ACTA MATHEMATICA VIETNAMICA
TOM 5, N°1 (1980)

THE BANACH — STEINHAUS THEOREM FOR MULTIVALUED M-CONVEX MAPPINGS

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The well-known Banach—Steinhaus Theorem asserts that a family of linear continuous and point bounded mappings from a barrel space into a Hausdorff locally convex space is equicontinuous [1]. More precisely, let X be a barrel space, Y a Hausdorff locally convex space and let $\mathcal{F} = \{f_v, v \in I\}$ be a family of linear continuous mappings from X into Y such that for every $x \in X$ the set $\{f_v(x), v \in I\}$ is bounded in Y. Then for every neighbourhood V of the origin of Y there is a neighbourhood U of the origin of X such that

$$f_v(U) \subset V \quad \text{for each } v \in I.$$

Our purpose in the present paper is to extend the above mentioned result to multivalued M-convex mappings.

First, let us introduce some notations and definitions. Throughout this paper we shall denote by X a barrel space, by Y a Hausdorff locally convex space and by M a closed convex cone in Y.

Definition 1. A multivalued mappings $F : X \to Y$ is said to be *M-convex* if for all points x and y of X and for all $\alpha \in [0,1]$

$$F(\alpha x + (1-\alpha)y) \subset \alpha F(x) + (1-\alpha)F(y) + M,$$

Definition 2. Given $x_0 \in X$, we say that a family of multivalued mappings $\mathcal{F} = \{F_v, v \in I\}$ from X into Y is *(M, x_0)-bounded* if for every neighbourhood V of the origin of Y and for every $x \in X$ there exist a positive number ρ and an index $v_0 \in I$ such that

$$F_v(x) \subset \rho V + F_{v_0}(x_0) + M,$$

for each $v \in I$.

Definition 3. We say that a set $A \subset Y$ is *M-bounded* if for every neighbourhood V of the origin of Y there exists a positive number ρ such that

$$A \subset \rho V + M$$

Definition 4. A family of multivalued mappings $\mathcal{F} = \{F_v, v \in I\}$ from X into Y is said to be *M-bounded at* $x_0 \in X$, if the set $\bigcup_{v \in I} F_v(x_0)$ is M-bounded.

Proposition 1. Let $\mathcal{F} = \{F_\nu, \nu \in I\}$ be a family of multivalued mappings from X into Y.

a) It $\mathcal{F}$ is (M, x_0)-bounded and if for every $\nu \in I$ the set $F_\nu(x_0)$ is M-bounded then $\mathcal{F}$ is M-bounded at every point of X.

b) If $\mathcal{F}$ is M-bounded at every point of X then $\mathcal{F}$ is (M, x)-bounded for every point $x \in X$, for which there exists $\nu_0 \in I$ such that $-F_{\nu_0}(x)$ in M-bounded.

Proof. a) Suppose that $\mathcal{F} = \{F_\nu, \nu \in I\}$ is (M, x_0)-bounded. By definition, for every neighbourhood V of the origin of Y and for every $x \in X$ there exists a positive number ρ and an index $\nu_0 \in I$ such that

$$F_\nu(x) \subset \rho V + F_{\nu_0}(x_0) + M \quad (\text{all } \nu \in I) \tag{1}$$

If $F_{\nu_0}(x_0)$ is M-bounded there is a positive number p sach that

$$F_{\nu_0}(x_0) \subset pV + M \tag{2}$$

From (1) and (2) it follows that for every V and for every $x \in X$,

$$F_\nu(x) \subset \rho^* V + M \qquad (\text{all } \nu \in I)$$

where $\rho^* = \rho + p$. This means that for every $x \in X$, $\bigcup_{\nu \in I} F_\nu(x)$ is M-bounded in Y.

