ACTA MATHEMATICA VIETNAMICA
TOM 5, N° 1 (1980)

ON THE KERNEL REPRESENTATION OF SOME POSITIVE LINEAR OPERATORS

NGUYỄN QUÝ HỶ
Hanoi University

§ 1. PRELIMINARIES

Let (T, Σ_T, μ) and (S, Σ_s, λ) be two spaces with completely additive σ-finite non-negative measures. For a real number p $(1 \leqslant p \leqslant \infty)$, by L^p (T, Σ_T, μ) (or L^p (T)) we shall denote the B-space of Σ_T-measurable real valued functions f defined on T for which $\|f\|_p < \infty$, where $\|f\|_p$ is the norm of the element $f \in L^p$ (T, Σ_T, μ) and defined as

$$\|f\|_p \equiv \|f\|_{L^p(T)} = \begin{cases} \left[\int\limits_T |f(t)|^p \mu(dt)\right]^{1/p} & \text{if} : 1 \leqslant p < \infty \\ \underset{\mu \ t \in T}{\text{vrai sup}} \{ |f(t)| \} & \text{if} : p = \infty \end{cases}$$

L^q (S, Σ_s, λ) (or L^q (S)) will denote a similar space for the space with measure (S, Σ_s, λ). Let X^p (T) be a certain dense in the space $L^p(T)$ linear manifold and let Y^q (S) be any normed linear space imbedded in the space $L^q(S)$, i.e. $Y^q(S) \subset L^q(S)$ and there exists a positive constant M_q such that

$$\|f\|_q \leqslant M_q \|f\|_{Y^q(S)} \ (\forall f \in Y^q(S)) \ (^*), \tag{1.1}$$

Suppose that F is a defined on X^p (T) linear operator transforming X^p (T) into Y^q (S) for which there is a constant M_{pq} such that

$$\|Ff\|_{Y^q(S)} \leqslant M_{pq} \|f\|_p \ (\forall f \in X^p(T)). \tag{1.2}$$

The class of such operators is denoted by $[X^p(T) \underset{p}{\rightarrow} Y^q(S)]$ In the special case: $X^p(T) = L^p(T)$, denote the class $[X^q(T) \underset{p}{\rightarrow} Y^q(S)]$ by $[X^p(T) \rightarrow Y^q(S)]$.

Let K^p (T) be the positive cone in the partial ordered linear space $L^p(T, \Sigma_T, \mu)$ i.e.

$$K^p(T) = \{f \in L^p(T, \Sigma_T, \mu) : f \geqslant \theta\} .$$

(*) If $Y^q(S) \equiv L^q(S)$, then inequality (1.1) becomes an equality with $M_q = 1$.

where the ordered relation: «$f \geqslant \theta$» means that $f(t) \geqslant 0$ ($\forall t \in T$ (mod μ)). By $K^p_o(T)$ we shall denote the positive cone in the linear manifold $X^p(T)$: $K^p_o(T) = K^p(T) \cap X^p(T)$. And $K^q(S), K^q_o(S)$ will denote the similar positive cones in $L^q(S, \Sigma_s, \lambda)$ and $Y^q(S)$, respectively. An operator F transforming $X^p(T)$ into $Y^q(S)$ is called positive (and written $F \geqslant \theta$) if $F[K^p_o(T)] \subset K^q_o(S)$.

In this paper we shall investigate the kernel representation of a positive operator F belonging to the class $[X^p(T) \underset{p}{\rightarrow} Y^q(S)]$ ($1 \leqslant p, q < \infty$)—namely investigate the existence of a real valued function $k(s, t)$ defined on $S \times T$ such that

$$[Ff](s) = \int_T k(s, t) f(t) \mu(dt) \; (\forall s \in S \; (\text{mod } \lambda), \; \forall f \in X^p(T)). \tag{1.3}$$

