ACTA MATHEMATICA VIETNAMICA
TOM 5, N° 1 (1980)

INFINITE MATRICES AND TENSORIAL TRANSFORMATIONS

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1. Introduction: In recent years there has emerged a great deal of interest in the study of infinite matrix transformations from sequence spaces to sequence spaces. Various infinite matrices have been constructed to transform one sequence space into another, for instance one may refer to [6] and [7] and numerous references cited therein. We consider here several classes of these transformations and initiate the attempt of exploring the basic inner structure of these collections after equipping them with suitable natural locally convex topologies. One of the important outcome of our investigations is the representation theorem of these infinite matrix transformations, carried on with the help of weak topology. The last section is concerned with the precise form of tensorial transformations acting on a given collection of infinite matrices into another; for such classical ideas connected with the summability, one may refer to Cooke [1].

2. Notations and Terminology: An *infinite matrix* shall be denoted by $a = (a_{mn})$:

$$a = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} & \cdots \\ a_{10} & a_{11} & \cdots & & a_{1n} & \cdots \\ a_{20} & a_{21} & & & & \\ \vdots & \vdots & & & & \\ a_{m0} & a_{m1} & \cdots & & a_{mn} & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \\ \cdots & \cdots & \cdots & & \cdots & \end{pmatrix}$$

where a'_{mn} s belong to the field $\mathbf{K}$ of scalars. Denote by $\mathbf{N}$ the set of all non-negative integers and by e^{mn} $(m, n \in \mathbf{N})$ an infinite matrix whose element at the cross of m-th row and n-th column is one and other elements are zeros. Let Ω

be the family of all infinite matrices endowed with usual operations of pointwise addition and scalar multiplication. Thus Ω is a vector space over $\mathbf{K}$. By a *matrix space* Δ we mean any subspace of Ω. The matrix space generated by $\{e^{mn} : m,n \in \mathbf{N}\}$ shall be denoted by Φ. If $N \in \mathbf{N}$ and $a \in \Omega$, we define

$$a^N = \sum_{0 < m+n < N} \sum a_{mn}\, e^{mn}$$

and call it as the N-th *plane section* of the matrix a.

For a matrix space Δ, we define $\Delta^{\#}$ by

$$\Delta^{\#} = \{ b = (b_{mn}) : b \in \Omega \text{ with } \Sigma\, \Sigma\, |a_{mn}\, b_{mn}| < \infty,\ \forall\, a \in \Delta \}$$

where

$$\Sigma\, \Sigma\, a_{mn}\, b_{mn} = \lim_{N \to \infty} \sum_{0 < m+n < N} \sum a_{mn}\, b_{mn},$$

and term it as the *K-dual of* Δ. Clearly $\Delta^{\#}$ is a vector space over $\mathbf{K}$ and contains Φ. In the rest of this paper we assume that each matrix space Δ contains Φ. Under this assumption, Δ and $\Delta^{\#}$ form a dual system which we express as $\langle \Delta, \Delta^{\#} \rangle$. Hence, we may talk about the weak topology $\sigma(\Delta, \Delta^{\#})$, the Mackey topology $\tau(\Delta, \Delta^{\#})$, the strong topology $\beta(\Delta, \Delta^{\#})$ etc. Observe that $\sigma(\Delta, \Delta^{\#})$ is generated by the family $\{ p_b : b \in \Delta^{\#} \}$ of semi-norms on Δ, where

$$p_b(a) = |\, \Sigma\, \Sigma\, a_{mn}\, b_{mn} \,| .$$

We also have a natural locally convex topology on Δ called the K-th *normal topology*. This is denoted by $\eta(\Delta, \Delta^{\#})$ and is generated by the family $\{ q_b : b \in \Delta^{\#} \}$ of semi-norms on Δ, where

$$q_b(a) = \Sigma\, \Sigma\, |\, a_{mn}\, b_{mn} \,| .$$

Let us note that all the topologies on Δ discussed above are Hausdorff. Similarly, we may talk about such topologies on $\Delta^{\#}$ by interchanging the roles of Δ and $\Delta^{\#}$.