b) Suppose now that $\mathcal{F}$ in M-bounded at every point of X and consider an arbitrary point x_0 of X for which there exists $\nu \in I$ such that $-F_{\nu_0}(x_0)$ is M-bounded in Y. For every neighbourhood V in Y and for every $x \in X$ there is a positive number ρ satisfying

$$F_\nu(x) \subset \rho V + M \qquad (\text{all } \nu \in I) \tag{3}$$

Since $-F_{\nu_0}(x_0)$ is M-bounded there is a positive number γ such that

$$-F_{\nu_0}(x) \subset \gamma V + M. \tag{4}$$

Setting $\rho^* = \rho + \gamma$ we obtain from (3) and (4) that for any $x \in X$, there are $\rho^* > 0$ and $\nu_0 \in I$ such that:

$$F_\nu(x) \subset \rho V + M \subset \rho V + F_{\nu_0}(x_0) + M - F_{\nu_0}(x_0)$$
$$\subset (\rho + \gamma)V + F_{\nu_0}(x_0) + M = \rho^* V + F_{\nu_0}(x_0) + M$$

(all $\nu \in I$). Therefore $\mathcal{F}$ is (M, x_0)-bounded. The proof is complete.

Definition 5. A family $\mathcal{F} = \{F_\nu, \nu \in I\}$ of multivalued mappings from X into Y is said to be *M-equicontinuous at $x_0 \in X$* if for every neighbourhood V of the origin of Y there exists a neighbourhood U of the origin of X such that $F_\nu(x_0 + U) \subset V + F_\nu(x_0) + M$ for all $\nu \in I$.

Definition 6. A family $\mathcal{F} = \{F_\nu, \nu \in I\}$ of multivalued mappings from X into Y is said to be *(M, x_0)-equicontinuous* if for every neighbourhood V of the origin of Y, there exists a neighbourhood U of the origin of X such that

$$F_\nu(x_0 + U) \subset V + \mathit{conv}\,(\bigcup_{\mu \in I} F_\mu(x_0)) + M,$$

for all $\nu \in I$.

Clearly, if $\mathcal{F}$ is M-equicontinuous at x_0 then it is also (M, x_0)-equicontinuous. Conversely, we have the following

Proposition 2. Let $\mathcal{F} = \{ F_\nu, \nu \in I \}$ be a family of M-convex, multivalued mappings which are (M, x_0)-equicontinuous, and satisfy the following conditions:

i) $F_\nu (x_0)$ is a convex set for every $\nu \in I$,

ii) $\mathcal{F}$ and $-\mathcal{F}$ are M-bounded at x_0

Then $\mathcal{F}$ is M-equicontinuous at the point x_0.

(Here $-\mathcal{F}$ denote: the family $\{ -F \mid F \in F \}$)

Proof. Without loss of generality it can be assumed that $x_0 = 0$ (otherwise one could use the family of mappings

$$\mathcal{F} = \{ \overline{F}_\nu, \nu \in I \} \quad \text{where} \quad \overline{F}_\nu(x) = F_\nu (x_0 + x).$$

Since $\mathcal{F}$ is $(M. O)$-equicontinuous, for every neighbourhood V (which we can assume to be convex) of the origin of Y, there exists a neighbourhood U in X such that

$$F_\nu(U) \subset V + conv\, (\bigcup_{\mu \in I} F_\mu (O)) + M, \tag{5}$$

(all $\nu \in I$).

By Condition (ii), one can find a positive number ρ_0 such that

$$F_\nu (O) \cup -F_\nu (O) \subset \rho_0 V + M \tag{6}$$

(all $\nu \in I$).

Therefore

$$conv\, (\bigcup_{\mu \in I} F\mu (O)) \subset conv\, (\rho_0 V + M) = \rho_0 V + M. \tag{7}$$

From (5) and (7) we deduce

$$F_\nu(U) \subset (1 + \rho_0) V + M \qquad (\text{all } \nu \in I). \tag{8}$$

This together with (6) yields

$$F_\nu(U) \subset (1 + \rho_0) V + M \subset (1 + \rho_0) V + M - F_\nu(O) + F_\nu(O)$$
$$\subset (1 + 2\rho_0) V + F_\nu(O) + M \qquad (\text{all } \nu \in I).$$

Setting $U' = \frac{1}{1+2\rho_0} U$ we have for each $x' \in U'$ $x = (1 + 2\rho_0)\, x' \in U$, hence:

$$F_\nu(x') = F_\nu \left(\frac{1+2\rho_0}{1+2\rho_0} x' \right) = F_\nu \left(\frac{1}{1+2\rho_0} x + \left(1 - \frac{1}{1+2\rho_0}\right) O \right)$$

$$\subset \frac{1}{1+2\rho_0} F_\nu(x) + \left(1 - \frac{1}{1+2\rho_0}\right) F_\nu(O) + M$$

$$\subset \frac{1}{1+2\rho_0} ((1+2\rho_0)V + F_\nu(O) + M) + \left(1 - \frac{1}{1+2\rho_0}\right) F_\nu(O) + M.$$

$$\subset V + F_\nu(O) + M \quad (\text{all } \nu \in I)$$

This shows that $\mathcal{F}$ is M-equicontonuous at O and so concludes the proof.