It is well known that if $p = 1$, $X^1(T) = L^1(T, \Sigma_T, \mu)$ and $Y^q(S) = L^q(S, \Sigma_s, \lambda)$ then this problem has been studied under some assumptions for the linear operator F. For example, F is compact from $L^1(T, \Sigma_T, \mu)$ to $L^q(S, \Sigma_s, \lambda)$ (see [4; p. 379]) or is separable bounded from $L^1(T, \Sigma_T, \mu)$ to $L^q(S, \Sigma_s, \lambda)$ ($1 < q$) (see [5; p. 559]) and for $q = 1$, F is weakly compact from $L^1(T, \Sigma_T, \mu)$ to a separable subspace of $L^1(S, \Sigma_s, \lambda)$ (see [5; p. 547]). In the case when $T = S = (0, 1)$, μ and λ are the Lebesgue measures on the number line, $X^p(T) = L^p(0, 1)$, one considered the kernel representation of the form (1.3) for an operator $F \in [L^p(0, 1) \rightarrow Y^q(0, 1)]$, where $Y^q(0, 1) = C(0, 1)$ (*) (see [5; p. 557]) or $Y^q(0, 1) = C^r(0, 1)$ (*) (see) [7; p. 277]). The kernel representation of F belonging to some other classes of linear operators has been also considered in Refs [9], [10], [1] [2], [17].

§ 2. THE KERNEL REPRESENTATION OF A POSITIVE CONTINUOUS LINEAR OPERATOR FROM $L^p(T)$ TO $Y^q(S)$

By $V^q(S, \Sigma_s, \lambda)$ (or $V^q(S)$) we shall denote the B-space of all completely additive set functions on Σ_s of bounded q-variation. The norm of an element $u \in V^q(S, \Sigma_s, \lambda)$ is defined as

$$\|u\|_{V^q(S)} = \underset{e_s \in \Sigma_s}{q\text{-}\mathrm{var}}\, u(e_s) \equiv \sup_{\pi} \left\{ \sum_{i=1}^{n} \frac{|u(e_s^{(i)})|^q}{\lambda(e_s^{(i)})^{q-1}} \right\}^{1/q},$$

where the supremum is taken with respect to all finite families $\pi = \{e_s^{(i)}\}$ of disjoint Σ_s-measurable sets of finite nonzero measure. It is known (see [8]) that if $F \in [L^p(T, \Sigma_T, \mu) \rightarrow L^q(S, \Sigma_s, \lambda)]$ ($1 \leqslant p, q < \infty$), then there exists a real-valued function $K(e_s, t)$ defined on $\Sigma_s \times T$ such that

(*) If Ω is a domain of an Euclidean space R^n, $C^0(\Omega) \equiv C(\Omega)$ is the space of bounded continuous functions on Ω and $C^r(\Omega)$ ($r = 1, 2, \ldots$) is the space of functions possessing all bounded continuous partial dirivatives on Ω until r-th order.

$$Ff = \frac{d}{d\lambda} \int_T K(.,t) f(t) \mu(dt) \; (\forall f \in L^p (T, \Sigma_T, \mu)) \; (*) \tag{2.1}$$

$$\int_{e_T} K(.,t) \mu(dt) \in V^q (S, \Sigma_s, \mu) \text{ for each } e_T \in \Sigma_T ; \mu(e_T) < \infty, \tag{2.2}$$

$$K(e_s,.) \in L^{p'} (T, \Sigma_T, \mu) \text{ for each } e_s \in \Sigma_s \text{ and } \frac{1}{p} + \frac{1}{p'} = 1 \tag{2.3}$$

Hence we have

Theorem 2.1 Suppose that (T, Σ_T, μ), (S, Σ_s, λ) and $Y^q(S)$ satisfy the conditions quoted in the preliminaries and that

$$F \geqslant \theta, \; F \in [L^p (T, \Sigma_T, \mu) \to Y^q (S)] \; (1 \leqslant p, q < \infty)$$

Then there exists a $\lambda \times \mu$ - essentially unique, $\lambda \times \mu$ - measurable real - valued function $k(s,t)$ on $S \times T$, which fulfils the following condition:

$$[Ff](s) = \int_T k(s,t) f(t) \mu(dt) \; (\forall s \in S \; (\text{mod} \lambda), \forall f \in L^p (T)) \tag{2.4}$$

Besides we have

$$k(s,t) \geqslant 0 \; (\forall s \in S \; (\text{mod } \lambda), \forall t \in T \; (\text{mod } \mu)), \tag{2.5}$$

$$k(s,.) \in L^{p'} (T, \Sigma_T, \mu) \; (\forall s \in S \; (\text{mod } \lambda)), \tag{2.6}$$

$$\|F\|_{[L^p(T) \to L^q(S)]} \leqslant q - \operatorname*{var}_{e_s \in \Sigma_s} \{ \| \int k(s,.) \lambda(ds) \|_{L^{p'}(T)} \}. \tag{2.7}$$