3. The Role of e^{mn} : We begin from

Proposition 3.1 : For any $a \in \Delta$,

$$\sigma(\Delta, \Delta^{\#})\text{-}\lim_{N \to \infty} a^N = a;$$

moreover, the transformations $e^{mn} : \Delta \to \mathbf{K}$, given by $e^{mn}(a) = a_{mn}$ are continuous on $(\Delta, \sigma(\Delta, \Delta^{\#}))$ for all $m, n \in \mathbf{N}$.

Proof : For any $b \in \Delta^{\#}$,

$$p_b(a - a^N) = p_b((a_{mn} : m+n > N))$$

where $(a_{mn} : m+n > N)$ is an infinite matrix whose all coefficients a_{mn} are zeros provided $0 \leq m+n \leq N$. Thus

$$p_b(a - a^N) = \Big| \sum_{m+n > N} \sum a_{mn}\, b_{mn} \Big| \to 0 \text{ as } N \to \infty.$$

Thus we get the first part. For the second part, let us observe that for $a \in \Delta$,

$$| e^{mn}(a) | = | a_{mn} | = p_{e^{mn}}(a).$$

Corollary 3.2: The pair $\{ e^{mn} ; e^{mn} \}$ is a Schauder base for $(\Delta, \sigma(\Delta, \Delta^{\#}))$.

Remark: In Proposition 3.1 (and hence also in Corollary 3.2) the topology $\sigma(\Delta, \Delta^{\#})$ can be replaced by $\eta(\Delta, \Delta^{\#})$.

K-normal and K-perfect matrix spaces: A matrix space is called *K-normal* provided $a=(a_{mn}) \in \Delta$ whenever $| a_{mn} | \leq | b_{mn} |$ for $m+n \geq 0$, for some $b = (b_{mn}) \in \Delta$. Clearly $\Delta^{\#}$ is K-normal for any matrix space Δ. A matrix space Δ is said to be *K-perfect*, if $\Delta = \Delta^{\#\#} = (\Delta^{\#})^{\#}$; observe that $\Delta \subset \Delta^{\#\#}$ is always true. Let us introduce the following matrix spaces:

$$C = \{ a : a \in \Omega, \lim_{m+n \to \infty} a_{mn} \text{ exists} \} ;$$

$$C_{oo} = \{ a : a \in \Omega, a_{mn} \to 0 \text{ as } m + n \to \infty \}$$

$$l_{pp} = \{ a : a \in \Omega, \Sigma \Sigma | a_{mn} |^p < \infty \} , 0 < p < \infty ;$$

$$l_{\infty\infty} = \{ a : a \in \Omega, \sup_{m, n \geqslant 0} | a_{mn} | < \infty \}.$$

For $1 \leqslant p < \infty$, equip l_{pp} with the norm

$$\| a \|_{pp} = \{ \Sigma \Sigma | a_{mn} |^p \}^{1/p}$$

and for $p = \infty$, equip $l_{\infty\infty}$ with the norm $\| a \|_{\infty\infty}$, where

$$\| a \|_{\infty\infty} = \sup_{m, n \geqslant 0} | a_{mn} |.$$

The spaces C and C_{oo} are equipped with the norms inherited from $\| . \|_{\infty\infty}$. Unless otherwise stated, the matrix spaces discussed above, are endowed with these norms with respect to which they are complete. It is not difficult to verify that

$$C_{oo}^{\#} = C^{\#} = l_{11} ; \quad l_{11}^{\#} = l_{\infty\infty}, \; l_{\infty\infty}^{\#} = l_{11} ;$$

$$l_{pp}^{\#} = l_{qq} \left(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \right).$$

Thus

$$l_{pp} (1 \leqslant p \leqslant \infty) \text{ is } K\text{-perfect.}$$

In the sequel we also need the following two spaces

$$\delta = \{ a : a \in \Omega, | a_{mn} |^{\frac{1}{m+n}} \to 0 \text{ as } m + n \to \infty \} ;$$

$$d = \{ a : a \in \Omega, \lim_{m+n \to \infty} \sup | a_{mn} |^{\frac{1}{m+n}} < \infty \},$$

and endow these spaces with the total paranorm topology given by $\| a \| = \sup \{ | a_{oo} | ; | a_{mn} |^{\frac{1}{m+n}}, m + n > 0 \}$ where $a \in \delta$ or d. The space δ can be then regarded as the space of entire functions of two complex variables [2] equipped with the topology of uniform convergence on compact sets in $\mathbf{C} \times \mathbf{C}$, **where** $\mathbf{C}$ is the complex plane (cf. [3]). These spaces are known to be Fréchet spaces.