Definition 7. A multivalued mapping F from X into Y is said to be M-closed if for every closed set A in Y, the set

$$F^-(A) = \{x \in X, \quad F(x) \subset A + M\}$$

is closed in X.

Theorem I. Let $\mathcal{F} = \{F_\nu, \nu \in I\}$ be a family of M-convex, M-closed multivalued mappings from X into Y Let $x_0 \in X$. If $\mathcal{F}$ is (M, x_0)-bounded then it is also (M, x_0)-equicontinuous

Proof. We shall assume $x_0 = 0$. Let V be a convex balanced and closed neighbourhood of the origin of Y.

Set:

$$A = \bigcap_{\nu \in I} F_\nu^- \left(\overline{\frac{1}{2} V + \text{conv} \cdot (\bigcup_{\mu \in I} F_\mu(O)) + M}\right)$$

$$= \bigcap_{\nu \in I} \left\{x \in X, \ F_\nu(x) \subset \left(\overline{\frac{1}{2} V + \text{conv} \ (\bigcup_{\mu \in I} F_\mu(O)) + M}\right)\right\}$$

where the bar denotes the topological closure.

Obviously $A \neq \phi$ (at least $O \in A$). For any two elements x_1, x_2 of A and for any $\alpha \in [0,1]$ we have

$$F_\nu(\alpha x_1 + (1-\alpha) x_2) \subset \alpha F_\nu(x_1) + (1-\alpha) F_\nu(x_2) + M$$

$$\subset \alpha \left(\overline{\frac{1}{2} V + conv \ (\bigcup_{\mu \in I} F_\mu(o)) + M}\right) + (1-\alpha) \left(\overline{\frac{1}{2} V + conv \ (\bigcup_{\mu \in I} F_\mu(o)) + M}\right) + M$$

$$= \left(\overline{\frac{1}{2} V + conv \ (\bigcup_{\mu \in I} F_\mu(o)) + M}\right) \quad (all \ \nu \in I).$$

Consequently $\alpha x_1 + (1-\alpha) x_2 \in A$ which shows the convexity of A. Since F_ν is M-closed, A is closed in X.

Let x be an arbitrary point of X. As $\mathcal{F}$ is M-bounded relatively to O, for $\frac{1}{2} V$ there exists positive numbers ρ_1, ρ_2 and indices $\nu_1 \in I$, $\nu_2 \in I$ such that:

$$F_\nu (x) \subset \frac{1}{2} \rho_1 V + F_{\nu 1} (O) + M \quad (\text{all } \nu \in I).$$

$$F_\nu (-x) \subset \frac{1}{2} \rho_2 V + F_{\nu 2} (O) + M$$

Without loss of generality we may suppose $\rho_1 \geqslant 1$, $\rho_2 \geqslant 1$. Then we have:

$$F_\nu \left(\frac{1}{\rho_1} x\right) = F_\nu \left(\frac{1}{\rho_1} x + \left(1 - \frac{1}{\rho_1}\right) O\right) \subset \frac{1}{\rho_1} F_\nu (x) + \left(1 - \frac{1}{\rho_1}\right) F_\nu (O) + M$$

$$\subset \frac{1}{\rho_1} \left(\frac{1}{2} \rho_1 V + F_\nu (O) + M\right) + \left(1 - \frac{1}{\rho_1}\right) F_\nu (O) + M$$

$$\subset \overline{\left(\frac{1}{2} V + \operatorname{conv} \left(\bigcup_{\mu \in I} F_\mu (O)\right) + M\right)}$$

