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MULTIDIMENSIONAL QUANTIZATION I. THE GENERAL CONSTRUCTION

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Introduction.

Let G be a connected and simply connected Lie group. In order to find irreducible unitary representations of G, the famous Kirillov's orbit method has proposed some procedure of quantization, starting from a linear (i.e., one-dimensional vector) bundle over a G-homogeneous sympletic manifold (see [2], § 15). We remark that the quantum systems can be constructed, starting from an irreducible representation, in general, of an arbitrary dimension of the stabilizer G_F of a point F in a K-orbit Ω (see [2], [4]).

The goal of this paper is the exposition of the new procedure of quantization in the general case, starting from an arbitrary irreducible G-bundle associated with the given Hamiltonian mechanical system. The new procedure of quantization gives us a large number of irreducible representations of the group G. In particular, we obtain a lot of known irreducible unitary repre sentations of Lie groups and groups of diffeomorphisms.

In § 1 we will generalize the usual construction of holomorphically induced representations. The representations thus obtained will be called the *partially invariant holomorphically induced representations (in L^2-coho-mologies)* and denoted by (L^2-coh_k) Ind (G; p. ρ. F, σ) In § 2 these representations will then be illustrated as representations obtained from the natural generalization of the Kirillov's procedure of quantization, which we call the *multidimensional quantization*.

§ 1. CONSTRUCTION OF UNITARY REPRESENTATIONS

In this section we construct unitary representations of the Lie group G in the space of partially invariant, partially holomorphic square integrable sections of multidimensional G-bundles over the K-orbits, or in the spaces of higher L^2-cohomologies of suitable sheafs.. To do this we must generalize the construction of holomorphically induced representations. The principle difference of our construction from the usual one (see [1] — [4]) is the another meaning of the concept of polarization. A polarization in our sense is consists of a Lie subalgebra p and some of its irreducible representation ρ of some dimension.

Let us denote by G a connected Lie group, by g its Lie algebra, by $g_C = g \underset{R}{\otimes} C$ the complexification of g. It is clear that g_C is a complex Lie algebra, there exists the natural anti-automorphism of complex conjugation in which. For the element $z = x + iy \in g_C$ we denote its complex conjugate element by $\bar{z} = x - iy$. If a is a subset of g_C, we define:

$$\bar{a} = \{ \bar{z} ; z \in a \}$$

Suppose that D is a closed subgroup (not necessarily connected) of G with Lie algebra d, $\tilde{\sigma}$ is some fixed irreducible unitary representation of D in a separable Hilbert space $\tilde{V}$.

Definition 1.1. A $(D, \tilde{\sigma})$ – *polarization* is consists of a collection (p, H, ρ) such that:

1) H is a closed Lie subgroup of G, containing D, p is a complex Lie subalgebra of g_C such that $p \cap g = h$ is the Lie algebra of the Lie group H.

2) The Lie subalgebra p is invariant under the operators $Ad_{g_C} x$, $x \in D$.

3) There exists a closed Lie subgroup M in G, containing H such that $m = (p + \bar{p}) \cap g$ is its Lie algebra.

4) σ is an irreducible unitary representation of the group H in a separable Hilbert space V such that the restriction $\sigma | D$ is a multiple of the representation $\tilde{\sigma}$, i.e. the space V is a tensor product $V' \otimes V'$, where V' is a Hilbert space, and

$$\sigma | D = I_V \otimes \tilde{\sigma}$$

Let us denote by $d\sigma$ the corresponding [2] representation of the Lie algebra h.

5) ρ is a representation, satisfying all the conditions of E. Nelson (see, for example, [2], 10.5, Th. 3) of the complex Lie algebra p by hermition operators (in general, unbounded) in the Hilbert space V such that

Proposition 1.1. If (p, H, ρ, σ) is any $(D, \tilde{\sigma})$ – polarization, then we have:

$$p \cap \bar{p} = h_C$$
$$p + \bar{p} = m_C$$

where h_C, m_C are the complexications of h, m, respectively.

Indeed, from the definition 1.1., we have

$$h_C = h + ih = p \cap g + ip \cap g \subset p \cap \bar{p},$$
$$m_C = m + im = (p + \bar{p}) \cap g + i(p + \bar{p}) \cap g \subseteq p + \bar{p}$$

Conversely, assume that $z = x + iy$. If $z \in p \cap \overline{p}$ then $\overline{z} \in p \cap \overline{p}$,

$$x = \frac{1}{2}(z + \overline{z}) \in p \cap \overline{p} \cap g \subseteq h$$

$$y = \frac{-i}{2}(z - \overline{z}) \in p \cap \overline{p} \cap g \subseteq h.$$