Proposition 3.3: We have $\delta^{\#}=d$ and $d^{\#}=\delta$. Thus δ and d are K-perfect.

Proof: We prove only $\delta^{\#}=d$; the proof of $d^{\#}=\delta$ is similar. Now observe that $d \subset \delta^{\#}$ is obvious. For $\delta^{\#} \subset d$, we follow ([2] Lemma 2.1) with minor modifications. Indeed, let $a \in \delta^{\#}$ and $a \notin d$. For each integer $i \geqslant 1$ there exist sequences $\{m_i\}$ and $\{n_i\}$ (at least one of which tends to infinity with i) such that

$$|a_{m_i n_i}| > i^{2(m_i+n_i)}.$$

Define the matrix b by

$$b_{mn} = \begin{cases} i^{-m_i-n_i}; & m = m_i,\ n = n_i, \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Then $b \in \delta$. However

$$\Sigma\ \Sigma\ |a_{mn} b_{mn}| = \infty,$$

and so $a \notin \delta^{\#}$, a contradiction. This proves the result.

The matrix e^{mn} *and associated operators*: The importance of the matrices e^{mn}'s does not lie only in representing the elements of a matrix space Δ in which they are present; but is also exhibited in obtaining the information concerning the structures of Δ and $\Delta^{\#}$. Let us recall that a sequence $\{x_n\}$ of elements in a locally convex space $(X, \mathcal{F})$ is called *regular* (resp. *bounded*) if for some continuous semi-norm p (resp. for every continuous semi-norm p) we have a constant $M = M(p) > 0$ with $p(x_n) \geqslant M$ (resp. $p(x_n) \leqslant M$) for all n. We prove

Proposition 3.4: (i) $\{e^{mn}\}$ is $\sigma(\Delta, \Delta^{\#})$-bounded if and only if $\Delta^{\#} \subset l_{\infty\infty}$; (ii) $\Delta \subset l_{\infty\infty}$ if and only if $\{e^{mn}\}$ is $\sigma(\Delta^{\#}, \Delta)$-bounded; and (iii) $\{e^{mn}\}$ is $\sigma(\Delta, \Delta^{\#})$-regular if and only if $l_{\infty\infty} \subset \Delta^{\#}$.

Proof: (i) and (ii) are straightforward and their proofs are, therefore, omitted. To prove (iii) assume that $\{e^{mn}\}$ is regular. Let $y \in l_{\infty\infty}$ and $x \in \Delta$ be arbitrary. Now there exist $z = \{z_{ij}\} \in \Delta^{\#}$ and $\alpha > 0$ with $p_z(e^{mn}) \geqslant \alpha$. Therefore

$$\Sigma\ \Sigma\ |y_{ji}\, x_{ji}| \leqslant \frac{M}{\alpha}\ \Sigma\ \Sigma\ |x_{ji}\ z_{ji}| < \infty$$

and so $y \in \Delta^{\#}$. The other part is obvious.

Now consider the dual system $\langle \Delta, \Delta^{\#} \rangle$. For any integer $N \geqslant 0$, define $S_N : \Delta \to \Delta$ by

$$S_N(x) = \sum_{0 \leqslant m+n \leqslant N} \sum x_{mn}\, e^{mn} = x^N.$$

Clearly each S_N is an $\eta(\Delta, \Delta^{\#}) - \eta(\Delta, \Delta^{\#})$ or a $\sigma(\Delta, \Delta^{\#}) - \sigma(\Delta, \Delta^{\#})$ continuous linear operator. Indeed,

$$p_y(S_N(x)) = p_y N(x); \quad q_y(S_N(x)) = q_y N(x),$$

where $y \in \Delta^{\#}$ and $x \in \Delta$. It is natural to investigate the equicontinuity of $\{S_N : N \geqslant 0\}$. In this direction we have

Theorem 3.5: *The sequence* $\{S_N : N \geqslant 0\}$ *is* $\sigma(\Delta; \Delta^{\#})$*-equicontinuous on* Δ *if and only if* $\Delta^{\#} = \Phi$.

