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THE CLASS OF GENERALIZED PERIODIC FUNCTIONS

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Introduction

Let a function $k(.) : [0, \infty) \to (0, \infty)$ and a number $\alpha > 0$ be given; A function $x(.) : [0, \infty) \to R$ is called k, α-generalized periodic function (k, α-g.p.f.) iff

$$x(k(t)) = \alpha x(t), \ \forall t \in [0, \infty) \equiv R_+. \tag{0.1}$$

We denote the class of all k, α-g.p.f. by k, α-GPF. In the paragraphs II and III we will only consider the continuous functions.

Let $\tau(t) = k(t) - t$. Then (0. 1) may be written in the form

$$x(t + \tau(t)) = \alpha x(t), \ \forall t \in R_+.$$

$\tau(.)$ is called a function-period of $x(.)$. Thus, a g.p.f. (g.p. oscillation) is different from the ordinary periodic function ($\alpha = 1$, $\tau(t) \equiv \tau = \text{const}$) in that its « period » changes in time and the difference of the values of amplitudes in the corresponding points is defined in the coefficient α.

Let us denote $\underbrace{k(k(...(k(t))...))}_{\nu \text{ times}} = k^{[\nu]}(t)$, $\nu = 1, 2, ...$; $k^{[0]}(t) \equiv t$; $k^{[\nu]}(0) = k_\nu$;

$[k_\nu, k_{\nu+1}) = K_\nu$; $k_{\nu+1} - k_\nu = l_\nu$.

For simplicity we will only consider the function $k(.)$ having the following property.

k-Property

1. $k(.) \in C(R_+)$, $k(0) > 0$.
2. If t is increasing from $k_{\nu-1}$ to k_ν, then $k(t)$ is increasing from k_ν to $k_{\nu+1}$, for every $\nu \geqslant 1$. Hence there exists the inverse function $h(.)$ of $k(.)$.
3. There exists the derivative $k'(t) > 0$ for every $t \in R_+ \setminus \{ k_\nu, \nu = 0, 1, 2, ... \}$ ($\exists h'(t) > 0 \ \forall t \in [k_1, \infty) \setminus \{ k_\nu, \nu = 1, 2, ... \}$).

The study of the class k, α-GPF has a great importance because we have different classes of functions which are periodic in different sense if $k(.)$ and α are chosen to be different. Partly, taking $\alpha = 1$, $k(t) \equiv t + \tau$, we get the case of ordinary periodic oscillations; if $l_{v+1} > l_v$ and $0 < \alpha < 1$, we have the oscillations retarded in time and damped in amplitude. This class, as we will see later in §§ I, III, extends our knowledge on some classic questions, for example, on the totally continuity of functions of the class L_p (Lemma 3.1). Besides that (see §§ II, IV) it turns out that this class is independent of all well-known classes of periodic or almost periodic functions. We note that k, α-g.p.f'. s were studied and used in [1] ($\alpha = 1$, with the viewpoint of generalization of the Floquet theory).

§ I. THE UNIFORM CONTINUITY OF G.P.F.

It is clear that a continuous periodic function (in usual sense) is continuous in R_+. This problem is not so simple for k, α-g.p.f. 's. Note that the region D_x of definition of a k, α-g.p.f. $x(.)$ is not always coincident with R_+ ($D_x = [0, L)$, $L = \sum_v l_v$). That is why when speaking of uniform continuity of $x(.)$, we have to consider the uniform continuity on D_x.

Theorem 1.1. *A k, α-g.p.f. is uniformly continuous if at least one of the following conditions is satisfied:*

a) $0 < \alpha < 1$.

b) $\alpha = 1$ *and* $k^{[v]'} \geqslant \beta > 0$ *for every* $v = 1,2,\dots$; *where* $k^{[v]'} = \inf\limits_{s \in (0,k_1)} (k^{[v]})'(s)$.

c) $\alpha > 1$ *and* $|h^{[v]}(t_1) - h^{[v]}(t_2)| < \overline{\delta_o}(\varepsilon/\alpha^v)$ *for every pair* $t_1, t_2 \in K_v$, $|t_1 - t_2| < \gamma_v \overline{\delta_o}(\varepsilon)$, $\gamma_v \geqslant \gamma > 0$, $\forall v$, *where* $\overline{\delta_o}(\varepsilon) = \sup \{\delta > 0 : |f(s_1) - f(s_2)| < \varepsilon$, $\forall s_1, s_2 \in K_o$, $|s_1 - s_2| < \delta\}$.

d) $x(.) \in Lip(K_o)$ *and* $\alpha^v / k^{[v]'} \leqslant A < \infty$, $\forall v$.

Proof. We prove, for example, the case c). Let $x(.) \not\equiv$ const. Choose $\varepsilon_o > 0$ such that $\overline{\delta_o}(\varepsilon) < l_o$. Then $l_v \geqslant \gamma\, \overline{\delta_o}(\varepsilon_o)$, $\forall v = 1, 2, \dots$ since on the contrary for those v for which $l_v < \gamma\, \overline{\delta_o}(\varepsilon_o) \leqslant \gamma_v\, \overline{\delta_o}(\varepsilon_o)$ we should have

$$l_o = h^{[v]}(k_{v+1}) - h^{[v]}(k_v) < \overline{\delta_o}(\varepsilon_o/\alpha^v) \leqslant \overline{\delta_o}(\varepsilon_o)$$

(because of the condition c) and the monotony of function $\overline{\delta_o}(.)$), that is a contradiction with the choosing of ε_o.

For $\varepsilon > 0$ let $\delta(\varepsilon) = \min\{\overline{\delta_o}(\varepsilon_o), \gamma\,\overline{\delta_o}(\varepsilon_o), \gamma\,\overline{\delta_o}(\varepsilon)\}$. Now if $|t_1 - t_2| < \delta(\varepsilon)$, then only two following cases are possible: or t_1 and t_2 belong to the same interval K_v, or $t_1 \in K_v$, $t_2 \in K_{v+1}$. Let $t_1 \in K_v$ and $t_2 \in K_v$. Since $|t_1 - t_2| < \gamma\,\overline{\delta_o}(\varepsilon) \leqslant \gamma_v\,\overline{\delta_o}(\varepsilon)$, from c) we have

$$|h^{[v]}(t_1) - h^{[v]}(t_2)| < \overline{\delta_o}(\varepsilon/\alpha^v).$$

Therefore from the definition of $\overline{\delta_o}(\varepsilon)$ we get

$$|x(t_1) - x(t_2)| = \alpha^v\, |x(h^{[v]}(t_1)) - x(h^{[v]}(t_2))| < \alpha^v \cdot \frac{\varepsilon}{\alpha^v} = \varepsilon.$$

If $t_1 \in K_\nu$ and $t_2 \in K_{\nu+1}$, then $|x(t_1) - x(t_2)| \leqslant |x(t_1) - x(k_{\nu+1})| + $
$$+ |x(k_{\nu+1}) - x(t_2)| < 2\varepsilon.$$

The point c) is proved. The proofs of b) and d) are analogous, a) is obvious Q. E. D.