Hence $z \in h_C$, and $h_C = p \cap \overline{p}$. If

$z \in p + \overline{p}$, then

$\overline{z} \in p + \overline{p}$,

$$x = \frac{1}{2}(z + \overline{z}) \in (p + \overline{p}) \cap g = m$$

$$y = \frac{-i}{2}(z - \overline{z}) \in (p + \overline{p}) \cap g = m$$

Thus, $z \in m_C$, and $p + \overline{p} = m_C$

Theorem 1. Suppose that (p, H, ρ, σ) is any $(D, \tilde{\sigma})$ — polarization. Then:

1) There exists a structure of a mixed manifold of type (k, l) in the space $x = D \setminus G$, the space of all right cosets clases, where:

$$k = \dim G - \dim M + \dim H - \dim D$$

$$l = \frac{1}{2}(\dim M - \dim H)$$

2) In the smooth G — bundle $\mathscr{E}_{\sigma|D} = G \underset{\sigma|D}{\times} V$, associated with the representation $\sigma|D$, there exists a structure of a partially invariant (see its exact definition in the proof of this theorem) G — bundle $\mathscr{E}_{\sigma,\rho}$ such that the natural representation of the group G, arising in the space of partially invariant and partially holomorphic sections of $\mathscr{E}_{\sigma,\rho}$ is equivalent to the representation of this group by right translations in the space $C^\infty(G; p, H, \rho, \sigma)$ of smooth functions f on G with values in V and satisfying the following system of equations:

$$f(hx) = \sigma(h) f(x) \;; \quad \forall h \in H, \forall x \in G,$$

$$L_X f + \rho(X) f = 0 \;; \quad \forall X \in p$$

where L_X is the Lie derivative along the vector field ξ_X on G, corresponding to X.

Proof. From the first assertion of the theorem 1 in § 13.4 of [2] we have that there exists a structure of a mixed manifold of type (k', l) on the right G—space $H \setminus G$, where $k' = \dim G - \dim M$, $l = \frac{1}{2}(\dim M - \dim H)$. Obviously, $D \setminus G$ is a smooth fibre bundle over the base $H \setminus G$ with the typical fibre $D \setminus H$. From the direct constructing of Charts of manifolds $D \setminus H, H \setminus G, D \setminus G$, we see that the structure of the mixed manifold on the base $H\backslash G$ induces the structure of a mixed manifold on $D\backslash G$. The easy computing of dimensions shows that $D\backslash G$ is equiped by the structure of a mixed manifold of type (k, l), where

$$k = k' + \dim(D\backslash H) = \dim G - \dim M + \dim H - \dim D,$$

$$l = \frac{1}{2}(\dim M - \dim H)$$

The first assertion of our theorem is thus proved.

Let $\tilde{\rho}$ be the representation of the local group $\tilde{P}$, corresponding to the representation ρ of the Lie algebra p, and let $\tilde{M_C}$ be the local Lie group corresponding to the Lie algebra m_C. Then $\mathcal{E}_\rho = \tilde{M}_C \underset{\tilde{P}}{\times} V$ is a holomorphic M_C-bundle over $P\backslash M_C$. Obviously $\mathcal{E}_\rho$ is equivalent to the restriction of the bundle $\mathcal{E}_\sigma = M \underset{H}{\times} V$ to the corresponding neighborhood W of the initial point $z_0 = \{H\} \in H\backslash M$.

To define the structure of a partially holomorphic bundle on $\mathcal{E}_\sigma$, we use the identification of its restriction to subsets of the form W_g with the holomorphic bundles what are obtained from $\mathcal{E}_\rho$ by translating. Thus we have the structure of partially holomorphic bundle on $\mathcal{E}_\sigma$. The given bundle is denoted by $\overline{\mathcal{E}}_{\delta,\rho}$. Now we consider the natural projection map $p: D\backslash G \to H\backslash G$. The inverse image bundle $p^*\mathcal{E}_{\delta,\rho}$ is a bundle over $D\backslash G$ which we call the *partially holomorphic partially invariant G-bundle.*

Obviously in the category of smooth vector bundles $p^*\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\sigma|D}$ are equivalent.

The sections of the bundle $\mathcal{E}_{\sigma|D}$ are identified with the functions on G with values in V satisfying the equations

$$f(hx) = \sigma(h)f(x); \quad h \in D, \quad x \in G.$$

If the functions satisfied the stronger conditions

$$f(hx) = \sigma(h)f(x); \quad h \in H, \; x \in G,$$
$$L_X f - \rho(x)f = 0; \; X \in p,$$

then they are called *partially invariant partially holomorphic sections* of the partially invariant partially holomorphic G-bundle $p^* \mathcal{E}_{\sigma,\rho}$. Clearly the space of all partially invariant and partially holomorphic sections of G-bundle $p^*\mathcal{E}_{\sigma,\rho}$ is the image in the natural embedding of the space of partially holomorphic sections of the partially holomorphic G-bundle $\mathcal{E}_{\sigma,\rho}$ into the space of all smooth sections of the G-bundle $\mathcal{E}_{\sigma|D}$ Thus the theorem is proved. Q.E.D.