Proof. $F \in [L^p(T) \to Y^q(S)]$ and the space $Y^q(S)$ is imbedded in $L^q(S)$, therefore (see (1.1): $F \in [L^p(T) \to Y^q(S)]$ and

$$\|F\|_{[L^p(T) \to L^q(S)]} \leqslant M_q \cdot \|F\|_{[L^p(T) \to Y^q(S)]} \tag{2.8}$$

Hence, there exist a real - valued function $K(e_s, t)$ defined on $\Sigma_s \times T$ and fulfiling conditions (2.1) — (2.3). Let

$$[Uf](e_s) = \int_T K(e_s, t) f(t) \mu(dt) \; (\forall f \in L^p(T), \forall e_s \in \Sigma_s). \tag{2.9}$$

From (2.1) we have

$$[Uf](e_s) = \int_{e_s} [Ff](s) \lambda(ds) \; (\forall f \in L^p(\boldsymbol{T}), \forall e_s : \lambda(e_s) < \infty) \tag{2.10}$$

For a Σ_T–measurable set e_T of finite measure $\chi_{e_T}(.) \in K^p(T)$ and because $F \geqslant \theta$ then $[F\chi_{e_T}](.) \in K^q_\circ(S)$ i.e.

$$[F\chi_{e_T}](s) \geqslant 0 \; (\forall s \in S \; (\text{mod } \lambda), \; \forall e_T : \mu(e_T) < \infty) \tag{2.11}$$

By (2.9) — (2.11), it is clear that

$$\int_{e_T} K(e_S, t) \mu(dt) = \int_{e_S} [F\chi_{e_T}](s) \lambda(ds) \geqslant 0 \; (\forall e_T, e_S : \mu(e_T), \mu(e_S) < \infty) \tag{2.12}$$

(*) The symbol $\frac{d}{d\lambda}$ applied to a completely additive and absolutely continuous set function denotes the integrable point function associated with it by the Radon – Nikodym theorem.

It is known that (T, Σ_T, μ) is a space with σ-finite measure, hence from (2.12) it is easy to see that

$$K(e_S, t) \geqslant 0 \quad (\forall t \in T \ (\text{mod}\ \mu),\ \forall e_S:\ \lambda(e_S) < \infty) \tag{2.13}$$

Now we prove that for all $t \in T$ (mod μ), $K(., t)$ is a completely additive set function on Σ_S, i.e.

$$K\left(\bigcup_{n=1}^{\infty} e_S^{(n)}, t\right) = \sum_{n=1}^{\infty} K\left(e_S^{(n)}, t\right) \ (\forall t \in T \ (\text{mod}\ \mu)), \tag{2.14}$$

where $\{e_S^{(n)}\}$ (n = 1, 2,...) is a family of disjoint Σ_S - measurable sets. Since (S, Σ_S, λ) is a space with σ-finite measure, so proving (2.14) we can suppose that the sets $e_S^{(n)}$ have finite measures. By (2.2), (2.13) we have

$$\int_{e_T} K\left(\bigcup_{n=1}^{\infty} e_S^{(n)}, t\right) \mu(dt) = \sum_{n=1}^{\infty} \int_{e_T} K\left(e_S^{(n)}, t\right) \mu(dt) = \int_{e_T} \sum_{n=1}^{\infty} K\left(e_S^{(n)}, t\right) \mu(dt)$$

$(\forall e_T:\ \mu(e_T) < \infty)$.

Therefore (2.14) is proved.

Let $\bar{e}_S$ be a $\Sigma_{\bar{S}}$ measurable set of zero measure, by (2.9) (2.10) we conclude:

$$\int_{e_T} K(\bar{e}_S, t)\, \mu(dt) = \int_{\bar{e}_S} [F \chi e_T](s)\, \lambda(ds) = 0 \ (\forall e_T: \mu(e_T) < \infty).$$

Then it is clear that

$$K(\bar{e}_S, t) = 0 \quad (\forall t \in T \ (\text{mod}\ \mu),\ \forall \bar{e}_S:\ \lambda(\bar{e}_S) = 0) \tag{2.15}$$

From (2.3) it is easy to deduce that the completely additive set function $K(., t)$ $(\forall t \in T$ (mod $\mu))$ is finite on Σ_S, therefore by (2.15) $K(., t)$ is absolutely continuous with respect to the measure λ (see [5: p. 147]).