Theorem 1. 2. *A k, α - g. p. f. $x(.)$ is not uniformly continuous in the following cases:*

a) $\alpha = 1$ and $\inf_\nu l_\nu = 0$.

b) $\alpha > 1$ and $\bar{k}^{[\nu]'} \leqslant \bar{k} < +\infty$ for some sequence $\nu_1, \nu_2, \ldots, \nu_i, \ldots$,

$$\bar{k}^{[\nu]'} = \sup_{s \in K_0} (k^{[\nu]})'(s).$$

Proof. The point a) is clear. Let us consider b) For every $\delta > 0$ there exist s_1 and $s_2 \in K_0$ such that $|s_1 - s_2| < \delta/\bar{k}$ and $\Delta_x[s_1, s_2] = |x(s_1) - x(s_2)| > 0$.

Choose ν_i so large that $\alpha^{\nu_i} \Delta_x[s_1, s_2] > 1$ and put $t_j = k^{[\nu_i]}(s_j)$, $j = 1, 2$. Then though

$$|t_1 - t_2| = (k^{[\nu_i]})'(s) \, . \, |s_1 - s_2| \leqslant \bar{k} \, |s_1 - s_2| < \delta,$$

we have $|x(t_1) - x(t_2)| = \alpha^{\nu_i} \Delta_x[s_1, s_2] > 1$.

Thus $x(.)$ cannot be uniformly continuous. Q. E. D.

From the theorems proved above the question arises: It is possible or not to introduce a concept of uniform continuity so that any k, α-g. p. f. will be uniformly continuous?

Definition 1. 1. Let a function $k(.)$ have the k-property. A function $f(.)$: $[0, L] \to R$ is called k-uniformly continuous (k-u. c. f.) iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ for every pair t_1, t_2, $t_1 \in K_\nu$, $t_2 \in K_\mu$

$$(\mu \geqslant \nu), \ |t_2 - t_1| < \delta \text{ and } |t_1 - h^{[\mu-\nu]}(t_2)| < \delta \, \underline{k}^{[\nu]'}. \tag{1.1}$$

The class of all k-u. c. f. is denoted by k-UC.

Theorem 1. 3. *For any $0 < \alpha \leqslant 1$ we have k, α-GPF $\subset$ k-UC.*

Proof. It is clear that the class UC of all uniformly continuous functions belongs to k-UC for any $k(.)$. Therefore we must only consider the case $\alpha = 1$ (because we have k, α-GPF $\subset$ UC for any $0 < \alpha < 1$).

For an arbitrary $\varepsilon > 0$ we choose $\delta > 0$ so that $|x(s_1) - x(s_2)| < \varepsilon$ for every pair $s_1, s_2 \in K_0$, $|s_1 - s_2| < \delta$. Let t_1, t_2 be arbitrary points, for which $0 < t_2 - t_1 < \delta$ and (1.1) holds. Then $h^{[\nu]}(t_1) \in K_0$, $h^{[\mu]}(t_2) \in K_0$ and

$$|h^{[\nu]}(t_1) - h^{[\nu]}(t_2)| = (h^{[\mu]})'(t) \, . \, |t_1 - h^{[\mu-\nu]}(t_2)| \leqslant \frac{1}{\underline{k}^{[\nu]'}} |t_1 - h^{[\mu-\nu]}(t_2)| < \delta$$

because of (1.1), where $t \in K_\nu$. Hence for the chosen values of δ we get

$$|x(t_1) - x(t_2)| = |x(h^{[\nu]}(t_1)) - x(h^{[\mu]}(t_2))| < \varepsilon. \qquad \text{Q.E.D.}$$

§ II. THE CONNECTION WITH THE CLASSES APF AND AAPF

It is possible to show that the class k, α-GPF intersects all well-known classes of periodic or almost periodic functions such as the class APF of almost periodic functions defined by Bohn (see [2]), the class AAPF of asympt tic almost periodic functions defined by Frechet (see [3]), the class of $\varepsilon\pm$ - periodic functions defined by Massera (see[4]), the class of semiperiodic functions (see[5]),...

For example, we will consider two cases: the connection between k, α-GPF and APF, and between k, α-GPF and AAPF.

It is possible to consider k, α-g.p.f.s of more general form, but for simplicity we often take k, α-g.p.f.s of the following form F. 1. For $t \in K_\nu$

$$k(t) = \frac{l_{\nu+1}}{l_\nu}(t - k_\nu) + k_{\nu+1}$$

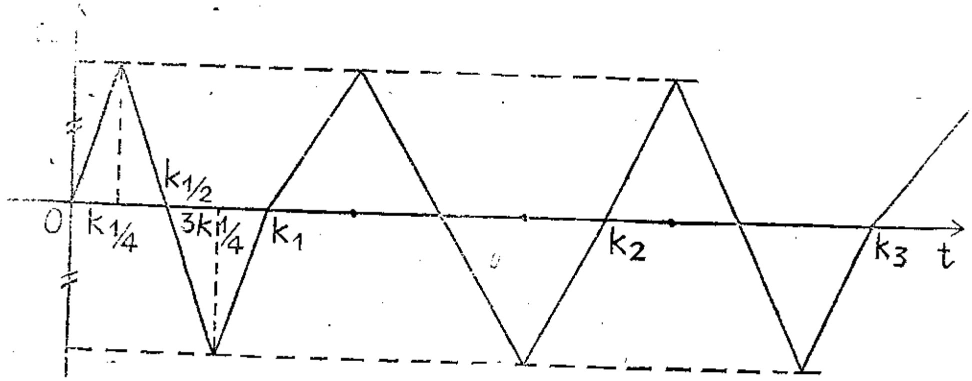

Fig. 1

$$(\text{then } k^{[\mu-\nu]}(t) = \frac{l_\mu}{l_\nu}(t - k_\nu) + k_\mu, \ \forall \nu, \mu).$$

Furthermore we suppose $L = +\infty$.

Theorem 2. 1. *a) For $0 < \alpha \neq 1$ a k, α-g.p.f. $x(.)$ ($x(.) \not\equiv 0$) cannot be almost periodic.*

b) Let $\alpha = 1$. If a k, α-g.p.f. $x(.)$ of form F. 1 belongs to the class APF, then $\inf_\nu l_\nu > 0$ and for each $\nu = 0, 1, 2, \dots$ there exists a sequence $i^{(\nu)}$, $i = \overline{1, \infty}$, $i^{(\nu)} > \nu \ \forall i$ and $l_{i^{(\nu)}} \to l_\nu$ when $i \to \infty$.

Proof. a) Let $0 < \alpha < 1$. Since $x(.) \not\equiv 0$ there exists $t_0 \in K_0$ such that $x(t_0) \neq 0$; for example, $x(t_0) = x_0 > 0$. We take $\varepsilon = x_0/2$ and choose ν as large that $x(t) < \varepsilon$ for $t \geq k_\nu$. Then in $[k_\nu - t_0, \infty)$ there exists no ε-almost period because for any $l \in [k_\nu - t_0, \infty)$ we have $|x(t_0 + l) - x(t_0)| > x(t_0) - \varepsilon > x_0/2$. This shows $x(.)$ cannot be almost periodic. The case $\alpha = 1$ may be proved analogously.

b) If $\inf_{\nu} l_\nu = 0$ then $x(.)$ cannot even be uniformly continuous, consequently $x(\cdot) \overline{\in} \mathrm{APF}$.

Suppose the condition (2.1) is not satisfied. That is there exists ν_0 that satisfies $|l_\nu - l_{\nu}| \geqslant \delta > 0$ for any $\nu > \nu_0$.