Proof. If $\Delta^{\#} = \Phi$, the set $\{S_N : N \geqslant 0\}$ is obviously $\sigma(\Delta, \Delta^{\#})$-equicontinuous on Δ. For converse, consider $y \in \Delta^{\#}$.

Then the sequence $\{y \circ S_N : N \geqslant 0\}$ of functionals on Δ is $\sigma(\Delta, \Delta^{\#})$—equicontinuous. Therefore $\{y \circ S_N : N \geqslant 0\}$ is finite dimensional (cf. Proposition 7.9, p. 24 [4] or [5], p. 161). Hence there exist integers $k_1, k_2, \ldots, k_p$ with $0 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_p$ such that $\{y \circ S_{k_1}, y \circ S_{k_2}, \ldots, y \circ S_{k_p}\}$ is linearly independent and $\{y \circ S_N : N \geqslant 0\} = \mathrm{sp}\{y \circ S_{k_1}, \ldots, y \circ S_{k_p}\}$. Thus for $x \in \Delta$,

$$(y \circ S_N)(x) = \sum_{i=1}^{p} \alpha_i^N (y \circ S_{k_i})(x)$$

or, $y(x^{(N)}) = \sum_{i=1}^{p} \alpha_i^N y(x^{k_i})$.

Hence, on replacing x by e^{mn}'s for $m+n > k_p$, we find $y_{mn} = 0$, for $m+n > k_p$. Thus $y \in \Phi$ and the result follows.

4. Tensorial Transformations: In this section we consider transformations resulting from a tensor of order four, which relate various matrix spaces. Indeed, if $g = \delta_{mn}^{pq}$ is a tensor of order four having values in the field of scalars for fixed pair of integers p, q and m, n, we assume that its multiplication with any preassigned matrix $b = (b_{pq})$, is defined for all indices $m, n \geqslant 0$, namely

$$g \cdot b = \sum_{p+q} \sum_{\geqslant 0} \delta_{mn}^{pq} b_{pq} = a_{mn}, \text{ say} \tag{4.1}$$

is well defined for all $m, n \geqslant 0$. In the following result we impose conditions on the tensor g so that it becomes a tensorial transformation from the matrix space δ to the matrix space C.

Theorem 4.2: *Suppose (4.1) is true for each* $b \in \delta$. *Then* $a = (a_{mn}) \in C$ *if and only if there exists a constant* $M > 0$ *such that*

$$|\delta_{mn}^{oo}|, \; |\delta_{mn}^{pq}|^{\frac{1}{p+q}} \leqslant M, \text{ for all } m, n; \, p, q \in \mathbf{N}, \tag{4.3}$$

and

$$\lim_{m+n\to\infty} \delta_{mn}^{pq} = d_{pq} \textit{ exists for every } p, q \geqslant 0 \qquad (4.\ 4)$$

Proof : The proof of the sufficiency part is straightforward and is therefore omitted.

For converse, let $a \in C$ where $a = (a_{mn})$ is given by (4.1). For $b \in \delta$, define the matrix $f = (f_{mn})$ of functionals by

$$f_{mn}(b) = a_{mn} = \sum_{p+q \geqslant 0} \sum \delta_{mn}^{pq} b_{pq}.$$

Since the set $\{|\delta_{mn}^{oo}, |\delta_{mn}^{pq}|^{\frac{1}{p+q}}, p+q \geqslant 1\}$ is bounded for fixed pair of integers m, n; it follows that the functionals f'_{mn}s are continuous (cf. [2], p. 15). Moreover, these functionals are poinwise bounded. Therefore by uniform boundedness principle. there exists a ball $B_\varepsilon(z)$ such that for all b $\in B_\varepsilon(z)$

$$|f_{mn}(b)| \leqslant M, \text{ for all } m, n \geqslant 0,$$

where M is a constant $\geqslant 1 \Rightarrow |f_{mn}(b)| \leqslant M$ for all $m, n \geqslant 0$ and all b with $b\| \leqslant \varepsilon$. Choosing b to be the matrices b^{pq} for $p+q \geqslant 0$ respectively where $b^{pq} = (\varepsilon_{ij})$,

$$\varepsilon_{ij} = \begin{cases} \varepsilon^{p+q}, & i = p, \ j = q; \\ 0, & \text{otherwise}, \end{cases}$$

when $p+q > 0$ and $b^{oo} = (\varepsilon_{ij})$, $\varepsilon_{oo} = \varepsilon$, $\varepsilon_{ij} = 0$, $i+j \geqslant 1$, we obtain