Hence by the Radon — Nikodym theorem, it is clear that for each $t \in T$ (mod μ) there exists a function $k(., t) \in L^1(S)$ such that

$$K(e_S, t) = \int_{e_S} k(s, t)\, \lambda(ds) \ (\forall e_S \in \Sigma_S,\ \forall t \in T \ (\text{mod}\ \mu)). \tag{2.16}$$

By (2.13), (2.16) we have

$$\int_{e_S} k(s, t)\, \lambda(ds) \geqslant 0 \quad (\forall t \in T \ (\text{mod}\ \mu),\ \forall e_S:\ \lambda(e_S) < \infty) \tag{2.17}$$

Because $(S, \Sigma_S; \lambda)$ in a space with σ-finite measure, then from (2.17) we easily obtain (2.5).

For all $f \in L^p(T)$ and $e_S \in \Sigma_S$, by (2.5) (2.16), (2.3) we have

$$\int_T \left\{ \int_{e_S} k(s, t)\, f(t)\, \lambda(ds) \right\} \mu(dt) = \int_T f(t)\, K(e_S, t)\, \mu(dt) < \infty$$

Therefore (see [12; p. 299]) the function $k(s,t)f(t)$ is $\lambda \times \mu$-integrable on $e_S \times T$. Hence by (2.16), (2.9), (2.10) and the Fubini theorem, it is easy to deduce that.

$$\int_{e_S} \{ \int_T k(s,t) f(t) \mu(dt) \} \lambda(ds) = \int_T K(e_S, t) f(t) \mu(dt) =$$

$$= \int_{e_S} [Ff](s) \lambda(ds), \ \forall e_S : \lambda(e_S) < \infty, \ \forall f \in L^p(T)).$$

Then (2.4) is evident.

Since $F \in [L^p(T) \to L^q(S)]$, so

$$|[Ff](s)| < \infty \ (\forall s \in S \ (\mathrm{mod}\ \lambda), \ \forall f \in L^p(T)) \tag{2.18}$$

By (2.4), (2.18) it deduces that for all $f \in L^p(T)$ and $s \in S \,(\mathrm{mod}\ \lambda)$: $k(s,.) f(.) \in L^1(T)$. Therefore (see [5; p. 380]) we have (2.6). It is known (see [8; p. 187]) that

$$\|F\|_{[L^p(T) \to L^q(S)]} \leqslant q\text{-}\underset{e_S \in \Sigma_S}{\mathrm{var}} \{ \| K(e_S,.)\|_{L^{p'}(T)} \}$$

Hence by (2.16) we obtain (2.7).

In order to prove the rest of the theorem, first we regard that the function $k(s,t) f(t)$ is $\lambda \times \mu$ - integrable on $e_S \times T$ $(\forall e_S \in \Sigma_S, \ \forall f \in L^p(T))$. Therefore $k(s,t) f(t)$ is $\lambda \times \mu$ - measurable on $S \times T$ $(\forall f \in L^p(T))$. Hence $k(s,t)$ is also $\lambda \times \mu$ - measurable on $S \times T$. Now we consider the uniqueness of the kernel $k(s,t)$. Suppose that there exists an other $\lambda \times \mu$ - measurable function $\overline{k}(s,t)$ on $S \times T$ such that it fulfils the condition of the form (2.4):

$$[Ff](s) = \int_T \overline{k}(s,t) f(t) \mu(dt) \ (\forall f \in L^p(T), \ \forall s \in S \ (\mathrm{mod}\lambda)). \tag{2.19}$$

It is known that $\chi_{e_T}(.) \in L^p(T)$ for all Σ_T - measurable sets e_T of finite measure. Then by (2.4), (2.19) we have:

$$\int_{e_T} k(s,t) \mu(dt) = \int_{e_T} \overline{k}(s,t) \mu(dt) \ (\forall s \in S \ (\mathrm{mod}\lambda). \ \forall e_T : \mu(e_T) < \infty).$$

Therefore $k(s,t) = \overline{k}(s,t)$ $(\forall s \in S)$ $(\mathrm{mod}\lambda)$, $\forall t \in T$ $(\mathrm{mod}\mu)$). Q.E.D.