Give $\varepsilon > 0$ and let $\overline{\delta}_{\nu_0}(\varepsilon)$ be the maximal number such that

$$|x(k_{\nu_0+1} - \overline{\delta}_{\nu_0})| = |x(k_{\nu_0} + \overline{\delta}_{\nu_0})| \leqslant \varepsilon$$

($\overline{\delta}_{\nu_0}(\varepsilon) \to 0$ when $\varepsilon \to 0$). Furthermore put $\delta(\varepsilon) = \min_{\mu > \nu_0} |k_{\nu_0} + l(\varepsilon) - k_\mu|$ where $l(\varepsilon)$ is a sufficiently large ε — almost peoriod. We also have $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$. Indeed, assume $\delta(\varepsilon) = k_{\mu_0} - (k_{\nu_0} + l(\varepsilon))$. We have $|x(k_{\nu_0} + \delta(\varepsilon)) - x(k_{\nu_0} + \delta(\varepsilon) + l(\varepsilon))| = |x(k_{\nu_0} + \delta(\varepsilon)) - x(k_{\mu_0})| = |x(k_{\nu_0} + \delta(\varepsilon))| < \varepsilon$. From this it follows that $\delta(\varepsilon) < \overline{\delta}_{\nu_0}(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Then, put $\delta_\nu = |l_\nu - l_{\nu_0}|$. Suppose $k_{\nu_0} + l(\varepsilon) \equiv k_\nu$ for some ν and $l_\nu > l_{\nu_0}$. we have

$$|x(k_{\nu_0+1}) - xk_{\nu_0+1} + l(\varepsilon))| = |x(k_{\nu+1} - \delta_\nu)| < \varepsilon.$$

But $x(k_{\nu+1} - \delta_\nu) = x(h^{[\nu - \nu_0]}(k_{\nu+1} - \delta_\nu))$. The interval $[k_{\nu+1} - \delta_\nu, k_{\nu+1}]$ of length δ_ν goes into the interval $[k_{\nu_0+1} - \delta_{\nu\nu_0}, k_{\nu_0+1}]$ of length $\delta_{\nu\nu_0} = \frac{l_{\nu_0}}{l_\nu}\delta_\nu$ by the mapping $h^{[\nu - \nu_0]}(.)$. Therefore $\delta_{\nu\nu_0} < \overline{\delta}_{\nu_0}$, i.e. $\frac{l_{\nu_0}}{l_\nu}\delta_\nu = \frac{l_{\nu_0}}{l_\nu}(l_\nu - l_{\nu_0}) =$

$$= l_{\nu_0}\left(1 - \frac{l_{\nu_0}}{l_\nu}\right) < \overline{\delta}_{\nu_0}.$$

However $l_\nu \geqslant l_{\nu_0} + \delta$, hence from this it follows that $l_{\nu_0}\left(1 - \frac{l_{\nu_0}}{l_{\nu_0} + \delta}\right) =$

$= \frac{\delta l_{\nu_0}}{l_{\nu_0} + \delta} < \delta_{\nu_0}$, which cannot be when $\varepsilon \to 0$ ($\delta_{\nu_0} \to 0$).

This means that $x(.)$ could not be almost periodic.

The case $k_{\nu_0} + l(\varepsilon) = k_\nu \pm \delta(\varepsilon)$ is proved analogously we must only note that in this case there will be $\delta_\nu \pm \delta(\varepsilon)$ instead of δ_ν, where $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$. Q.E.D.

Theorem 2.2. a) *For $0 < \alpha < 1$ we have k, α - GPF $\subset$ AAPF. If $\alpha > 1$ then a $k, \alpha - g.\ p.\ f.\ x(.) \in$ AAPF ($x(.) \not\equiv 0$) b) Let $\alpha = 1$ and suppose that there exists ν such that $\Sigma\, |l_\nu - l_{\nu_0}| < +\infty$. Then k, α-GPF $\subset$ AAPF.*

Proof. a) This point is proved analogously as Theorem 1.1a).

a) Taking into account the asymptotic condition in the definition of the class AAPF, it is sufficient to consider the values $t \geqslant k_{\nu_0}$. Let $\varepsilon > 0$ be arbitrary and choose $\delta(\varepsilon) > 0$ such that $|x(t_1) - x(t_2)| < \varepsilon/2$ for every pair $t_1, t_2 \in K_\nu$,

$l_\nu = l_{\nu_0}$, $|t_1 - t_2| < \delta$. For this δ there exists an integer $i(\varepsilon)$ such that $\sum_{\nu > i} | l - l_{\nu_0} | < \delta/2$. Then we have $| x(t + nl_{\nu_0}) - x(t) | < \varepsilon$ for any $t > k_i$ and $n = 1,2,...$ This follows from the following remark:

$$\text{Suppose } | k_i + nl_{\nu_0} - k_j | + | l_i - l_{\nu_0} | + | l_{j-1} - l_{\nu_0} | + | l_j - l_{\nu_0} | + + | l_{j+1} - l_{\nu_0} | < \delta/2$$

(thanks to the condition b) for every n and i there always exists such j). Then $| x(t + nl_{\nu_0}) - x(t) | < \varepsilon$ for every $t \in K_i$.

Indeed, for example, suppose
$k_j - (k_i + nl_{\nu_0}) = a_1 > 0$, $l_{\nu_0} - l_i = a_2 > 0$, $l_{j-1} - l_{\nu_0} = a_3 > 0$, $l_j - l_{\nu_0} = a_4 > 0$, $l_{j+1} - l_{\nu_0} = a_5 > 0$, $l_i < l_{\nu_0} < l_{j-1} < l_j < l_{j+1}$. If $t \in [k_i, k_i + a_1]$ then $t + nl_{\nu_0} \in [k_j - a_1, k_j]$, $x(t) = x(h^{[i-\nu_0]}(t))$, $x(t + nl_{\nu_0}) = x(h^{[j-\nu_0-1]}(t + nl_{\nu_0}))$.

We have $h^{[i-\nu_0]}(t) \in \left[k_{\nu_0}, k_{\nu_0} + \frac{l_{\nu_0}}{l_i} a_1 \right]$, where

$$\frac{l_{\nu_0}}{l_i} a_1 = \frac{l_{\nu_0}}{l_{\nu_0} - a_2} a_1 = \left(1 + \frac{a_2}{l_{\nu_0} - a_2}\right) a_1 < 2a_1 < \delta$$

(a_2 is sufficiently small). Hence

$$\left| x(h^{[i-\nu_0]}(t)) \right| = \left| x(h^{[i-\nu_0]}(t)) - x(k_{\nu_0}) \right| < \varepsilon/_2$$

by the manner of choosing δ. Analogously $\left| x(h^{[j-\nu_0-1]}(t+nl_{\nu_0}) \right| > \varepsilon/_2$.
Consequently $| x(t+nl_{\nu_0}) - x(t) | < \varepsilon$.

The remained interval $[k_i + a_1, k_{i+1}]$ is also considered by the manner of mapping two points t and $t + nl_{\nu_0}$ into k_{ν_0} and comparing the distance between two corresponding images. Q. E. D.