$|\delta_{mn}^{oo}| \varepsilon \leqslant M$ for all $m, n \geqslant 0$, and

$|\delta_{mn}^{pq} \varepsilon^{p+q} \leqslant M$, for all $m, n \geqslant 0$ and $p+q \geqslant 1$. Thus

$$|\delta_{mn}^{oo} \leqslant \frac{M}{\varepsilon}, \ |\delta_{mn}^{pq}|^{\frac{1}{p+q}} \leqslant \frac{M^{\frac{1}{p+q}}}{\varepsilon}, \text{ for } m+n \geqslant 0, \text{ and } p+q > 0.$$

Since $M^{\frac{1}{p+q}} \leqslant M$ for all $p+q > 0$ it follows that

$$|\delta_{mn}^{oo}|, \delta_{mn}^{pq}|^{\frac{1}{p+q}} \leqslant \frac{M}{\varepsilon}, \text{ for all } m+n \geqslant 0 \text{ and } p+q > 0.$$

This proves (4.3) The condition (4.4) obviously follows.

Theorem 4. 5: *Let (4. 1) be true for $b \in C_{oo}$. Then $a = (a_{mn}) \in \delta$ if and only if*

$$\Big\{\sum_{p+q} \sum_{\geqslant 0} |\delta_{mn}^{pq}|\Big\}^{\frac{1}{m+n}} \to 0 \textit{ as } m+n \to \infty. \qquad (4.6)$$

Proof: The proof follows on the lines of the preceding result and is therefore omitted.

For proving the next result, we need

Proposition 4.7 : If $A : l_{11} \to l_{11}$ is a linear transformation from l_{11} into itself defined by the relation (4.1), i.e. $A(b) = a = (a_{mn})$ where $b \in l_{11}$ and $a \in l_{11}$, then

$$\sup \left\{ \sum_{m+n} \sum_{\geqslant 0} | \delta^{pq}_{mn} | : p+q \geqslant 0 \right\} \leqslant M \tag{4.8}$$

where $M = \|A\|$ (operator norm of A).

Proof : From the convergence of the series $\sum_{p+q} \sum_{\geqslant 0} \delta^{pq}_{mn} \; b_{pq}$ for each $b = (b_{pq}) \in l_{11}$, one can easily derive its absolute convergence which in turn, implies that $|\delta^{pq}_{mn}| \leqslant k_{mn}$ for all $p, q \geqslant 0$ where k_{mn} is a constant depending on m and n. Now A, being the pointwise limit of a pointwise bounded sequence $\{A_r : r \geqslant 1\}$ of continuous linear operators A_r' s from l_{11} to itself such that

$$A_r(b) = a = (a_{mn}), \text{ where } a_{mn} = \sum_{p+q} \sum_{\geqslant 0} \delta^{pq}_{mn} \; b_{pq}$$

for all m, n with $m+n \leqslant r$ and 0 otherwise, is continuous. Therefore A is bounded. Let $\|A\| = M$. Then for all p, q, the following inequality

$$\|A(e^{pq})\| \sum_{p+q} \sum_{\geqslant 0} | \delta^{pq}_{mn} | \leqslant \|A\| = M,$$

gives the required result.

Theorem 4.9 : *Let (4.1) be true for $b \in l_{11}$. Then $a = (a_{mn}) \in \delta$, if and only if*

$$| \delta^{pq}_{mn} |^{\frac{1}{m+n}} \to 0 \text{ as } m+n \to \infty, \tag{4.10}$$

uniformly in p and q.