(all $\nu \in I$). Consequently $\frac{1}{\rho_1} x \in A$. Similarly $-\frac{x}{\rho_1} \in A$. Put $\rho_o = \max \{\rho_1, \rho_2\}$ we have $\frac{x}{\rho_o} \in A \cap (-A)$ i.e. $A \cap -A$ is an absorbing set in X. Then $U = A \cap (-A)$ is a non-empty convex balanced and absorbing set. Remembering that X is a barrel space, we conclude that U is a neighbourhood of the origin of X. We have:

$$F_\nu (U) \subset \overline{\left(\frac{1}{2} V + \operatorname{conv} \left(\bigcup_{\mu \in I} F_\mu (O)\right) + M\right)} + M$$

$$\subset \frac{1}{2} V + \operatorname{conv} \left(\bigcup_{\mu \in I} F_\mu (O)\right) + M + M + \frac{1}{2} V$$

$$= V + \operatorname{conv} \left(\bigcup_{\mu \in I} F_\mu (O)\right) + M. \qquad (\text{all } \nu \in I).$$

Thus $\mathcal{F}$ is (M, O)-equicontinuous, which concludes the proof.

Corollary 1. Let $\mathcal{F} = \{F_\nu, \nu \in I\}$ be family of M-convex, M-closed multivalued mappings from X into Y. Assume that for some $x_o \in X$:

(i) $F_\nu (x_o)$ is convex for each $\nu \in I$,

(ii) $\mathcal{F}$ and $-\mathcal{F}$ are M-bounded at every point of X. Then $\mathcal{F}$ is M-equicontinuous at x_o.

Proof. This follows at once from Proposition 2, since in view of part b) of Proposition 1, the family $\mathcal{F}$ is (M, x_o)-bounded.

Corollary 2. Let $\mathcal{F} = \{f_\nu, \nu \in I\}$ be a family of M-convex, M-closed single-valued mappings. Assume that $\mathcal{F}$ and $-\mathcal{F}$ are M-bounded at x_o Then $\mathcal{F}$ is M-equicontinuous at x_o.

Proof. Obvious.

Definition 8. A multivalued mapping F from X into Y is said to be M-upper semicontinuous at $x_0 \in X$ if for any neighbourhood V of the origin of Y there exists a neighbourhood U of the origin of X such that

$$F(x_0 + U) \subset V + F(x_0) + M$$

Definition 9. A family $\mathcal{F} = \{F_\nu, \nu \in I\}$ of multivalued mappings from X into Y is said to be M-converging to the multivalued mapping F at $x_0 \in X$ if for any neighbourhood V of the origin in Y there exists $\nu_0 \in I$ such that

$$F_\nu(x_0) \subset F(x_0) + V + M$$

and $F(x_0) \subset F_\nu(x_0) + V + M$ whenever $\nu \geqslant \nu_0$

Theorem 2. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of M-convex, M-closed, multivalued mappings from X into Y which are (M, x_0)-bounded.

Assume:

i) $F_n(x_0)$ is M-bounded in Y for every n,

ii) $\{F_n\}$ is M-converging to F at every point of X,

iii) $F(x)$ is compact for all $x \in X$ and $F(x_0)$ is convex.

Then F is M-convex, M-upper semi-continuous at x_0.

Proof. It is clear that F is a M-convex mapping. So we have only to prove the M-upper semicontinuity of F at x_0. Without loss of generality we may assume $x_0 = 0$. By Theorem 1, for any neighbourhood V (which we may assume to be convex) of the origin of Y there exists a neighbourhood U in X such that

$$F_n(U) \subset V + conv\left(\bigcup_{m=1}^{\infty} F_m(O)\right) + M, \text{ for every } n. \tag{9}$$

Since $\{F_m\}$ is M-converging to F at O there exists m_0 with

$$F_m(O) \subset F(O) + V + M \text{ for all } m \geqslant m_0. \tag{10}$$

But by hypothesis $F_m(O)$ is M-bounded in Y, therefore one can find a number $\rho_0 > 0$ such that

$$\bigcup_{m=1}^{m_0} F_m(O) \subset \rho_0 V + M. \tag{11}$$

From (10) and (11) we have:

$$conv\left(\bigcup_{m=1}^{\infty} F_m(O)\right) = conv\left(\left(\bigcup_{m=1}^{m_0} F_m(O)\right) \cup \left(\bigcup_{m=m_0}^{\infty} F_m(O)\right)\right) \tag{12}$$

$$\subset (1 + \rho_0) V + E(O) + M.$$

Relations (9) and (12) imply:

$$F_n(U) \subset (2+\rho_0)V + F(O) + M \qquad \text{(all } n\text{)}.$$

Let x be an arbitrary point in U. Since $\{F_n\}$ is M-convergent to F at x, there exists a number $n_0 \geqslant m_0$ such that:

$$F(x) \subset F_n(x) + V + M \subset (3+\rho_0)V + F(O) + M,$$

for all $n \geqslant n_0$.