We see that if we have a $(D, \tilde{\sigma})$ – polarization (p, H, ρ, σ) on $D\setminus G$, then we have a partially invariant partially holomorphic G — bundle $p^* \mathscr{E}_{\sigma, \rho}$. To obtain an unitary representation we apply the usual construction of *unitary G — bundle* ([4], p.44).

Suppose that Δ_G (resp., Δ_D) is the modular function of the group G (resp., D), $\delta^2(h) = \Delta_D(h)\setminus\Delta_G(h)$, $h \in D$, is the non-unitary character of D. We consider the G — bundle $\mathscr{M} = G \underset{D}{\times} C$, associated with the non-unitary character δ^2 of the subgroup D. The positive measurable sections of the bundle $\mathscr{M}$ and the quasiinvariant measures on $X = D\setminus G$ are in a bijection.

We denote by $\mathscr{M}^{1/2}$ the bundle, associated with the character $\delta = (\Delta_D/\Delta_G)^{1/2}$. Thus the bundle $\widetilde{\mathscr{E}}_{\sigma, \rho} = p^* \mathscr{E}_{\sigma, \rho} \otimes \mathscr{M}^{1/2}$ is an G — bundle over $D\setminus G$. If s is a section of the bundle $\widetilde{\mathscr{E}}_{\sigma, \rho}$ then $\| s \|_v^2$ is a setion of the bundle $\mathscr{M}$. This the integral:

$$\| s \|^2 = \int_{D\setminus G} \| s \|_v^2$$

is defined and we can define a scalar product of a pair of sections of this type by the formula

$$(s_1, s_2) = \int_{D\setminus G} (s_1(x), s_2(x))\, dx$$

Now we fix a section $\mu = \mu_X$ of the bundle $\mathscr{M}$, i.e., a quasiinvariant measure on X. Let $L^2(G; p, H, \rho, \sigma)$ be the Hilbert space, which is the completion with respect to the written above scalar product of the space $\Gamma(\widetilde{\mathscr{E}}_{\sigma; \rho})$ of sections of $\widetilde{\mathscr{E}}_{\sigma, \rho}$ of type $\tilde{s} = s \,.\, \mathscr{M}^{1/2}$, where s is a partially and invariant partially holomorphic section of $p^* \mathscr{E}_{\sigma, \rho}$. In this Hilbert space $L^2(G; p.H, \rho, \sigma)$ on ehas the natural unitary representation of the group G, which e will be denoted by:

$$\mathrm{Ind}\,(G;\, p, H, \rho, \sigma)$$

and will called the *partially invariant holomorphically induced representation.*

The further generalization of this construction gives us representations of the group G in L^2 – cohomologies (see also [3], §§6, 7).

Suppose that $\mathscr{F}$ is the sheaf of germs of partially invariant sections of the bundle $p^* \mathscr{E}_{\sigma, \rho}$ over the mixed manifold $D\setminus G$. We denote by $\mathscr{F}^q$ the sheaf of differential forms of type (o, q) on $D\setminus G$ with values in the bundle $p^* \mathscr{E}_{\sigma, \rho}$. Here we call a form of type (o, q) any expression of type:

$$\Sigma \varphi_{i_1 \ldots i_q}(y, z, t)\, dz_{i_1} \wedge \ldots \wedge dz_{i_q}$$

in a local system of coordinates on X (the real coordinates $y = (y_1, \ldots, y_{k'})$ define the projection of the point $x \in X$ in the manifold $M \setminus G$, the complex coordinates $z = (z_1, \ldots, z_l)$ define the place of the point x in the fibre lying over the point y of the bundle $H \setminus G$, and the real coordinates $l = (l, \ldots, l_{k-k'})$ define the place of the point x in the fibre lying over the point (y, z) of the bundle $D \setminus G$), where $\varphi_{i_1 \ldots i_q}(y, z, l)$ are functions in a neighborhood of the point $x \in X$ with values in V. In the transition from the local chart (y, z, l) to the other chart (y', z', l') this expression is changed by:

$$\Sigma \psi_{j_1 \ldots j_q}(y', z', l')\, dz'_{j_1} \wedge \ldots \wedge dz'_{j_q},$$

where

$$\psi_{j_1 \ldots j_q}(y', z'\, l') = u(y', z', l')\ \Sigma \varphi_{i_1 \ldots i_q}(y(y'), z(z'\, y'), l'\ (y\, z', l')\ \frac{\partial z_{i_1}}{\partial z'_{j_1}} \cdots \frac{\partial z_{i_1}}{\partial z'_{j_q}}$$

and $u(y'\, z', l)$ is the transition function of our bundle.