The following example shows that there exist the k, α-g. p, f'. s not belonging to AAPF. Suppose $x(.)$ has the form F.1 and $\{\nu_i\}$ is a sequence for which $\nu_i - \nu_{i-1} \to \infty$ when $i \to \infty$, $\nu_1 > 0$.

Putting $l_{\nu i} = l + \varepsilon_0$, $i = 1,2,...$, and all the other intervals equal to l, we get the desired function.

Indeed, all the ε-almost periods $l(\varepsilon)$ have to tend to nl when $\varepsilon \to 0$. Then, however, since $\nu_i - \nu_{i-1} \to \infty$ ($\sum_{\nu=\nu_{i-1}}^{\nu_i} l_\nu \to \infty$), when $i \to \infty$ for every arbitrary large $l(\varepsilon)$ there always exists i such that

$$\left| x(k_{\nu_i} + l(\varepsilon)) - x(k_{\nu_i}) \right| = \left| x(l - \varepsilon_0 \pm \min_n (nl - l(\varepsilon)) \right| \to \left| x(l - \varepsilon_0) \right| \text{ when } \varepsilon \to 0.$$

This shows that for sufficiently small $\varepsilon > 0$ there is no ε-almost (asymptotic) period $l(\varepsilon)$.

§III. THE TOTAL CONTINUITY OF FUNCTIONS OF THE CLASS L_p, k, α — GPF

First we introduce the following concept generalizing that of total continuity of functions of the class L_p (see [6]).

Let us fix a subset Δ of the set $C[0, T]$.

Definition 3. 1. We say that a function $p(.) \in L_p[0, T]$ is Δ,p-totally continuous in $[0, T]$ iff

$$\int_0^T |f(t+\delta(t)) - f(t)|^p dtp \to 0 \text{ when } \|\delta(.)\|_C \to 0,\ \delta(.) \in \Delta\ (f(t)=0 \text{ for } t \overline{\in} [0, T]).$$

For example, if $\Delta \equiv C[0,T]$ then any Δ,p-totally continuous function is totally continuous (in usual sense), but not reversely.

We denote the set of all Δ, p-totally continuous functions in $[0,T]$ by Δ, $p\text{-}C_t[0,T]$.

In what follows we only need the following class

$$\Delta = \{\delta(.) \in C : \inf_t \delta(t) > 0,\ \exists\ \delta'(t) \text{ and}$$

$$-1 \leqslant \underline{\delta'} \leqslant \delta'(t) \leqslant \overline{\delta'} < +\infty \quad \forall t\}$$

($\underline{\delta'}$ and $\overline{\delta'}$ are general for whole class Δ). This Δ corresponds to the class of functions $\tau(t) = t + \delta(t)$ for which $\inf_t (\tau(t) - t) > 0$, $\exists\ \tau'(t)$ and

$$0 < \underline{\tau'} \leqslant \tau'(t) \leqslant \overline{\tau'} < \infty \quad \forall t.$$

Speaking about Δ, we will mean the very this class.

Lemma 3. 1. The following correlation holds

$$L_p[0, T] \supset \Delta,\ p\text{-}C_t[0, T].$$

Proof. We use the same arguments as in [6]. We must show the following: for any $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that $\int_0^T |f(t+\delta(t)) - f(t)|^p dt < \varepsilon$ for every $\delta(.) \in \Delta$ and $\|\delta(.)\|_C < \delta_\varepsilon$.

From absolute continuity of Lebesgue integral for arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every measurable subset $e \subset [0, T]$, $\text{mes}\, e < \delta_1$, we have $\int_e |f(t)|^p dt < \varepsilon/M$ (the number M will be fixed below).

We set $\tau(t) = t + \delta(t)$ and denote the inverse function of $\tau(.)$ by $\eta(.)$. From the definition of Δ is follows that

$$0 < \underline{\eta'} \leqslant \eta'(t) \leqslant \overline{\eta'} < +\infty \quad \forall t.$$

Therefore $$\int_e |f(t+\delta(t))|^p dt = \int_e |f(\tau(t))|^p dt =$$
$$= \int_{\tau(e)} |f(t)|^p \eta'(t)\, dt \leqslant \overline{\eta}' \int_{\tau(e)} |f(t)|^p dt$$
(the formula of substitution of the argument in an integral on a measurable set may be established by approximating the given set by open sets and representing these sets by union of denumerable number of open intervals, then applying the well-known formula of substitution of the argument in the integral on every this interval).

However, mes $\tau(e) = \int_e \tau'(t)\, dt \leqslant \overline{\tau}'$. mes e. Consequently, taking $\delta_2 = \min(\delta_1/\overline{\tau}', \delta_1)$, we get
$$\int_e |f(t)|^p dt < \varepsilon/M, \qquad \int_e |f(t+\delta(t))|^p dt < \overline{\eta}'/M \cdot \varepsilon,$$
where e is an arbitrary measurable set, $e \subset [0, T]$, mes $e < \delta_2$ and $\delta(.) \in \Delta$.

Since $f(.) \in L_p \subset L_1$ by the Luzin's theorem there exists a closed set $F \subsetneq [0, T]$ such that mes $([0, T] \setminus F) < \min(\delta_2, \delta_2/\overline{\eta}')$ and $f|_F(.) \in C(F)$. Hence for $\varepsilon > 0$ there exists $\delta_3 > 0$ such that $|f(t_1) - f(t_2)| < (\varepsilon/M)^{\frac{1}{p}}$ for every pair $t_1, t_2 \in F$, $|t_1 - t_2| < \delta_3$.

Now we put $\delta_\varepsilon = \min\{\delta_2, \delta_3\}$ and let $\delta(.)$ be an arbitrary function from Δ, $\|\delta(.)\|_C < \delta_\varepsilon$. Applying the Cauchy — Bunhiakowski inequality, we get
$$\int_0^T |f(t+\delta(t)) - f(t)|^p dt = \int_0^{T-\delta_2} \cdot + \int_{T-\delta_2}^T \cdot \leqslant \int_{F\cap[0,T-\delta_2]} \cdot + \int_{[0,T]\setminus F} \cdot + \int_{T-\delta_2}^T \cdot \leqslant$$
$$\leqslant \int_{F_1} \cdot + \int_{F_2} \cdot + 2\left[(\varepsilon/M)^{\frac{1}{p}} + \left(\varepsilon \cdot \frac{\overline{\eta}'}{M}\right)^{\frac{1}{p}}\right]^p,$$
where
$$F_1 = \{t \in F \cap [0, T-\delta_2] : t + \delta(t) \in F\},$$
$$F_2 = (F \cap [0, T-\delta_2]) \setminus F_1.$$
Note that
$$\text{mes}\, F_2 \leqslant \text{mes}\{t : \tau(t) \in [0, T] \setminus F\} = \text{mes}\, \eta([0, T] \setminus F) =$$
$$= \int_{[0,T]\setminus F} \eta'(t)\, dt \leqslant \overline{\eta}' \cdot \text{mes}([0, T] \setminus F) < \delta_2$$
by the way of choosing F. Therefore
$$\int_{F_2} |f(t+\delta(t)) - f(t)|^p dt \leqslant \left[(\varepsilon/M)^{\frac{1}{p}} + (\varepsilon \cdot \eta'/M)^{\frac{1}{p}}\right]^p.$$

At last, since for $t \in F_1 \subset F$, $t + \delta(t)$ also belongs to F and $|\delta(t)| < \delta_\varepsilon$ we have

$$\int_{F_1} |f(t + \delta(t)) - f(t)|^p \, dt \leqslant (\varepsilon/M) \, . \, T.$$

$$\text{Thus} \int_0^T |f(t + \delta(t)) - f(t)|^p \, dt \leqslant \varepsilon/M \, . \, T + 3 \left[(\varepsilon/M)^{\frac{1}{p}} + (\varepsilon \, . \, \overline{\eta}'/M)^{\frac{1}{p}} \right]^p =$$

$$\frac{\varepsilon}{M} \left\{ T + 3(1 + \overline{\eta}'^{\frac{1}{p}})^p \right\} < \varepsilon,$$

if M has been chosen sufficiently large. Q.E.D.