§ 3. THE KERNEL REPRESENTATION OF SOME POSITIVE LINEAR OPERATORS FROM $X^p(T)$ TO $Y^q(S)$.

Now we consider the kernel representation of a positive linear operator defined on a certatin dense in $L^p(T)$ linear manifold $X^p(T)$.

Theorem 3.1. Suppose that (T, Σ_T, μ) (S, Σ_S, λ) and $X^p(T)$, $Y^q(S)$ satisfy the conditions quoted in the preliminaries and that $F \geqslant \theta$, $F \in [X^p(T) \underset{p}{\to} Y^q(S)]$, where $1 \leqslant p, q < \infty$ is a B-space. Let the positive cone $K_0^p(T)$ of $X^p(T)$ be dense in the positive cone $K^p(T)$ of $L^p(T)$.

Then there exists a $\lambda \times \mu$ - essentially unique, $\lambda \times \mu$ - measurable real - valued function $k(s,t)$ on $S \times T$, which fulfils the following conditions

$$[Ff](s) = \int_T k(s,t) f(t) \mu(dt) \ (\forall s \in S \ (\mathrm{mod}\, \lambda). \ \forall f \in X^p(T)). \tag{3.1}$$

$$\widetilde{F} \in [L^p(T) \to Y^q(S)], \text{ where } \widetilde{F} f \equiv \int_T k(.,t) f(t) \mu(dt), \tag{3.2}$$

Besides, we have (2.5), (2.6) and

$$\| F \|_{[X^p(T) \underset{p}{\to} L^q(S)]} = \| \widetilde{F} \|_{[L^p(T) \to L^q(S)]} \leqslant q \text{-} \underset{e_S \in \Sigma_S}{\mathrm{var}} \{ \| \int_{e_S} k(s,.) \lambda(ds) \|_{L^{p'}(T)} \} \tag{3.3}$$

Proof. Because $F \in [X^p(T) \underset{p}{\to} Y^q(S)]$, $Y^q(S)$ is a B - space and $X^p(T)$ is dense in $L^p(T)$, then (see [11; p. 124]) there is uniquely a defined on $L^p(T)$ extension $\widetilde{F}$ of F such that $\widetilde{F} \in [L^p(T) \to Y^q(S)]$ and

$$\widetilde{F} f = Ff \quad (\forall f \in X^p(T)), \tag{3.4}$$

$$\| \widetilde{F} \|_{[L^p(T) \to Y^q(S)]} = \| F \|_{[X^p(T) \underset{p}{\to} Y^q(S)]}. \tag{3.5}$$

Let $f+$ be a function belonging to $K^p(T)$. Since $\underset{o}{K}{}^p(T)$ is dense in $L^p(T)$, so there exists a sequence $\{ f_n^+ \} \subset \underset{o}{K}{}^p(T)$ such that

$$\| f^+ - f_n^+ \|_{L^p(T)} \to 0 \quad (n \to \infty). \tag{3.6}$$

We knew that $\widetilde{F} \in [L^p(T) \to Y^q(S)]$ and $\{ f_n^+ \} \subset \underset{o}{K}{}^p(T) \subset X^p(T)$, hence by (3.4) — (3.6). it follows

$$\| \widetilde{F} f^+ - F f_n^+ \|_{L^q(S)} \leqslant M_q \| \widetilde{F} f^+ - \widetilde{F} f_n^+ \|_{Y^q(S)} \leqslant M_q \| F \|_{[X^p(T) \underset{p}{\to} Y^q(S)]}$$
$$\| f^+ - f_n^+ \|_{L^p(T)} \to 0, \quad (n \to \infty). \tag{3.7}$$

Because $F \geqslant \theta$ (*i.e.* $F f_n^+ \in \underset{o}{K}{}^q(S) \subset K^q(S)$ and the positive cone $K^q(S)$ of the B-space $L^q(S)$ is closed in $L^q(S)$, then by (3.7) it is easy to see that $\widetilde{F} f^+ \in K^q(S)$. Since $\widetilde{F} f^+ \in Y^q(S)$, so $\widetilde{F} f^+ \in K^q(S) \cap Y^q(S) = \underset{o}{K}{}^q(S)$ $(\forall f^+ \in K^p(T))$, i. e. $\widetilde{F} \geqslant \theta$. Therefore, applying Theorem (2.1) for $\widetilde{F} \in [L^p(T) \to Y^q(S)]$ we deduce that there is a $\lambda \times \mu$ - measurable real - valued function $k(s,t)$ on $S \times T$, which fulfils (2.5), (2.6) and the following conditions

