) $\sup_{t \in T} \{ \| k(., t) \|_{Y^q_E(S)} \} \; \| F = _{[L^1(T) \to Y^q_E(S)]}$,

3) if $E = \Phi$ for a fixed $s \in S$ (mod λ): $k(s,.) \in L^\infty(T, \mu)$ (*)

(*) It is well known (see [4; p. 379]) that this theorem has been formulated in the particular case that T is Euclidean, μ is Lebesgue measure, $q=1$, $E=\Phi$ and $Y^1_\Phi(S) = L^1(S, \lambda)$

Proof. Since F is a weakly compact linear operator from $L^1(T, \mu)$ to the B-space $Y_E^q(S)$ and T is a separated locally compact space with positive Radon measure μ, so (see [6; p. 882]) there exists a μ-measurable function x on T to $Y_E^q(S)$ such that

$$\| x(t) \|_{Y_E^q(S)} \leqslant \| F \|_{[L^1(T) \to Y_E^q(S)]} \quad (\forall t \in T), \tag{2.3}$$

$$Ff = \int_T f(t)\, x(t)\, \mu(dt) \quad (\forall f \in L^1(T, \mu)). \tag{2.4}$$

Because $Ff \in Y_E^q(S) \subset L_E^q(S, \lambda)$, hence by (2.4):

$$[Ff](s) = [\int_T f(t)\, x(t)\, \mu(dt)](s) \quad (\forall s \in S \ (\mathrm{mod}\lambda),\ \forall f \in L^1(T, \mu)). \tag{2.5}$$

For each $f \in L^1(T, \mu)$, the functional $f(.)$ is μ-measurable on T, therefore the function $f(.)\, x(.)$ transforming T into $Y_E^q(S)$ is also μ-measurable (see [5; p 120]) Besides, by (2.3) we have

$$\| f(t).\, x(t) \|_{Y_E^q(S)} = | f(t) | . \| x(t) \|_{Y_E^q(S)} \leqslant | f(t) | . \| F \|_{[L^1(T) \to Y_E^q(S)]} (\forall t \in T)$$

Hence (see [5; pp. 131 – 132]) the function $f(.)\, x(.)$ transforming T into $Y_E^q(S)$ is μ-integrable. But the space $Y_E^q(S)$ is imbedded in $L_E^q(S, \lambda)$, thus the function $f(.)\, x(.)$ transforming T into $L_E^q(S, \lambda)$ is also μ-integrable. It is known that μ and λ are σ-finite positive measures, then (see [5; p. 218]) there is a function g_f corresponding to the μ-integrable function $f(.)\, x(.)$, such that g_f is a $\lambda \times \mu$-measurable function on $S \times T$ to E and

$$g_f(., t) = f(t) . x(t) \quad (\forall t \in T \ (\mathrm{mod}\mu),\ \forall f \in L^1(T, \mu)). \tag{2.6}$$

Moreover, for a fixed $s \in S$ (mod λ): $g_f(s,.)$ is μ-integrable on T and the integral $\int_T g_f(s, t)\, \mu(dt)$, as a function of s, is equal to the element $\int_T x(t) f(t)\, \mu(dt) \in$ $\in L_E^q(S, \lambda)$:

$$\int_T g_f(s, t)\, \mu(dt) = [\int_T x(t) f(t)\, \mu(dt)](s), \quad (\forall s \in S \ (\mathrm{mod}\lambda)). \tag{2.7}$$

Since $x(t) \in Y_E^q(S) \subset L_E^q(S, \lambda)$ $(\forall t \in T)$, the function k defined by

$$k(s, t) = [x(t)](s) \quad (\forall (s, t) \in S \times T) \tag{2.8}$$

transforms $S \times T$ into E. Then (see (2.6)):

$$g_f(s, t) = f(t) k(s, t), \quad (\forall s \in S \ (\mathrm{mod}\lambda),\ \forall t \in T \ (\mathrm{mod}\mu),\ \forall f \in L^1(T, \mu)). \tag{2.9}$$