As in [7] we introduce

Definition 3.2. A function $f(.) : R \to R$, $f\Big|_{[-T,\, T]}(.) \in L_p[-T, T]$ for any fixed finite T, is called p-totally continuous (p-t.c.f) on R iff

$$\lim_{\delta \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t + \delta) - f(t)|^p \, dt \right\} = 0.$$

We denote the set of all p-t.c.f'.s on R by $p\text{-}C_t(R)$. The set $p\text{-}C_t(R)$ is defined in the same way $\left(\text{when there is an integral} \frac{1}{T}\int_0^T . \text{ instead of } \frac{1}{2T}\int_0^T .\right)$.

At last we specify the general definition for class L_p.

Definition 3.3. A function $x(.) : R_+ \to R$ is called L_p, k, α-generalized periodic (L_p, k, α—g.p.f.) iff $x\Big|_{K_\nu}(.) \in L_p(K_\nu)$ for every $\nu = 0, 1, 2, \ldots$ and $x(k(t)) = x(t)$ for a.e. $t \in R_+$.

In the case $\alpha = 1$ we will also consider such functions $x(.) : R \to R$.

The class of all L_p, k, α-g.p.f'.s is denoted by L_p, k, α-GPF.

Theorem 3.1. *Assume* $0 < k' \leqslant (k^{[\nu]})'(t) \leqslant \overline{k}' < \infty$, $\forall t \in K_0$,

$$\nu = 0, 1, 2, \ldots \textit{ and } \sum_{\nu=0}^{m} \alpha^{p\nu} \leqslant A \sum_{\nu=0}^{m-1} l_\nu, \ \forall m = 1, 2, \ldots, A < \infty.$$

Then L_p, k, α-$GPF(R_+) \subset p\text{-}C_t(R_+)$.
(This also holds in the case R*).*

Proof. We first note that $l_\nu = k^{[\nu]}(k_1) - k^{[\nu]}(0) = (k^{[\nu]})'(s) \, . \, l_0 \geqslant \underline{k}' l_0 \geqslant \underline{l} > 0$, $\forall \nu = 0, 1, 2, \ldots$

For $\varepsilon > 0$ let $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ be the positive numbers corresponding to ε in the definitions of Δ, p-t.c.f. and of absolute continuity of Lebesgue integral for $|x(.)|^p$ on K_0.

Put $\delta(\varepsilon) = \min \{\underline{l}/2, \delta_1(\varepsilon)\underline{k}', \delta_2(\varepsilon)\underline{k}'\}$ and let $0 < \delta < \delta(\varepsilon)$. Then ($k_m < T \leqslant k_{m+1}$):

$$J(T, \delta) = \frac{1}{T}\int_0^T |x(t+\delta) - x(t)|^p\, dt \leqslant \frac{1}{T}\int_0^{k_{m+1}} \cdot =$$

$$= \frac{1}{T}\sum_{\nu=0}^{m}\int_{k_\nu}^{k_{\nu+1}} \cdot = \frac{1}{T}\sum_{\nu=0}^{m}\left\{\int_{k_\nu}^{k_{\nu+1}-\delta} \cdot + \int_{k_{\nu+1}-\delta}^{k_{\nu+1}} \cdot\right\} \leqslant$$

$$\leqslant \frac{1}{T}\sum_{\nu=0}^{m}\left\{\left[\left(\int_{k_{\nu+1}}^{k_{\nu+1}+\delta} |x(t)|^p\, dt\right)^{\frac{1}{p}} + \left(\int_{k_{\nu+1}-\delta}^{k_{\nu+1}} |x(t)|^p\, dt\right)^{\frac{1}{p}}\right]^p + \right.$$

$$\left. + \overline{k}'\alpha^{\nu p}\int_0^{k_1}\left|x(h^{[\nu]}(k^{[\nu]}(t)+\delta)) - x(t)\right|^p dt.\right.$$

Note that the interval of length δ in K_ν goes into the interval of length $< \delta/\underline{k}' < \delta_2$ in K_0 by mapping $h^{[\nu]}(.)$. Therefore

$$\int_{k_{\nu+1}}^{k_{\nu+1}+\delta} |x(t)|^p\, dt \leqslant \alpha^{p(\nu+1)}\varepsilon\underline{k}', \quad \int_{k_{\nu+1}-\delta}^{k_{\nu+1}} |x(t)|^p\, dt < \alpha^{p\nu}\varepsilon\underline{k}'.$$

Moreover, since

$$h^{[\nu]}(k^{[\nu]}(t)+\delta) - t = h^{[\nu]}(k^{[\nu]}(t)+\delta) - h^{[\nu]}(k^{[\nu]}(t)) =$$
$$= \delta \cdot (h^{[\nu]})'(\tilde{t}) \leqslant \delta/\underline{k}' < \delta_1,$$

from the way of choosing δ we have

$$\int_0^{k_1}\left|x(h^{[\nu]}(k^{[\nu]}(t)+\delta)) - x(t)\right|^p dt < \varepsilon$$

(pay our attention on the fact that here the function $h^{[\nu]}(k^{[\nu]}(.)+\delta)$ belongs the class Δ!).

Thus
$$J(T, \delta) \leqslant \frac{1}{T}\sum_{\nu=0}^{m}\left\{\varepsilon\overline{k}'\alpha^{p\nu} + \varepsilon\overline{k}'\alpha^{p\nu}(1+\alpha)\right\} \leqslant$$

$$\leqslant c\varepsilon \cdot \frac{1}{T}\cdot\sum_{\nu=0}^{m}\alpha^{p\nu} \leqslant c\,\varepsilon\,\frac{1}{\sum_{\nu=0}^{m-1} l_\nu}\cdot\sum_{\nu=0}^{m}\alpha^{p\nu} \leqslant cA\varepsilon.$$

Therefore
$$\lim_{\delta\to 0}\ \limsup_{T\to\infty} J(T, \delta) = 0. \qquad \text{Q.E.D.}$$

Remark. In the case $0 < \alpha < 1$ the conditions of the Theorem 3.1 are satisfied immediately. For $\alpha = 1$ these conditions have the form

$$\frac{m+1}{\sum_{\nu=0}^{m-1} l_\nu} \leqslant \frac{m+1}{ml} \leqslant A < \infty,$$

hence they also hold.

In order to prove Theorem 3.2 we have now

Lemma 3.2. a) Let $f(.) \in L_p[0, T]$ and exist two intervals U_1, U_2 (arbitrary small) $\subset [0, T]$ such that $f(t_1) \neq f(t_2)$ for a.e. $t_1 \in U_1$ and $t_2 \in e_2 \subset U_2$, mes $e_2 > 0$.