$$[F f](s) = \int_T k(s,t) f(t) \mu(dt). \ (\forall s \in S \ (\mathrm{mod}\, \lambda), \forall f \in L^p(T)), \tag{3.8}$$

$$\| \widetilde{F} \|_{[L^p(T) \to L^q(S)]} \leqslant q \text{-} \underset{e_s \in \Sigma_s}{\mathrm{var}} \{ \| \int_{e_s} k(s,.) \lambda(ds) \|_{L^{p'}(T)} \}; \tag{3.9}$$

By (3.8), (3.4) it follows (3.1). And by (3.8) we have (3.2).

It is known that $F \in [X^p(T) \underset{p}{\to} Y^q(S)]$, $\widetilde{F} \in [L^p(T) \to Y^q(S)]$ and the space $Y^q(S)$, is imbedded in $L^q(S)$ then $F \in [X^p(T) \underset{p}{\to} L^q(S)]$, $\widetilde{F} \in [L^p(T) \to L^q(S)]$, It is clear (see (3.4)) that the operator $F \in [L^p(T) \to L^q(S)]$ is a continuous linear extension of $F \in [X^p(T) \underset{p}{\to} L^q(S)]$, hence (see [11 ; p. 124]).

$\| \widetilde{F} \|_{[L^p(T) \to L^q(S)]} = \| F \|_{[X^p(T) \underset{p}{\to} L^q(S)]}$. Therefore by (3.9) we obtain (3.3).

In order to prove the unicity of the kernel $k(s, t)$ fulfiling (3.1), (3.2), suppose that there exists another $\lambda \times \mu$-measurable function $\overline{k}(s, t)$ on $S \times T$ such that

$$[Ff](s) = \int_T \overline{k}(s, t) f(t) \mu(dt) \quad (\forall s \in S(\mathrm{mod}\ \lambda), \quad \forall f \in X^p(T)), \tag{3.10}$$

$$\overline{F} \in [L^p(T) \to Y^q(S)], \text{ where } \overline{F} f = \int_T \overline{k}(\cdot, t) f(t) \mu(dt) \tag{3.11}$$

By (3.10), (3.11) it is evident that $Ff = \overline{F} f (\forall f \in X^p(T))$, i. e. $\overline{F}$ is also a defined on $L^p(T)$ continuous linear extension of F. Since the such extension is unique, so $\overline{F} = \widetilde{F}$. Therefore (see (3.8), (3.11)):

$$\int_T k(s, t) f(t) \mu(dt) = \int_T \overline{k}(s, t) f(t) \mu(dt) \ (\forall s \in S(\mathrm{mod}\ \lambda), \ \forall f \in L^p(T)) \tag{3.12}$$

Let e_T be any Σ_T - measurable set of finite measure, by (3.12) it follows

$$\int_{e_T} k(s, t) \mu(dt) = \int_{e_T} \overline{k}(s, t) \mu(dt) \ (\forall s \in S(\mathrm{mod}\ \lambda), \ \forall e_T : \mu(e_T) < \infty).$$

Hence $k(s, t) = \overline{k}(s, t) (\forall s \in S(\mathrm{mod}\ \lambda), (\forall t \in T(\mathrm{mod}\ \mu))$. Q.E.D.

Now we consider the case that μ is the Lebesgue measure on R^n and that $T = \Omega_T$, where Ω_T is a bounded domain of R^n. Let

$$w_\varepsilon(t, \zeta) = \begin{cases} C_\varepsilon^{-1} \cdot \exp\left\{ \dfrac{|t-\zeta|^2}{|t-\zeta|^2 - \varepsilon^2} \right\}, & \text{if } |t-\zeta| < \varepsilon, \\ 0 & , \text{if } |t-\zeta| \geqslant \varepsilon, \end{cases} \tag{3.13}$$

where $t = (t_1, \ldots, t_n) \in R^n$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in R^n$, $|t - \zeta| = \sqrt{\sum_{i=1}^n (t_i - \zeta_i)^2}$, -