Therefore from (2.5), (2.7) we get (2.1). And by (2.8) it follows that $k(.,t)= =x(t) \in Y_E^q(S)$ $(\forall t \in T)$. By (2.4) it is easy to see that

$$\| F \|_{[L^1(T) \to Y_E^q(S)]} \leqslant \sup_{t \in T} \{ \| \varkappa(t) \|_{Y_E^q(S)} \} = \sup_{t \in T} \{ \| k(.,t) \|_{Y_E^q(S)} \}.$$

Hence from (2.3) we get (2.2).

Because the measure μ is σ - finite, then we can write $T = \bigcup_n T_n$, where $\mu(T_n) < \infty$ $(\forall n)$. Put $f_n(t) = \chi_{T_n}(t)$ $(\forall t \in T)$, it is clear that $f_n \in L^1(T, \mu)$ $(\forall n)$. Therefore (see (2.9))

$$k(s,t) = g_{f_n}(s,t), \text{ if } (s,t) \in S \times T_n \,(\text{mod}\, \lambda \times \mu). \tag{2.10}$$

It is known that g_{f_n} is $\lambda \times \mu$ - measurable on $S \times T$ $(\forall n)$. Hence by (2.10) it follows that k is also $\lambda \times \mu$ - measurable on $S \times T$

In order to prove that the function k is $\lambda \times \mu$-essentially unique on $S \times Y$, suppose that there is an other $\lambda \times \mu$-measurable function $\widetilde{k}$ on $S \times T$ to E such that

$$[Ff](s) = \int_T \widetilde{k}(s,t) f(t) \mu(dt), \; (\forall s \in S (mod \lambda), \; \forall f \in L^1(T, \mu). \tag{2.11}$$

Then, from (2.1) we have

$$\int_A [k(s,t) - \widetilde{k}(s,t)] \, \mu(dt) = 0 \; (\forall s \in S \; (mod \lambda), \; \forall A \subset T : \mu(A) < \infty)$$

Therefore $k(s,t) = \widetilde{k}(s,t)$ $(\forall (s,t) \in S \times T$ $(mod \lambda \times \mu)$. This completes the proof of conclusions 1 and 2.

Now, we consider Conclusion 3. Here, suppose that $E = \Phi$. Since $F \in [L^1(T) \to \to Y_\Phi^q]$ and $Y_\Phi^q(S) \subset L^q(S, \lambda)$, so

$$| [Ff](s) | < \infty \; (\forall s \in S \, (mod \lambda), \forall f \in L^1(T, \mu)). \tag{2.12}$$

By (2.1) and (2.12) it follows that for a fixed $s \in S (\text{mod} \lambda)$: $k(s,\cdot) f(\cdot) \in L^1(T, \mu)$ $(\forall f \in L^1(T, \mu))$. Therefore (see [5; p. 380]) $k(s,.) \in L^\infty(T, \mu)$ $(\forall s \in S(\text{mod}\lambda))$. Q. E. D.

Now, we study the case that the domain of the operator F is a linear manifold $X(T)$ dense in $L^1(T, \mu)$.

Corollary 2.2 Let $F \in [X(T) \underset{1}{\to} Y_E^q(s)]$ $(1 \leqslant q \leqslant \infty$, where $T, S, E, Y_E^q(s)$, $X(T)$ satisfy the conditions quoted in the preliminaries.

