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A CHARACTERIZATION OF MIXED – STABLE LAWS

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The aim of this note is to prove the following theorem:

A probability measure on R^n is mixed-stable if and only if it is completely self-decomposable.

Throughout this paper the symbol R^n $(n = 1, 2,...)$ will denote the n-dimensional Euclidean space, with the usual norm $\| . \|$ and the inner product $< , >$. For every positive number a we define a transform T_a on the class of Borel probability measures on R^n by

$$T_a \mu (A) = \mu (\{a^{-1} x : x \in A\}) \ (A \subset R^n).$$

A probability measure μ on R^n is called stable if for every positive interger k there exists a positive constant a_k and a vector b_k in R^n such that

$$\mu^{*k} = T_{a_k} \mu * \delta_{b_k}$$

Further, μ is called mixed-stable if there exists a triangular array $\{\mu_{mk}\}$ of stable measures μ_{mk} on R^n $(k = 1, 2,..., k_m ; m = 1, 2,...)$ such that the sequence

$$\mu_{m1} * \mu_{m2} * \cdots * \mu_{mk_m}$$

converges weakly to μ as $m \to \infty$.

The concept of completely self decomposable measures has been introduced in [3]. Namely, a probability measure μ on R^n is called completely self-decomposable if for every sequence $c_1, c_2,...$ of numbers from the interval $(0, 1)$ there exists a sequence $\mu_{c_1}, \mu_{c_1, c_2}...$ of probability measures on R^n such that

$$\mu = T_{c_1} \mu * \mu_{c_1}, \ \mu_{c_1} = T_{c_2} \ \mu_{c_1} * \mu_{c_1, c_2}, ...$$

Proof of the theorem. It is well-known (see, for example [1]) that if μ is a stable measure on R^n, then it is a Gaussian measure or there is a number $0 < \alpha < 2$, a finite Borel measure σ on the unit sphere $S = \{x \in R^n : \| x \| = 1\}$ and a vector $a \in R^n$ such that

$$\widehat{\mu} (y) = \exp \{i < a, y > + \int_S \int_0^\infty K (\rho x, y) \frac{d\rho}{S^{1+\alpha}} \sigma (dx) \} \tag{1}$$

where
$$K (x, y) = \exp (i < x, y >) - 1 - \frac{i < x, y >}{1 + \| x \|^2} \tag{2}$$

On the other hand, from the general form of characteristic functionals of completely self-decomposable measures on Banach spaces ([3], Theorems 6.4 and 7.2), it follows that if μ is completely self-decomposable on R^n then $\mu = \rho * \mu_1$, where ρ is a Gaussian measure and

$$\widehat{\mu_1}(y) = \exp\left\{i < a, y > + \int_B \left(\int_0^\infty K(sx, y) \frac{ds}{S^{2\|x\|+1}} \right) \frac{\sin \pi \|x\|}{\|x\|^{2\|x\|}} \, m(dx)\right\} \qquad (3)$$

($y \in R^n$), where a is a vector in R^n, m is a finite Borel measure on the open unit ball $B = \{x \in R^n : \|x\| < 1\}$ vanishing at 0 and the kernel K is given by the formula (2). Consequently, for a degenerate measure $m = \delta_{x_0}$ where $x_0 \in B$ the formula (3) is of the form (1) and then μ_1 is a stable measure on R^n. Since for every finite measure m on B there exists a sequence of measures of the form $\sum_{k=1}^{u} \lambda_k \delta_{x_k}$ where $\lambda_k \geqslant 0$ and $x_k \in B$ $(k = 1, 2, \ldots u.)$ converging to m, it follows, by virtue of the formulas (1) and (3), that every completely self-decomposable measure is mixed-stable.

Conversely, let μ be a mixed-stable measure on R^n. Then it is a weak limit of a sequence $\mu_{m1} * \mu_{m2} * \ldots * \mu_{mk_m}$ where μ_{mk} $(k \leqslant k_m;\ m = 1, 2, \ldots)$ are some stable measures on R^n. By virtue of Proposition 1.9 [2] it follows that every stable measure on R^n is completely self-decomposable. Consequently, every finite convolution of stable measures is completely self-decomposable. Since the class of all completely self-decomposable probability measures on R^n is closed under the weak convergence it follows that every mixed-stable measure is completely self-decomposable, which completes the proof of the theorem.

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