$C_\varepsilon = \varepsilon^n \int_{|\zeta|<1} \exp\left\{ \dfrac{|\zeta|^2}{|\zeta|^2 - 1} \right\} d\zeta$ and ε is a positive constant. Let $f_\varepsilon(t)$ be the average function for $f \in L^p(\Omega_T)$ on the sphere of radius ε with center t, i. e.

$$f_\varepsilon(t) = \int_{R^n} \omega_\varepsilon(t, \zeta) \widetilde{f}(\zeta) d\zeta, \text{ where } \widetilde{f}(t) = \begin{cases} f(t), \text{ if} : t \in \Omega_T, \\ 0, \text{ if} : t \in R^n \setminus \Omega_T. \end{cases} \tag{3.14}$$

It is known (see [16; p. 19]) that

$$\|f_\varepsilon - f\|_{L^p(\Omega_T)} \to 0 \quad (\varepsilon \to 0) \quad (\forall f \in L^p(\Omega_T)), \tag{3.15}$$

$$f_\varepsilon \in C^\infty(\Omega_T) \quad (\forall f \in L^p(\Omega_T)), \tag{3.16}$$

where $C^\infty(\Omega_T)$ is the space of all infinitely differentiable functions on Ω_T. Suppose that $X^p(\Omega_T)$ is a linear manifold in $L^p(\Omega_T)$ such that $X^p(\Omega_T) \supset C^\infty(\Omega_T)$. By (3.15), (3.16) it is clear that $X^p(\Omega_T)$ is dense in $L^p(\Omega_T)$. Besides, for each $(f \in K^p(\Omega_T)$ by (3.13), (3.14), (3.16) we have $f_\varepsilon \in K^p(\Omega_T) \cap X^p(\Omega_T) = \overset{\circ}{T}{}^p(\Omega_T)$ Hence by (3.15) it follows that $\overset{\circ}{K}{}^p(\Omega_T)$ is also dense in $K^p(\Omega_T)$. Therefore, from Theorem (3.1) it is easy to deduce the following corollary.

Corollary 3.2: Under the assumptions of Theorem (3.1) for (S, Σ_s, λ) and $Y^q(S)$, let μ be the Lebesgue measure on R^n, Ω_T be a limited domain of R^n. Suppose that $F \in [X^p(\Omega_T) \underset{p}{\rightarrow} Y^q(S)]$, $F \geqslant \theta$ where $1 \leqslant p,\ q < \infty$, $X^p(\Omega_T)$ is a linear manifold in $L^p(\Omega_T)$ such that $X^p(\Omega_T) \supset C^\infty(\Omega_T)$

Then there exists a $\lambda \times \mu$-essentially unique, $\lambda \times \mu$-measurable real-valued function $k(s, t)$ on $S \times \Omega_T$, which fulfils the following conditions

$$[Ff](s) = \int_{\Omega_T} k(s, t)\, f(t)\, dt \ (\forall s \in S \ (\text{mod}\ \lambda),\ \forall f \in X^p(\Omega_T), \tag{3.17}$$

$$\widetilde{F} \in [L^p(\Omega_T) \rightarrow Y^q(S)], \text{ where } \widetilde{F} f = \int_{\Omega_T} k(., t)\, f(t)\, dt. \tag{3.18}$$

Besides, we have

$$k(s, t) \geqslant 0 \ (\forall s \in S \ (\text{mod}\ \lambda),\ \forall t \in \Omega_T \ (\text{mod}\ \mu)), \tag{3.19}$$

$$k(s,.) \in L^{p'}(\Omega_T) \ (\forall s \in S \ (\text{mod}\ \lambda)), \tag{3.20}$$

$$\|F\|_{[X^p(\Omega_T) \underset{p}{\rightarrow} L^p(S)]} = \|\widetilde{F}\|_{[L^p(\Omega_T) \rightarrow L^p(S)]} \leqslant q\text{-}\underset{es \in \Sigma_S}{\text{var}} \left\{ \left\| \int_{es} k(s,.)\, \lambda(ds) \right\|_{L^{p'}(\Omega_T)} \right\} \tag{3.21}$$