Suppose that $Y_E^q(S)$ is a reflexive space. Then,

1) there exists a $\lambda \times \mu$ - essentially unique, $\lambda \times \mu$ - measurable function k on $S \times T$ to E such that

$$[Ff](s) = \int_T k(s,t) f(t) \mu(dt) \; (\forall s \in S \, (\text{mod} \; \lambda), \; \forall f \in X(T)), \tag{2.13}$$

$$\widetilde{F} \in [L^1(T) \to Y^q_E(S)], \text{ where } \widetilde{F} f \equiv \int_T k(.,t) f(t)\, \mu(dt), \tag{2.14}$$

2) for a fixed $t \in T : k(.,t) \in Y^q_E(S)$ and

$$\sup \{ \| k(.,t) \|_{Y^q_E(S)} \} = \| F \|_{[X(T) \underset{1}{\to} Y^q_E(S)]}, \tag{2.15}$$

3) if $E = \Phi$, for a fixed $s \in S$ (mod λ): $k(s,.) \in L^\infty(T, \mu)$.

Proof. Because $F \in [X(T) \underset{1}{\to} Y^q_E(S)]$, $Y^q_E(S)$ is a B – space and $X(T)$ is dense in the space $L^1(T, \mu)$, then (see [11; p.124]) there is uniquely an extension $\widetilde{F} \in [L^1(T) \to Y^q_E(S)]$ of the operator $F : X(T) \to Y^q_E(S)$, i.e.

$$\widetilde{F} f = Ff \quad (\forall f \in X(T)), \tag{2.16}$$

$$\| \widetilde{F} \|_{[L^1(T) \to Y^q_E(S)]} = \| F \|_{[X(T) \underset{1}{\to} Y^q_E(S)]}. \tag{2.17}$$

Since the space $Y^q_E(S)$ is reflexive, so (see [5; p, 520]) $\widetilde{F}$ is a weakly compact linear operator from $L^1(T, \mu)$ to $Y^q_E(S)$ Then, by Theorem (2.1) it follows that there is a $\lambda \times \mu$ - measurable function k on $S \times T$ to E such that

$$[\widetilde{F} f](s) = \int_T k(s,t) f(t) \mu(dt) \ (\forall s \in S(\text{mod}\lambda),\ \forall f \in L^1(T, \mu)). \tag{2.18}$$

Besides, it fulfils Conclusion 3 and $k(.,t) \in Y^q_E(S)$ $(\forall t \in T)$ and

$$\sup_{t \in T} \{ \| k(.,t) \|_{Y^q_E(S)} \} = \| \widetilde{F} \|_{[L^1(T) \to Y^q_E(S)]}. \tag{2.19}$$

By (2.16) – (2.19) we have (2.13) — (2.15).

In order to prove the uniqueness of the function k defined by (2.13) – (2.14), suppose that there is another $\lambda \times \mu$ - measurable function h on $S \times T$ to E such that

$$[Ff](s) = \int_T h(s,t) f(t) \mu(dt) \ (\forall s \in S \text{ (mod } \lambda).\ \forall f \in X(T)), \tag{2.20}$$

$$H \in [L^1(T) \to Y^q_E(S)]. \text{ where } Hf \equiv \int_T h(.,t) f(t) \mu(dt) \tag{2.21}$$

By (2.13), (2.20) (2.21) it follows $Hf = Ff$ $(\forall f \in X(T))$ i.e. $H \in [L^1(T) \to Y^q_E]$ is also an extension of the operator $F \in [X(T) \underset{1}{\to} Y^q_E(S)]$.

From the uniqueness of a continuous linear extension, we have $\widetilde{F} = H$.

Therefore (see (2.21))

$$[\widetilde{F}f](s) = \int_T h(s,t) f(t) \mu(dt) \, (\forall s \in S (\mathrm{mod}\lambda), \forall f \in L^1(T, \mu)) \tag{2.22}$$

By (2.18), (2,22) and Conclusion 1 of Theorem (2.1) it is easy to see that $k(s, t) = h(s, t) \; (\forall (s, t) \in S \times T \; (\mathrm{mod} \; \lambda \times \mu))$. Q.E.D.

Corollary 2.3. In Corollary (2.2) if replace the assumption that $Y^q_E(S)$ is a reflexive space by the following assumption: the imbedding of $Y^q_E(S)$ in $L^q_E(S, \lambda)$ is compact. Then, in the conclusions, $Y^q_E(S)$ is replaced by $L^q_E(S, \lambda)$.