Then
$$J_\delta^0 = \int_0^T \left| f(t + \delta(t)) - f(t) \right|^p dt \neq 0$$

for any $\delta(.) \in \Delta$, $\|\delta(.)\|_C$ is sufficiently small.

b) Assume there exists an interval $U \subset [0, T]$, in which $f(.) \in C^1$ and $f(.) \not\equiv$ const. Then $J_\delta^0 \geqslant c\delta^{p+1}$ for any $\delta(.) \in \Delta$, $0 < \delta < \delta(t) \leqslant \bar{\delta}$, $\bar{\delta}$ is sufficiently small (This means that the lower bound of J_δ^0 may be taken independently of very $\delta(.)$, but only by its lower bound),

Proof. a) Suppose the statement don't hold.

Let exist a sequence $\delta_n(.) \in \Delta$, $\|\delta_n(.)\|_C \to 0$ when $n \to \infty$ and $J_{\delta_n}^0 = 0$. Put $\tau_n(t) = t + \delta_n(t)$. Then $0 < \tau_n^{[o]} < \tau_n^{[1]} < \ldots < \tau_n^{[N]} = T$, where $\tau_n^{[i]} = \tau_n(\tau_n^{[i-1]})$, $\max_i \left|\tau_n^{[i]} - \tau_n^{[i-1]}\right| < \delta_n$.

From
$$0 \leqslant \sum_i \int_{\tau_n^{[i-1]}}^{\tau_n^{[i]}} |f(\tau_n(t)) - f(t)|^p \, dt = J_{\delta_n}^0 = 0$$

it follows that $f(\tau_n^{[i]}(t)) = f(t)$ for a.e. $t \in [0, \tau_n^{[1]}]$, i.e. $f(\tau^{[i-j]}(t)) = f(t) =$

$$= \text{for a.e. } t \in [\tau^{[j-1]}, \tau^{[j]}],\ i > j.$$

Choose n such that $\delta_n < \min\{dU_1, dU_2\}$ (for example, suppose $U_1 < U_2$). Then there exist i and j such that

mes $\quad [\tau_n^{[j-1]}, \tau_n^{[j]}] \subset U_1$, $[\tau_n^{[i-1]}, \tau_n^{[i]}] \subset U_2$, $(e_2 \cap [\tau_n^{[i-1]}, \tau_n^{[i]}]) > 0$.

We have $f(\tau^{[i-j]}(t)) = f(t)$ for a.e. $t \in [\tau_n^{[j-1]}, \tau_n^{[j]}]$.

This is a contradiction with the condition of the Lemma, because if $t_2 \in e_2$ then $f(t_2) \neq f(t_1)$ for a.e. $t_1 \in U_1$.

b) From the condition it follows that there exists an interval $V \subset U$ in which $|f'(t)| \geqslant c_1 > 0$. Choose $\overline{\delta}$ sufficiently small so that V contains at least two intervals $[\tau^{[i-1]}, \tau^{[i]}]$, $[\tau^{[i]}, \tau^{[i+1]}]$. For every $\delta(.)$ satisfying the condition of the Lemma we have

$$\int_0^T |f(\tau(t)) - f(t)|^p \, dt \geqslant \int_{\tau^{[i-1]}}^{\tau^{[i]}} |f'(\widetilde{t})|^p \cdot |\delta(t)|^p \, dt \geqslant c_1^p \, \delta^{p+1}. \quad \text{Q.E.D.}$$

Theorem 3.2. *Suppose a L_p, k, α-g.p.f. $x(.)$ satisfies the condition of Lemma 3.2 b) on K_o, and moreover* $\sup \dfrac{\sum_{\nu=0}^{m-1} \alpha^{\nu p}}{\sum_{\nu=0}^{m} l_\nu} = \infty$. *Then* $x(.) \overline{\in}\, p\text{-}C_1(R_+)$.

Proof. For sufficiently small $\delta > 0$ we have

$$J(T, \delta) = \frac{1}{T} \int_0^T \Big| x(t+\delta) - x(t) \Big|^p dt \geqslant \frac{1}{T} \int_0^{k_m} \cdot =$$

$$= \frac{1}{T} \sum_{\nu=0}^{m-1} \int_{k_\nu}^{k_{\nu+1}} \cdot \geqslant \frac{1}{T} \sum_{\nu=0}^{m-1} \int_{k_\nu}^{k_{\nu+1}-\delta} |x(t+\delta) - x(t)|^p \, (dt) \geqslant$$

$$\geqslant \underline{k'} \cdot \frac{1}{T} \cdot \sum_{\nu=0}^{m-1} \alpha^{\nu p} \cdot \int_0^{h^{[\nu]}(k_{\nu+1}-\delta)} |x(h^{[\nu]}(k^{[\nu]}(t) + \delta)) - x(t)|^p \, dt.$$

Note that $h^{[\nu]}(k_{\nu+1} - \delta)$ tends to k_1 uniformly in ν when $\delta \to 0$ (since $k_1 - h^{[\nu]}(k_{\nu+1} - \delta) = h^{[\nu]}(k_{\nu+1}) - h^{[\nu]}(k_{\nu+1} - \delta) = (h^{[\nu]})'(\widetilde{t}) \cdot \delta \leqslant \delta / \underline{k'}$, $\forall \nu$).

Consider the functions $h_\delta^{[\nu]}(t) = h^{[\nu]}(k^{[\nu]}(t) + \delta)$, $t \in K_o$.

We have $h_\delta^{[\nu]}(h^{[\nu]}(k_{\nu+1} - \delta)) = h^{[\nu]}(k_{\nu+1}) = k_1$; $\quad (h_\delta^{[\nu]})'(t) > 0$, $\forall \nu, t$:
$h^{[\nu]}(k^{[\nu]}(t)) + \delta) - t = h^{[\nu]}(k^{[\nu+1]}(t) + \delta) - h^{[\nu]}(k^{[\nu]}(t)) = (h^{[\nu]})'(\widetilde{t}) \cdot \delta$,

Consequently $\quad \delta/\overline{K} \leqslant h_{\delta}^{[\nu]}(t) - t \leqslant \delta/k'$.

Therefore by Lemma 3.2b) we get

$$\int\limits_{0}^{h^{[\nu]}(k_{\nu+1}-\delta)} \left| x\,(h_{\delta}^{[\nu]}(t)) - x(t)\,\right|^{p} dt \geqslant c\bar{\delta}^{1+p}$$

(here $\bar{\delta}$ may be taken $= \delta/k'$) and

$$J(T,\,\delta) \geqslant c_1\delta^{1+p} \sum_{\nu=0}^{m-1} \alpha^{\nu p} \Big/ \sum_{\nu=0}^{m} l_\nu \,.$$

From the condition of the Theorem it follws that $\lim\limits_{T\to\infty} \sup J\,(T,\,\delta) = 0$ for every sufficiently small $\delta > 0$. Q. E. D.

§IV. THE CONNECTION WITH THE CLASS D^p-APF (B^p-APF)

For convenience we give the following definition that belongs to R. Doss (see[7]).