Proof : Sufficiency follows by straightforward calculations. For necessity, assume that (4.10) is not true. Then for $\varepsilon > 0$, and any $N \in \mathbf{N}$, there exist integers m, n and p, q such that $m+n > N$ and

$$| \delta^{pq}_{mn} |^{\frac{1}{m+n}} > \varepsilon. \tag{4.11}$$

Since A maps l_{11} into δ, it follows A transforms l_{11} into itself and therefore $\sup \left\{ \sum_{m+n} \sum_{\geqslant 0} | \delta^{pq}_{mn} | : p+q \geqslant 0 \right\} \leqslant M$, by the preceding result. Thus we write $\omega_{mn} = \sup_{p+q \geqslant 0} | \delta^{pq}_{mn} |$, we can find a constant $K > 0$ such that

$$|\omega_{mn}| \leqslant \frac{K}{2}, \text{ for all } m, n \geqslant 0. \tag{4.12}$$

We also have

$$| \delta^{pq}_{mn} |^{\frac{1}{m+n}} \to 0 \text{ as } m+n \to \infty \text{ for each fixed } p \text{ and } q. \tag{4.13}$$

By (4.11) we can find m_1, n_1 and p_1, q_1 such that

$$| \delta^{p_1 q_1}_{m_1 n_1} |^{\frac{1}{m_1 n_1}} > \frac{\varepsilon}{2} \tag{4.14}$$

Now from the relations (4.11) to (4.13), choose m_2, n_2 sufficiently large with $m_2 + n_2 > m_1 + n_1$ and p_2, q_2 with $p_2 + q_2 > p_1 + q_1$ such that

$$\left|\frac{K}{2^{m_2+n_2}}\right| < \left(\frac{\varepsilon}{8}\right)^{m_1+n_1}; \tag{4.15}$$

$$\left|\delta^{p_2 q_2}_{m_2 n_2}\right|^{\frac{1}{m_2+n_2}} > \frac{\varepsilon}{2}; \tag{.16}$$

and

$$\left|\delta^{p_1 q_1}_{m_2 n_2}\right|^{m_2+n_2} < \frac{\varepsilon}{16}. \tag{4.17}$$

Proceeding in this way, we get sequences $\{m_k\}$ $\{n_k\}$, $\{p_k\}$ and $\{q_k\}$ with $m_k + n_k > m_{k-1} + n_{k-1}$, $p_k + q_k > p_{k-1} + q_{k-1}$, $k \geqslant 2$ such that

$$\left|\frac{K}{2^{m_k+n_k}}\right| < \left(\frac{\varepsilon}{8(k-1)}\right)^{m_{k-1}+n_{k-1}}; \tag{4.18}$$

$$\left|\delta^{p_k q_k}_{m_k n_k}\right|^{\frac{1}{m_k+n_k}} > \frac{\varepsilon}{2}; \tag{4.19}$$

and

$$\left|\delta^{p_j q_j}_{m_k n_k}\right|^{\frac{1}{m_k+n_k}} < \frac{\varepsilon}{8k}, \text{ where } 1 \leq j \leq k-1. \tag{4.20}$$

Let us now introduce the matrix $b = (b_{pq}) \in l_{11}$ as follows

$$b_{qp} = \begin{cases} \dfrac{1}{2^{m_k-n_k}}, & \text{when } p = p_k,\ q = q_k,\ k = 1, 2, \ldots; \\ 0 & \text{, otherwise.} \end{cases}$$

It is easily verified that $a = (a_{mn}) \notin \delta$ where

$$a_{mn} = \sum_{p+q \geq 0} \sum \delta^{pq}_{mn} b_{pq}, \text{ for all } m, n \geq 0.$$

Indeed,

$$\left|a_{m_k n_k}\right|^{\frac{1}{m_k+n_k}} \geq \frac{1}{2}\left|\delta^{p_k q_k}_{m_k n_k}\right|^{\frac{1}{m_k+n_k}} - \left|\sum_{j<k} \delta^{p_j q_j}_{m_k n_k} b_{p_j q_j}\right|^{\frac{1}{m_k+n_k}}$$

$$- \left|\sum_{j>k} \delta^{p_j q_j}_{m_k n_k} b_{p_j q_j}\right|^{\frac{1}{m_k+n_k}}$$

$$> \frac{\varepsilon}{4} - \frac{(k-1)\varepsilon}{8k} - \frac{\varepsilon}{8k} = \frac{\varepsilon}{8},$$

for all $k \geq 1$. Hence we arrive at a contradiction and the result follows.

On similar lines, we can prove the following result:

Theorem 4.21: *Let (4.1) be true for $b \in l_{11}$. Then $a = (a_{mn}) \in d$ if and only if*

$$\left| \delta^{pq}_{mn} \right|^{\frac{1}{m+n}} \leq M, \tag{4.22}$$

uniformly in p, q and m, n; where M is a positive constant.

Received November 20th, 1978

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