The following sequence of sheafs is exact:

$$0 \to \mathcal{G} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \ldots \to \mathcal{G}^l \to 0,$$

where the mapping $\mathcal{G} \to \mathcal{G}^0$ is induced by inclusion of the spaee of partially invariant partially holomorphic sections of the bundle $\mathcal{E}_{\sigma_1 \rho}$ into the space of smooth sections, and the following mappings are induced by the usual operator d'', mapping a form of type (o, q) intoa form of type (o, $q+1$).

The exact sequence of sheafs defines a complex of spaces of sections

$$0 \to \Gamma(\mathcal{G}^0) \to \Gamma(\mathcal{G}^1) \to \ldots \to \Gamma(\mathcal{G}^l) \to 0$$

the coho mlogies of which coincide with the coho mologies of the sheaf $\mathcal{G}$ (the analogue of the Dolbeault's theorem).

The space of smooth forms of type (o, q) on $D \setminus G$ is isomorphic to the subspace $C^q(p, H; C^\infty((G) \otimes V)$ in the tensor product $C^\infty(G) \otimes \Lambda^q(p/h_C)^* \otimes V$ This subspace consists of H-invariant elements. If the subgroup H is connected then the H-invariance can be changed into h-invariance (or the samet h_C-invariance), The mentioned subspace coincides exactly with the space $C^q(p, h_C, C^\infty(G) \otimes V)$ of relative q-dimensional cocycle of the algebra p. Thus the cohomologies $H^q(D \setminus G; F)$ coincide with the relative cohomologies $H^q(p, H; C^\infty(G) \otimes V)$ of the Lie algebra p.

Assume that there exists a H – invariant Hermitian structure in p/h_C, Then the induced hermitian structure arises in $\Lambda^q(p/h_C)^\star$ Now we denote by σ^q the unitary representation of group H in $\Lambda^q(p/h_C)^\star$. The space $C^q(p, H$ $C^\infty(G) \oplus V)$ can be interpreted as the space $\Gamma(\mathcal{E}\sigma^q \oplus \sigma, \rho)$ of partially invarian, partially holomorphic sections of the G – bundle $\mathcal{E}\sigma^q \otimes \sigma, \rho$, associated with the unitary representation $\sigma^q \otimes \sigma$ of the group H.

Denote by $L^2(\widetilde{\mathscr{E}}\sigma^q \otimes \sigma, \rho)$ the complection of the space of partially-invariant partially – holomorphic square integrable sections with respect to the natural scalar product of sections. We have the complex

$$0 \to L^2(\widetilde{\mathscr{E}}\sigma, \rho) \xrightarrow{d_1} L^2(\widetilde{\mathscr{E}}\sigma^1 \sigma, \otimes \rho) \xrightarrow{d_2} \dots \xrightarrow{d_1} L^2(\widetilde{\mathscr{E}}\sigma^1 \oplus \sigma, \rho) \to 0$$

The orthogonal complement of $\mathrm{Im}(d_k)$ in $\mathrm{Ker}(d_{k+1})$ is a Hilbert space in which the natural unitary representation of the group G arises. We denote this representation by $(L^2 - Coh_k)$ Ind $G; p, H, \rho, \sigma)$ and we call it the *partially – invaaiant holomorphically induced representation in k – dimensional L^2 – cohomologies.*

Now we return to the problem of the construction of unitary representations of the group G, associated with the orbits of the coadjoint representation (shortly, K – orbits). As we see above, a choice of a subgroup D, its irreducible unitary representation $\widetilde{\sigma}$ and a $(D, \widetilde{\sigma})$ – polarization (p, H, ρ, σ) plays a principal role. The orbit method gives us a canonical choice of these objects.

Let $g^\star$ be the dual space of the Lie algebra g. It is easy to see that the coadjoint representation (shortly, K – representation) of the group G in $g^\star$ divides $g^\star$ into K – orbits. We denote by $O(G)$ the space of all K – orbits of the group G.

Now we fix an K – orbits $\Omega \in 0(G)$ and a point F in it. Assume that G_F is the stabilizer of the point F, g_F is the Lie algebra of G_F. It is known that in the category of hologeneous G–spaces the following isomorphism has place

$$\Omega \approx G_F \backslash G$$

Suppose that $(G_F)_0$ is the connected component of the identity of the group G_F and $(G_F)_0 = S.R$ is its E. Cartan – Levi – Malsev's decomposition. $\widetilde{\sigma}$ is an irreducible unitary representation of G_F such that $\widetilde{\sigma}|_R = I$, the identity representation ofa suitable dimension, $\widetilde{\rho} = d\widetilde{\sigma}$ is the corresponding representation of the Lie algebra g_F and hence, also of its complex hull.

$$(g_F)_C = g_F \otimes_R C$$

Definition 1.2. A $(\widetilde{\sigma}, F)$ – *polarizlion* of the K–orbits Ω is consists of a triple (p, ρ, σ_0) such that:

1) p is a complex Lie subalgebra of g_C, containing the Lie subalgebra g_F.