Definition 4 1. A function $x\,(.)\,:\,R \to R$ of class L_p is called D^p-almost periodic (D^p-a . p . f.) iff three following coditions are satisfied :

1) $$\limsup_{\delta\to 0} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |\,x\,(t+\delta) - x(t)\,|^p \; dt \right\} = 0\,;$$

2) for every $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods τ for which

$$\lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |\,x(t+\tau) - x\,(t)\,|^p \; dt < \varepsilon\,;$$

3) for every $a>0$ we can construct a periodic function $x_a(.)$ with period a for which

$$\lim_{n\to\infty} \left\{ \limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left| \frac{1}{n} \sum_{\nu=0}^{n-1} x(t+\nu a) - x_a(t) \right|^p dt \right\} = 0.$$

In [7] R. Doss had proved that the class D^p-APF of all D^p-a.p.f.s coincides with the class B^p-APF of all almost periodic functions defined by Besicovitch (see [8]).

We will consider the functions $x(.) \in L_p$, k, 1-a.p.f. Having the following form (Fig.2).
(That is Fig. 1 in which there is only one interval $l_1 = l + \delta$, and all others, including the intervals on the negative semiaxis, are equal to l).

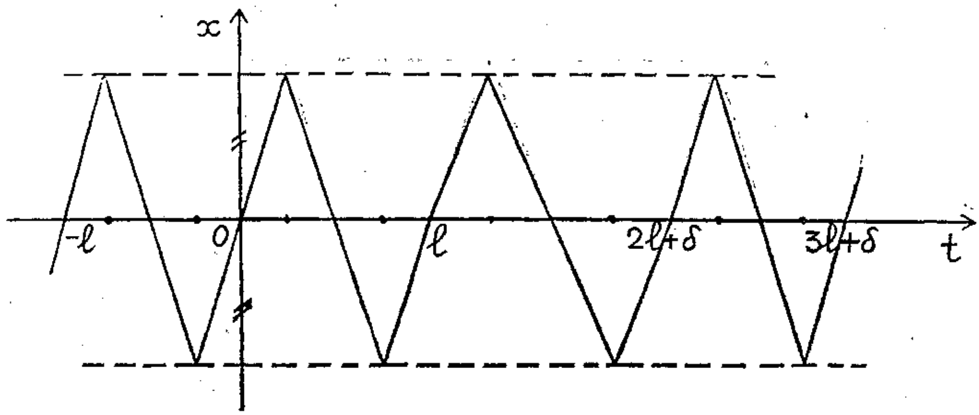

Fig.2

Theorem 4.1. *Suppose a L_p, k, 1-a.p.f. $x(.)$ has the form as in Fig.2. Then $x(.) \overline{\in} D^p$-APF.*

Proof. From Theorem 3; 1 we can see that the properties 1) and 2) in definition 4.1 are satisfied. We now verify 3).

Taking $a = l$, we whould get

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| \frac{1}{n} \sum_{\nu=0}^{n-1} x(t+\nu l) - x_1(t) \right|^p dt \underset{n \to \infty}{\longrightarrow} 0 ,$$

where $x_1(.)$ is a periodic function with period l.

Choosing $T = ml + \delta$, $m \geqslant 2$, we have

$$\frac{1}{2T} \int_{-T}^{T} \left| \frac{1}{n} \sum_{\nu=0}^{n-1} x(t+\nu l) - x_1(t) \right|^p dt \geqslant \frac{1}{2(ml+\delta)} \int_{2l+\delta}^{ml+\delta} \cdot =$$

$$= \frac{m-2}{2(ml+\delta)} \int_{2l+\delta}^{3l+\delta} \left| x(t) - x_1(t) \right|^p dt .$$

It follows that $x_1(t) = x(t)$ for a.e $t \in [2l + \delta, \infty)$. But $x_1(.)$ is a periodic function with period l, consequently $x_1(t)$ must differ from $x(t)$ at least on a set of positive measure which is contained in $[-l, 0]$. Therefore for any n, taking $T = ml$, $m > n$, we have

$$\frac{1}{2T}\int_{-T}^{0}\left|\frac{1}{n}\sum_{\nu=0}^{n-1}x(t+\nu l)-x_1(t)\right|^p dt \geqslant \frac{1}{2ml}\int_{-ml}^{-nl} \cdot =$$

$$=\frac{1}{2ml}\int_{-ml}^{-nl}\left|x(t)-x_1(t)\right|^p dt=\frac{m-n}{2ml}\int_{-1}^{0}\left|x(t)-x_1(t)\right|^p dt$$

$$\xrightarrow[T\to\infty]{}\frac{1}{2}\int_{-1}^{0}\left|x(t)-x_1(t)\right|^p dt>0.$$

This shows that the condition 3) cannot hold. Q.E.D.

§V. SMOOTHNESS OF FUNCTIONS OF CLASS APF

1. We first study smoothness of function $k(.)$.

In many cases we need the property that was mentioned in Theorem 3.1, i.e. $0<\underline{k}'\leqslant(k^{[\nu]})'(t)\leqslant\bar{k}'<\infty$, $\forall\nu=1,2,\dots,\ t\in K_0$, or even more: in every interval K_ν we can put

$$k'(t)=k'_\nu,\ (0<\inf_\nu k'_\nu\leqslant\sup_\nu k'_\nu<\infty,\ 0<\prod_\nu k'_\nu<\infty).$$

The following example shows that the class of these functions is sufficently wide.

Take

$$k'(t)=\begin{cases} c_i^- \text{ for } t\in[k_{i-1}+\delta_{i-1}^+,\ k_i-\delta_i^-]; \\ c_i^+ \text{ for } t\in[k_i+\delta_i^+,\ k_{i+1}-\delta_{i+1}^-]; \\ \text{interval connecting two points } [k_i-\delta_i^-,\ c_i^-] \\ \text{and } [k_i+\delta_i^+,\ c_i^+] \text{ for } t\in[k_i-\delta_i^-,\ k_i+\delta_i^+], \end{cases}$$

where $\delta_i^{\pm}$ are sufficiently small, $0<\inf_i c_i^{\pm}\leqslant\sup_i c_i^{\pm}<\infty$, $0<\prod_i c_i^{\pm}<\infty$.

Putting $k(t)=k(0)+\int_0^t k'(s)ds$, we have the desired function.

We can also point out such functions $k(.)$ from the class C^∞, i. e. connect two horizontal semilines so that the received curve belongs to C^∞. In fact, we solve this problem for two arbitrary semilines. It is obvious that we must only consider the case when one of these semilines is horizontal. For example, we must connect two semilines $y = 0$, $x \leqslant -1$ and $y = x$, $x \geqslant 1$ so that the mentioned above requirements are satisfied. Consider the following function

$$f(t) = \begin{cases} \exp\left(\frac{1}{t^2 - 1/4}\right), & |t| \leqslant 1/2 \\ 0, & |t| \geqslant 1/2 \end{cases}$$

It is well-known that $f(.) \in C^\infty$. Therefore the convolution $\varphi * f(.) \in C^\infty$ ($\varphi(.)$ will be fixed below). Take

$$F(x) = (cx) \cdot (\varphi * f)(x) = cx. \int_R \varphi(t) f(t - x)\, dt = cx. \int_{|t-x| \leqslant 1/2} \varphi(t) f(t - x)\, dt.$$

It is clear that $F(x) = 0$ for $x \leqslant -1$. $F(.) \in C^\infty$.