Therefore

$$F(U) \subset (3+\rho_0)V + F(O) + M.$$

Taking $U' = \frac{1}{3+\rho_0}U$. we have for every $x' \in U'$, an $x \in U$ such that $x' = \frac{1}{3+\rho_0}x$. Hence

$$\begin{aligned} F(x') &= F\left(\frac{1}{3+\rho_0}x\right) = F\left(\frac{1}{3+\rho_0}x + \left(1-\frac{1}{3+\rho_0}\right)O\right) \\ &\subset \frac{1}{3+\rho_0}F(x) + \left(1-\frac{1}{3+\rho_0}\right)F(O) + M \\ &\subset \frac{1}{3+\rho_0}(3+\rho_0)V + F(O) + M) + \left(1-\frac{1}{3+\rho_0}\right)F(O) + M \\ &= V + F(O) + M \end{aligned}$$

Consequently $\qquad F(U') \subset V + F(O) + M,$

so that F is M-upper semicontinuous at O, as was to be proved.

Proposition 3. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of multivalued mappings from X into Y. Suppose that $\{F_n\}$ is M-converging to F at x_0 and $F_n(x_0)$ is M-bounded for every n. Then the sequence $\{F_n\}_{n=1}^{\infty}$ is M-bounded at x_0.

Proof. Let V be an arbitrary neighbourhood of the origin of Y. By the M-convergence of $\{F_n\}$ to F, there is a number n_0 such that

$$F_n(x_0) \subset F(x_0) + V + M \text{ (all } n \geqslant n_0\text{) and} \tag{13}$$

$$F(x_0) \subset F_n(x_0) + V + M. \tag{14}$$

From the M-boundedness of $F_n(x_0)$ and from (14) it follows that $F(x_0)$ is M-bounded i.e. there exists a number $\rho_1 >$ with

$$F(x_0) \subset \rho_1 V + M. \tag{15}$$

Furthermore, $F_n(x_0)$ is M-bounded for every n. Thus we have:

$$\bigcup_{n=1}^{n_0} F_n(x_0) \subset \rho_2 V + M \tag{16}$$

for some $\rho_2 > 0$. Relations (13) and (15) give

$$F_n(x_0) \subset \rho_0 V + M \qquad \text{(all } n\text{)}$$

with $\rho_0 = max\ (1 + \rho_1, \rho_2)$. This proves the Proposition.

Theorem 5. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of M-convex, M-closed multivalued mappings from X into Y.

Suppose :

i) $\{F_n\}$ is M-converging to F at every point of X.

ii) for some $x_0 \in X$ $F_n(x_0)$ is M-bounded and convex in Y for every n.

iii) $F(x)$ is compact for every $x \in X$. Then F is M-upper semicontinuous at x_0.

Proof. Proposition 2 and Proposition 3 imply the M-equicontinuity of the family $\{F_M\}_{n=1}^{\infty}$ at x_0 (assuming $x_0 = O$) Thus, for any neighbourhood of the origin V of Y there is a neighbourhood U of the origin of X such that

$$F_n(U) \subset V + F_n(O) + M \text{ (all } n) \tag{17}$$

By Theorem 2, F is M-convex. For every $x \in U$, from the M-convergence of $\{F_n\}$ to F at x and at O, there exists n_0 such that

$$F_n(O) \subset F(O) + V + M \text{ and} \tag{18}$$

$$F(x) \subset F_n(x) + V + M \text{ for every } n \geqslant n_0. \tag{19}$$

Relations (17) and (19) imply :

$$F(x) \subset V + V + F_n(O) + M \qquad (\text{all } n \geqslant n_0)$$

and hence, by taking (18) into account :

$$F(x) \subset 3V + F(O) + M$$

for very $x \in U$

Taking $U' = \frac{1}{3} \cdot U$ we conclude

$$F(U') \subset V + F(O) + M$$

i.e. F is M-upper semicontinuous at O.

Received July 30, 1978

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