2) The subalgebra p is invariant with respect to the operators $\mathrm{Ad}_{g_C} x$, $x \in G_F$.

3) The space $p + \overline{p}$ is the complex hull of some real Lie subalgebra m, i, e.,

$$(p + \overline{p}) \cap g = m$$

4) All the subgroups M_o, H_o, M, H are closed in G, where M_o (respectively H_o) is the connected Lie subgroup of G with its Lie algebra m (resp.- $h = p\, g$) $M = G_F.M_o$, $H = G_F.H_o$.

5) σ_o is an irreducible unitary representation of the group H_o in the Hilbert space V such that:

a) The restriction $\sigma_o \mid G_F \wedge H_o$ is a multiple of $\chi_F . \widetilde{\sigma}$ where $\chi_F(\exp(.)) \stackrel{def}{=} e^{i2\pi<F,.>}$, i.e. $V = V' \otimes \widetilde{V}$, and $\sigma_o | G_F \wedge H_o = I_{V'} \otimes \chi_F . \widetilde{\sigma}$

6) ρ is a representation of the complex Lie algebra p by hermitian operators (in general, unbounded) in the Hilbert space $V = V' \otimes \widetilde{V}$ which satisfies the E. Nilson's condition and $\rho | h = d\sigma_o$, where $d\sigma_o$ is the representation of the Lie subalgebra h in V corresponding to σ_o.

Proposition 1. 2. Let G be a connected Lie group, then the following conditions are equivalent:

1) There exists a character $\chi_F : G_F \to T = S^1$ such that its differential $d\chi_F$,

$$(d\chi_F)(X) \stackrel{def}{=} \frac{d}{dt} (\chi_F(\exp tX))|_{t=0}, \quad x \in g_F \text{ is equal to } d\chi_F = i2\pi F|_{g_F}$$

2) The Kirillov's form B_Ω belongs to an integral de Rham's cohomology class.

Proof. See § 15.3 in [2] or [4]. Q.E.D

Definition 1.3. If one of the conditions of proposition 1.2. is satisfied, then we say that the K-orbit Ω is *integral*.

Theorem 2. Suppose that Ω is an integral K-orbit, $F \in \Omega$, $\chi_F: G_F \to T$ is a character of G_F, $\widetilde{\sigma} \in \widehat{G_F}$, the dual of G_F, such that $\widetilde{\sigma}|_R = I$, identity representation, (p, ρ, σ_o) is a $(\widetilde{\sigma}, F)$-polarization of the K-orbit Ω, H_o (resp., M_o) is the connected closed subgroup of G, the Lie algebra of which is $h = p \wedge g$ (resp., $m = (p + \overline{p}) \wedge g$), $H = G_F.H_o$, $M = G_F.M_o$. Then:

1) There exists a structure of mixed manifold of type (k, l) on the G-space $\Omega = G_F \setminus G$, where:

$$k = \dim G - \dim M + \dim H - \dim G_F$$

$$l = \frac{1}{2} (\dim M - \dim H)$$

2) There exists a unique unitary representation σ of the subgroup H such that the restriction σ / G_F is a multiple of the representation $\chi_F . \widetilde{\sigma}$ and $d\sigma = \rho | h$

3) On the smooth G-bundle $\mathscr{E}_{\sigma|G_F} = G \underset{\sigma|G_F}{\times} V$, associated with the representation σ / G_F there exists a structure of a partially-invariant partially

holomorphic G-bundle $\mathcal{E}_{\sigma,\rho}$ such that the representation of the group G arising in the space of partially invariant partially holomorphic sections of $\mathcal{E}_{\sigma,\rho}$ is equivalent to the representation of this group by right translations in the space $C^\infty(G; p, H, \rho, \sigma)$ of smooth functions f on G with values in V, and satisfying the following system of equations:

$$f(hx) = \sigma(h) f(x), \quad \forall h \in H \quad \forall x \in G,$$
$$L_X f + \rho(x) f = 0, \quad \forall X \in p,$$

where L_X is the Lie derivative along the vector field ξ_X on G, corresponding to X.

Proof. From the definition of a $(D, \tilde{\sigma})$ - polarization it follows that $(G_F)_0 \subset H_0$. Then $(G_F)_0$ is the connected component of identity in $H_0 \cap G_F$, and we have:

$$\sigma_0 \mid H_0 \cap G_F = I_{V'} \otimes (\chi_F . \tilde{\sigma}) \mid H_0 \cap G_F$$

On the other hand the subgroup G_F normaliser H_0. Thus G_F acts on the dual $\widehat{H_0}$ of subgroup H_0 However from the assumptions of our theorem, σ_0 is fixed over the action of G_F, then the formula

$$(x, h) \mapsto (I_{V'} \otimes \widetilde{\chi_F \sigma})(x)\, \sigma_0(h)$$

defines a representation of the product $G_F \times H_0$ in the space V.