Remark 3.3. It is known that the space $C^r(\Omega_T)$ $(r = 0, 1, 2,\ldots)$, the Sobolev space $W^l_p(\Omega_T)$, the Besov space $B^l_p(\Omega_T)$ $(1 < p < \infty, 0 < l < \infty)$ are linear manifolds in $L^p(\Omega_T)$ and contain $C^\infty(\Omega_T)$ (see [13; pp 79—81]) Besides, if Ω_S is a bounded domain of R^m with the boundary Γ_S of the class C^∞ (*), then the space $C^p(\Omega_S)$ $(p = 0, 1, 2\ldots)$, the Sobolev space $W^h_p(\Omega_S)$, the Besov space $B^l_p(\Omega_S)$ and the space of Bessel potentials $H^h_p(\Omega_S)$ $(1 < q < \infty,\ 0 \leqslant h < \infty)$ are B-spaces imbedded in $L^p(\Omega_S)$. Therefore, as examples of applying Corollary 3.2, we can show the existence of the kernel k, for which a positive operator F belonging to $[X^p(\Omega_T) \underset{p}{\rightarrow} Y^q(\Omega_S)]$. is represented in the form (3.17), where $X^p(\Omega_T)$ is $C^\infty(\Omega_T)$ (or $C^r(\Omega_T)$, $W^l_p(\Omega_T)$, $B^l_p(\Omega_T)$) and $Y^q(\Omega_S)$ is $C^p(\Omega_S)$ (or $W^h_q(\Omega_S)$, $B^h_q(\Omega_S)$, $H^h_q(\Omega_S)$). These spaces play

(*) i.e. Γ_S is a certain $(m-1)$ - dimensional infinitely differentiable orientable manifold, with respect to which Ω_S is locally in one side.

an important role in the theory of partial differential equations. Besides, with the aid of an overaging operator (see e.g. [3 ; pp. 39—41]), we can approximately determine above kernel k.

Then it may approximate some linear positive operators by integral operators and some linear equations with positive operators by integral equations.

As is well known (see [6], [14], [15]) various probability models have been constructed for estimating the values of an integral operator and for solving some integral equations of second type. Hence, with use of the results in this paper, we may estimate the values of some linear positive operators and solve some classes of linear equations with positive operators by the Monte—Carlo method.

The author wishes to express his gratitude to Prof. dr B. Bojarski and Doc. dr L. Kubik for many valuable discussions and remarks.

Received January 15th, 1978.

REFERENCES

[1] Bartle R.G., *On compactness in functional analysis*, Trans Amer. Math. Soc., 79 (1955), 35 — 37.

[2] Bartle R.G., Dunford N., Schwartz J., *Weak compactness and vector measures*. Can. J. Math., 7 (1955), 289—305.

[3] Berezanskij Ju. M. *Eigenfunction expansions of self adjoint operators*, Kiev, 1965 (in Russian)

[4] Dunford N., Pettis B.J., *Linear operations on summable functions*, Trans. Amer. Math. Soc., 47 (1940), 323—392.

[5] Dunford N., Schwartz J.T., *Linear operators — Part I*, Moscow 1962 (in Russian).

[6] Ermakov S.M. *The Monte — Carlo method and related problems* Moscow, 1971 (in Russian)

[7] Fullerton R.E. *Linear operators with range in a space of differentiable functions*. Duke Math. J., 13 (1946), 269—280.

[8] Fullerton R.E., *An inequality for linear operators between L^p spaces*, Proc. Amer. Math. Soc., 6 (1955), 186 — 190.

[9] Kantorovitch L., Vulich B., *Sur la représentation des opérations linéaires*, Compositio Math., 5 (1938), 119—165.

[10] Kantorovitch L., *Linear operations in semi — ordered space I*, Mat. Sib, 7(49), 1940, 209—284.

[11] Kantorovitch L.V., Akilov G.P. *Functional analysis in normed spaces*, Moscow (in Russian) 1959.

[12] Kolmogorov A.N., Fomin S.V.Elements of functions theory and functional analysis Moscow 1972. (in Russian)

[13] Krejn S.G. *Functional analysis*. Moscow 1972. (in Russian)

[14] Nguyen Quy Hy, *On the solution of an integral equation of second type by the Monte—Carlo method*, Ann. Polon. Math., 30 (1974), 175 — 190.

[15] Nguyen Quy Hy, *On the probability model for solving integral equations of second type*, Ann. Polon. Math., 31 (1975), 91—114.

[16] Sobolev S.L. *Some applications of functional analysis in mathematical physics*, Leningrad, 1950 (in Russian)

[17] Wada J., *Weakly compact linear operators on function spaces*, Osaka Math. J., 13 (1961), 169 — 183.