Put now $\varphi(t) = 1$ for $t \geqslant 1/2$ and $\varphi(t) \geqslant 0$, $\forall t$. Then for $x \geqslant 1$ we have

$$F(x) = cx \,. \int_{|t-x| \leqslant 1/2} f(t - x)\, dt = cx \,. \int_{-1/2}^{1/2} f(t)\, dt.$$

Hence, taking $c^{-1} = \int_{-1/2}^{1/2} f(t)dt$, we conclude that $F(.)$ is the connecting function.

2. Assume $k(.)$ is sufficiently smooth. We study now smoothness of a k, α — a.p.f. $f(.)$. Clearly we must only pay our attention on the points $k_i = k^{[i]}(0)$. We have

$$f'_-(k_i) = \lim_{t \to k_i - 0} \frac{f(t) - f(k_i)}{t - k_i} = \lim_{t \to k_i - 0} \frac{[f(h^{[i-1]}(t)) - f(k_1)] \,.\, \alpha^{i-1}}{[h^{[i-1]}(t) - k_1] \,.\, (k^{[i-1]})'(\xi)}$$

where $\xi \in (h^{[i-1]}(t), k_1)$.

Since $t \to k_i - 0 \Leftrightarrow h^{[i-1]}(t) \to k_1 - 0$ we get

$$f'_-(k_i) = f'_-(k_1) \,.\, \frac{\alpha^{i-1}}{(k^{[i-1]})'(k_1)} \,.$$

Analogously

$$f'_+(k_i) = f'_+(0) \,.\, \frac{\alpha_i}{(k^{[i]})'(0)} \,.$$

For $f(.) \in C^1$ at the point $t = k_i$ it is sufficient and necessary that

$$f'(k_1) \,.\, (k^{[i]})'(0) = \alpha f'_+(0) \,.\, (k^{[i-1]})'(k_1). \tag{5.1}$$

Considering $f|_{K_o}(.)$, we see that (5.1) is equivalent to

$$\frac{\alpha f'(0)}{f'(k(0))} = \frac{\prod_{\nu=o}^{i-1} k'(k^{[\nu]}(0))}{\prod_{\mu=o}^{i-2} k(k'^{[\mu]}(k(0)))} = k'(0),$$

i.e.
$$f'(k(0)) = \frac{\alpha}{k'(0)} \cdot f'(0). \tag{5.2}$$

For studying the derivatives of higher orders we put $f'(.)=f_{(1)}(.)$. We have

$$f_{(1)}(k(t)) = \frac{df}{dt}(k(t)) = \lim_{\bar{t} \to k(t)} \frac{\bar{f}(t) - f(k(t))}{\bar{t} - k(t)} = \alpha . \lim_{\tilde{t} \to t} \frac{f(\tilde{\bar{t}}) - f(t)}{(\bar{t} - t).k'(\xi)}$$

where $\tilde{t} = h(\bar{t})$, ξ is between t and $\tilde{t}$.

From this it follows that

$$f_{(1)}(k(t)) = \frac{\alpha}{k'(t)} \cdot f_{(1)}(t). \tag{5.3}$$

This explains why a condition of form (5.3) is often present in k; α-g.p. differential systems of the first order.

We can also prove that

$$f_{(n)}(k(t)) = \frac{1}{[k'(t)]^n} f_{(n)}(t) + \sum_{\nu=1}^{n-1} H_\nu(t) f_{(n-\nu)}(t),$$

where

$$H_\nu(t) \equiv H_\nu\left(\frac{1}{k'(t)}, \left(\frac{1}{k'(t)}\right)', \dots \left(\frac{1}{k'(t)}\right)^{(\nu)}\right).$$

§ VI. NORMALIZATION OF SPACES GPF AND L_p-GPF

1. Clearly in the space GPF we can consider the norm

$$\|f\|_\infty = \max_{t \in R_+} |f(t)| = \max_{K_0} |f(t)|, \ (\alpha = 1)$$

In general, every norm defined in $\mathrm{GPF}|_{K_0}$ may be extended on whole R_+. In the defined above norm k-UC (see Def. 1. 1) is a closed set in C, and k, α-GPF is a closed set in k-UC.

2. We can define the following seminorms in L_p-GPF $(\alpha = 1)$

$$\|f\|_1^p = \limsup_{T \to +\infty} \frac{1}{T} \int_0^T |f(t)|^p \, dt,$$

$$\|f\|_2^p = \limsup_{n \to \infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \int_{k_\nu}^{k_{\nu+1}} |f(t)|^p \, dt.$$

If $\limsup_{n\to\infty} \dfrac{\sum_{\nu=0}^{n-1} k^{[\nu]'}}{\sum_{\nu=0}^{n} l_\nu} > 0$, then the seminorm $\|.\|_1$ will be a norm.

If $\lim_{n\to\infty} \dfrac{1}{n} \sum_{\nu=0}^{n-1} k^{[\nu]'} > 0$, then the seminorm $\|.\|_2$ will be a norm.

It follows immediately, if we note that for every T there exists n such that $T \in K_n$ and then

$$\frac{1}{T}\int_0^T |f(t)|^p\, dt \geqslant \frac{1}{\sum_{\nu=0}^{n} l_\nu} \sum_{\nu=0}^{n-1} \int_{k_\nu}^{k_{\nu+1}} |f(t)|^p\, dt,$$

moreover

$$\int_{k_\nu}^{k_{\nu+1}} |f(t)|^p\, dt = \int_0^{k_1} |f(t)|^p \cdot (k^{[\nu]})'(t)\, dt.$$

Theorem 6.1. *Assume* $0 \leqslant k' \leqslant (k^{[\nu]})'(t) \leqslant \overline{k}' < \infty$, $\forall \nu = 1, 2, \ldots$, $t \in K_0$. *Then both norms* $\|.\|_1$ *and* $\|.\|_2$ *are equivalent to the norm*

$$\|f\|_0 = \left\| f\big|_{K_0} \right\|_{L_p}.$$

Proof. It is easy to see that thanks to the condition of the Theorem $\|.\|_1$, $\|.\|_2$ and $\|.\|_0$ are norms.

We now prove, for example, $\|.\|_1 \sim \|.\|_0$.

For every T there exists μ such that $T \in K_\mu$. We have

$$\frac{1}{T}\int_0^T |f(t)|^p\, dt \leqslant \frac{1}{T} \sum_{\nu=0}^{\mu} \int_{k_\nu}^{k_{\nu+1}} |f(t)|^p\, dt =$$

$$= \frac{1}{T} \sum_{\nu=0}^{\mu} \int_0^{k_1} |f(t)|^p \cdot (k^{[\nu]})'(t)\, dt \leqslant \frac{\|f\|_0^p}{T} (1 + \mu \overline{k}') \leqslant \frac{1 + \mu \overline{k}'}{l_0 [1 + (\mu - 1) k']} \|f\|_0^p$$

Analogously

$$\frac{1}{T}\int_0^T |f(t)|^p\, dt \geqslant \frac{1}{k_{\mu+1}} \int_0^{k_\mu} |f(t)|^p \geqslant \frac{1 + (\mu - 1) k'}{l_0 + (1 + \mu \overline{k}')} \|f\|_0^p.$$

From this the statement of the Theorem follows. Q. E. D.

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