In fact, on one hand we have

$$(x, h).(x', h') \mapsto (I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma})(x)\, \sigma_0(h)\, (I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma})(x')\, \sigma_0(h')$$
$$= (I_{V'} \otimes \widetilde{\chi_F \sigma})(xx') [(I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma})(x')]^{-1} \sigma_0(h) (I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma})(x') . \sigma_0(h')$$
$$= (I_{V'} \otimes \widetilde{\chi_F \sigma})(xx')\, (x' . \sigma_0)(h)\, \sigma_0(h')$$
$$= (I_{V'} \otimes \widetilde{\chi_F \sigma})(xx') . \sigma_0(hh')$$

On the other hand, according to the definition we have

$$(x, h)(x', h') = (xx', hh') \mapsto (I_{V'} \otimes \widetilde{\chi_F \sigma})(xx') \sigma_0(hh').$$

The representation $(x, h) \mapsto (I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma})(x)\, \sigma_0(h)$ is trivial on the kernel of the surjection

$$G_F \times H_0 \to G_F . H_0$$
$$(x, h) \mapsto x . h$$

Thus there exists a unique representation of the semidirect product $H = G_F.H_0$. We denote by σ this representation. Obviously that the given represen. tation σ is irreducible and $\sigma \mid H_0 = \sigma_0$,

$$\sigma \mid G_F = I_{V'} \otimes \widetilde{\chi_F^{\cdot} \sigma}$$

Now the complex (p, H, ρ, σ) is a $(G_F, \widetilde{\chi_F^{\cdot} \sigma})$ - polarization (see Def. 2. 1. above). Hence the theorem 2 follows from the theorem 1. Q. E. D.

We denote the corresponding unitary representation of the group G by Ind $(G; p, \rho, F, \sigma_0)$. If there exists an invariant hermitian structure on $p/h_{\mathbb{C}}$, then we also have unitary representations in L^2-cohomologies, which we denote by $(L^2\text{-}coh_k)$ Ind $(G; p, \rho, F, \sigma_0)$.

We remark that $(L^2\text{-}Coh_0)$ Ind $(G; p, \rho, F, \sigma_0)$ coincides with Ind $(G; p, \rho, F, \sigma_0)$.

In the set of all $(\widetilde{\sigma}, F)$-polarization of the K-orbit we introduce the following ordered relation,

Definition 1.4. We say that the $(\widetilde{\sigma}, F)$-polarization (p, ρ, σ_0) is smaller than the $(\widetilde{\sigma}, F)$-polarization (p', ρ', σ_0) and write $(p, \rho, \sigma_0) < (p', \rho', \sigma_0')$ if and only if:

a) $p \subseteq p'$.
b) $\sigma_0'|H_0 \simeq \sigma_0$
c) $\rho'|p \simeq \rho$

Remark 1.1. Our binary relation of $(\widetilde{\sigma}, F)$-polarizations satisfies all the axioms of a partially linear ordered relations. Thus there exists the maximal elements in the set of all $(\widetilde{\sigma}, F)$-polarizations.

Proposition 1.3. If the $(\widetilde{\sigma}, F)$-polarization (p, ρ, σ_0) is not maximal, then the partially invariant holomorphically induced representations (in L^2-cohomologies)

$$\text{Ind } (G; p, \rho, F, \sigma_0)$$
$$(L^2\text{-}coh_k) \text{ Ind } (G; p, \rho, F, \sigma_0)$$

are reducible.

Proof. If the $(\widetilde{\sigma}, F)$-polarization (p, ρ, σ_0) is not maximal then there exists a $(\widetilde{\sigma}, F)$-polarization $(p', \rho'\ \sigma_0')$ such that

$$(p, \rho, \sigma_0) < (p', \rho', \sigma_0')$$

In this case $L^2(G; p', \rho', F, \sigma_0')$ is included in $L^2(G; p, \rho, F, \sigma)$ as an invariant Hilbert space. Thus the representation Ind $(G; p, \rho, \sigma_0)$ is reducible. Analogously, the representation $(L^2\text{-}coh_k)$ Ind $(G; p, \rho, F, \sigma_0)$ is also reducible. Q.E.D.

§ 2. PROCEDURE OF THE MULTIDIMENSIONAL QUANTIZATION

In this section we shall show how the above described construction of unitary representation arises from the natural generalization of the usual Kirillov's procedure of quantization.

In general, a *quantization* means a procedure of constructing quantum systems from given classical systems. A majority of the existing methods of quantization are subsumed under the following schema ([2], § 15). Consider the *physical quantities* associated with a system. Among these we single out a certain set of *primary quantities* forming a Lie algebra under the *Poisson brakets.* We suppose that when we go over the quantum mechanics, the *commutation relations* among primary quantities are preserved in the following sense. Let h be the Planck's constant and $\hat{f}$ the *quantum mechanical operator*, corresponding to the primary classical quantity f. Then the following relation must be satisfied :

$$\widehat{\{f_1, f_2\}} = \frac{ih}{2\pi} [\widehat{f_1}, \widehat{f_2}] \tag{2.1}$$

This means that the correspondence

$$f \to \frac{ih}{2\pi} \widehat{f}$$

iis an operator representation of a Lie algebra of primary quantities. Ordinarily constants are included among the primary quantities, and one requires that the relation

$$\widehat{1} = I \text{ (identity operator)} \tag{2.2}$$

holds.