Proof. It is known (see the proof of Corollary (2.2)) that there is the extension $\widetilde{F} \in [L^1(T) \to Y^q_E(S)]$ of the operator $F \in [X(T) \underset{1}{\to} Y^q_E(S)$ for which we have (2.16). But $Y^q_E(S)$ is imbedded in $L^q_E(S)$, then $\widetilde{F} \in [L^1(T) \to L^q_E(S)]$ and $F \in [X(T) \underset{1}{\to} L^q_E(S)]$. Therefore, by (2.16) it follows that the operator $\widetilde{F} \in [L^1(T) \to L^q_E(S)]$ is also an extension of $F \in [X(T) \underset{1}{\to} L^q_E]$ and we have (see [11; p. 124])

$$\| F \|_{[X(T) \underset{1}{\to} L^q_E(S)]} = \| \widetilde{F} \|_{[L^1(T) \to L^q_E(S)]}. \tag{2.23}$$

On the other hand, $\widetilde{F} \in [L^1(T) \to Y^q_E(S)]$ and the imbedding of $Y^q_E(S)$ in $L^q_E(S)$ is compact. Then (see [5; p. 523]) is a compact linear operator from $L^1(T, \mu)$ to $L^q_E(S, \lambda)$. Therefore, similarly as in the proof of Corollary (2.2), using Theorem (2.1) for $\widetilde{F} \in [L^1(T) \to L^q_E(S)]$, by (2.16), (2.23) we get the conclusions of Corollary (2.2), where $Y^q_E(S)$ is replaced by $L^q_E(S, \lambda)$. Q.E.D.

§3. SOME EXAMPLES

Let μ and λ be the Lebesgue measures defined on Euclidean spaces R^n and R^m, let $T = \Omega_T$ and $S = \Omega_S$, where Ω_T and Ω_S are two bounded open domains of R^n and R^m with boundaries of the class C^∞, respectively. It is clear that T and S are separated locally compact spaces with finite positive Radon measures μ and λ. Let $Y^q(\Omega_S)$ be the space $L^q(\Omega_S)$ (or the Sobolev space $W^l_q(\Omega_S)$, the Besov space $B^l_q(\Omega_S)$ the space $H^l_q(\Omega_S)$ of Bessel potentials), where $1 < q < \infty$. $0 < l < \infty$. Then (see [12; pp. 79—81]) $L^q(\Omega_S)$ is a reflexive space imbedded in $L^q(\Omega_S)$. Hence, by Corollary (2.2) we can represent an operator $F \in [L^1(\Omega_T) \to Y^q(\Omega_S)]$ in the from (2.13).

Let $X(\Omega_T)$ be the space $W^h_p(\Omega_T)$, (or $B^h_p(\Omega_T)$, $L^p(\Omega_T)$, $C^r(\Omega_T)$, where $1 < p < \infty$; $h < \infty$; $r = 0, 1, 2, \ldots, \infty$). Then it is easy to deduce (see [12; pp. 81–83]) that $X(\Omega_T)$ is a linear manifold dense in $L^1(\Omega_T)$. Therefore, from Corollary (2.2) we can also consider the kernel representation of an operator $F \in [X(\Omega_T) \underset{\tau}{\rightarrow} Y^q(\Omega_S)]$.

We investigated the existence of the kernel k, for which the given operator F is represented in form (1.3) of an integral operator. With the aid of an overaging operator (see [3; pp. 39–41]) it can approximately determinate the kernel k. Therefore we may approximate the operator F by an integral operator and approximate the solution of some operational linear equations by the solution of Fredholm integral equations. As is well known, various probability models have been constructed for estimating the values of an integral operator and for solving a Fredholm integral equation of second type (see [7], [13], [14]) Then, with use of the results in this paper, we may estimate the values of operators and solve some classes of operational equations by the Monte-Carlo method.

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