We consider a fixed classical Hamilton system (Ω, B_Ω). Suppose that $F \in \Omega$, G_F is the stabilizer of F, and the K-orbit Ω is integral, $\chi_F^\bullet \widetilde{\sigma}$ is rreducible representation, described above, of the stabilizer G_F, (p, ρ, σ_0) is a $\widehat{(\sigma, F)}$-polarization of the K-orbit Ω.

Proposition 2.1. The choice of a $\widetilde{(\sigma, F)}$-polarization (p, ρ, σ_0) defines a giving of a integrable G-invariant distribution L of the complex hule of the tangent bundle $T\Omega$, such that $L + \overline{L}$ is also integrable. The maximality of the $\widetilde{(\sigma, F)}$-polarization is equivalent to the maximality of the distribution L.

Proof. Suppose that $p_0 : g_C \to (T_F\Omega)_C \cong (g/g_F)_C$ is the natural projection. We define the distribution L at the point $F \in \Omega$ by the formula

$$L_F = p_0(p)$$

The G-invariance defines our distribution at all the other points of the orbit. The remaining assertion is obvious. Q.E.D.

To construct a quantum system we choose the Hilbert space which is the completion of the space of all partially invariant square integrable sections of the unitary G-bundle

$$\widetilde{\mathcal{E}}_{\sigma, \rho} = p^\star \mathcal{E}_{\sigma, \rho} \otimes \mathcal{M}^{1/2} = p^\star(G \times_\sigma V) \otimes \mathcal{M}^{1/2}$$

From the local triviality of the bundle $\widetilde{\mathscr{E}}_{\sigma,\rho}$ we can choose a covering $\{U_j\}_{j\in I}$ of the K—orbit Ω by open sets U_i, $i \in I$, such that

$$\widetilde{\mathscr{E}}_{\sigma,\rho} \, | U_j \cong U_j \times V .$$

Then a section over U_j is a function φ_j on U_j with values in V, shortly, $\varphi = \{\varphi_j\}$, where $\varphi_j = \varphi \,|\, U_j$ and

$$\varphi_j(x) = g_{jk}(x)\varphi_k(x), \quad \forall x \in U_j \cap U_k ,$$

g_{jk} are the transition functions of $\widetilde{\mathscr{E}}_{\sigma,\rho}$.

Now to the smooth function $F \in C^\infty(\Omega)$ we let correspond the operator $\widehat{F}$, acting by formula:

$$\widehat{F} : \varphi = \{\varphi_j\} \to \widehat{F}\varphi = \{\widehat{F}_j \varphi_j\},$$

where:

$$\widehat{F}_j = I_V \otimes \left(\frac{2\pi}{ih} \xi_{F_j} + F_j\right) + \alpha_j(\xi_{F_j}),$$

$\alpha = \{\alpha_j\}$ is some defferential 1—form on Ω, the value of which at a point is an hermitian operator in the corresponding Hilbert fibre of the bundle $\widetilde{\mathscr{E}}_{\sigma,\rho}$

Remark 2.1. In the intersection $U_j \cap U_k$ we have:

$$g_{jk} \, . \, \widehat{F}_k + \widehat{F}_j \, . \, g_{jk}$$

In fact, $g_{jk} \, . \, \widehat{F}_k = g_{jk} \otimes \left(\frac{2\pi}{ih} \xi_{Fk} + F_k\right) + g_{jk} \circ \alpha_k(\xi_{Fk}) =$

$$= I_V \otimes \left(\frac{2\pi}{ih} \xi_{Fj} + F_j\right) \circ g_{jk} + \alpha_j(\xi_{Fj}) \circ g_{jk}$$

because $\alpha = \{\alpha_j\}_{j\in I}$ is a differential 1—form. Hence we have:

$$\begin{aligned} \widehat{F}_j \varphi_j &= \widehat{F}_j (g_{jk} \varphi_k) \\ &= (\widehat{F_j \, g_{jF}}) \varphi_k \\ &= g_{jk} \widehat{F}_k \varphi_k. \end{aligned}$$

This shows that $\{F_j \varphi_j\}_{j\in I}$ defines a section. Thus the action $\widehat{F}$ is an operator in the space of sections.

Proposition 2.2. The correspondence $F \to \widehat{F}$ defines a procedure of quantization if and only if the differential form α satisfies the relation

$$2(I_V \otimes B_\Omega)(\xi_F, \xi_{F'}) = \xi_F \alpha(\xi_{F'}) - \xi_{F'} \alpha(\xi_F) - \alpha([\xi_F, \xi_{F'}]) + \frac{ih}{2\pi} [\alpha(\xi_F), \alpha(\xi_{F'})]$$

Proof. It is clear that the correspondence $F \to \widehat{F}$ satisfies the relation 2.2). To prove that the operation $F \to \widehat{F}$ gives a procedure of quantization it s necessary and sufficient to verify the relation (2.1).

From the local triviality of the bundle we can restrict our proof to local case of trivial bundle. We have:

$$\frac{ih}{2\pi}\ [\widehat{F},\ \widehat{F}'] = \frac{ih}{2\pi}\left[I_V \otimes \left(\frac{2\pi}{ih}\xi_F + F\right) + \alpha(\xi_F),\ \ I_V \otimes \left(\frac{2\pi}{ih}\xi_{F'} + F'\right) + \alpha(\xi_{F'})\right]$$

$$= \frac{2\pi}{ih} I_V \otimes [\xi_F, \xi_{F'}] + I_V \otimes [\xi_F, F'] + I_V \otimes [F, \xi_{F'}] + [I_V \otimes \xi_F, \alpha(\xi_{F'})] + [\alpha(\xi_F),$$

$$I_V \otimes \xi_{F'}] + \frac{ih}{2\pi}\ [\alpha(\xi_F), \alpha(\xi_{F'})] = \frac{2\pi}{ih} I_V \otimes \xi_{\{F, F'\}} + I_V \otimes \{F,\ F'\} + \alpha(\xi_{\{F, F'\}}) -$$

$$- \alpha([\xi_F, \xi_{F'}]) - I_V \otimes \{F, F'\} + \xi_F\alpha(\xi_{F'}) - \xi_{F'}\alpha(\xi_F) + \frac{ih}{2\pi}[\alpha(\xi_F), \alpha(\xi_{F'})] = \widehat{\{F,F'\}}$$

$$- 2\,(I_V \otimes B_\Omega)\,(\xi_F, \xi_{F'}) + \xi_F\alpha(\xi_{F'}) - \xi_{F'}\alpha(\xi_F) - \alpha([\xi_F, \xi_{F'}]) + \frac{ih}{2\pi}\ [\alpha(\xi_F), \alpha(\xi_{F'})]$$

Thus the proposition is proved. Q.E.D.

Remark 2.2. If $[\alpha(\xi_F), \alpha(\xi_{F'})] = 0$ then we have the condition of proposition 2.2 in ordinary form:

$$2(I_V \otimes B_\Omega)\,(\xi_F, \xi_{F'}) = \xi_F\alpha(\xi_{F'}) - \xi_{F'}\,\alpha(\xi_F) - \alpha\,([\xi_F, \xi_{F'}]) = 2\,(d\alpha)\,(\xi_F, \xi_{F'})$$

Hence:

$$d\alpha = I_V \otimes B_\Omega$$

Now the space of partially invariant sections of the bundle $p^\star\mathscr{E}_{\sigma,\rho}$ is illustrated as the space of section, covariant derivative of which along every direction from the distribution L (see proposition 2.1) is equal to zero:

$$0 = \nabla_{\xi_X}\varphi = (\widehat{F}_X - I_V \otimes F_X)\varphi, \qquad \forall X \in p,$$

where F_X is the generating function of the field ξ_X. In fact $\nabla_{\xi_X} = \widehat{F_X} - I_V \otimes F_X$ enjoys all of the properties of a covariant derivative along the vector field ξ_X.

For simplicity in the sequel, we denote the generating function F_X of the field ξ_X, $X \in g$ by the same letter X. That is well, because $x \in g$ can be in the same time regarded as a function on the dual space $g^\star$.

The representation of the group G is defined by the formula

$$T(\exp X) = e^{-i\frac{2\pi}{h}\widehat{X}}$$

The relation (2.1) and the self – adjointness operators X guarantee that the condition

$$T(g_1 \cdot g_2) = T(g_1)\ T(g_2)$$

holds and the operator $T(g)$ are unitary in a certain neighborhood of the identity. This «local» representation admits a unique extension to a multi – valued representation of G, which will be single – valued on the simply connected covering group $\widetilde{G}$ of the group G. In a following paper of this series we will show:

Theorem 3. The partially invariant holomorphically induced representation of connected and simply connected Lie group G coincides with the representation of this group, arising in the procedure of multidimensional quantization.

This theorem gives a physical illustration of our construction of partially invariant holomorphically induced representation. Thus it shows the inverse application of physical ideas in the representation theory as one of the purely mathematical field.

Added in Proof. Some notations and statements just have been precised in the note «Construction des représentations unitaires par les K-orbites et quantification» C.R. Acad. Sc. Paris, t. 291 (1980), pp 295—298.

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