ACTA MATHEMATICA VIETNAMICA
TOM 4, №1 (1979)

STABLE RANDOM MEASURES

NGUYỄN VĂN THU
Institute of Mathematics, Hanoi

SUMMARY. In this note we prove a representation theorem for stable random measures on a locally compact second coutable, Hausdorff topological space.

Through the paper we shall preserve the terminology and notation in [1] Recall some of them : Let б denote a locally compact second coutable Hausdorff topological space. Further, let $\mathfrak{F}$ denote the class of all Borel functions $\sigma :\to [0, \infty]$, $\mathcal{M}$ the class of all Random measures on б, $\mathcal{M}_b$ the class of all totally bounded measures in $\mathcal{M}$. In the sequel we shall consider $\mathcal{M}$ as a separable metric space with the vague topology. By a random measure on б we mean a probability measure on the б — algebra of all Borel subsets of $\mathcal{M}$. Given a function $g \in \mathfrak{F}$ and a random measure ξ we define a random measure $Tg\,\xi$ on б by

$$(Tg\,\xi)(E) = \xi(\{\mu : g\mu \in E\})$$

(E — a Borel subset of $\mathcal{M}$). In particular, if g is identically equal to a constant $c > 0$ the transform Tg will be denoted by T_c

A random measure ξ is called stable if for every $k = 1,2,...$ there is a positive number a_k such that

$$\underbrace{\xi * \ldots * \xi}_{k\text{ — times}} =: \xi^{*k} = T a_k \xi \tag{1}$$

where the asterisk $*$ denotes the convolution operation.

From this definition it follows that every stable random measure is infinitely divisible. Let L_ξ denote the Laplace transform of an infinitely divisible random measure ξ. By virtue of Theorem 6.1 [1] we get the formula

$$-\log L_\xi(f) = \alpha f - \lambda(1 - e^{-\pi} f)\ (f \in \mathfrak{F}) \tag{2}$$

where $\alpha \in \mathcal{M}$, λ is a measure on $\mathcal{M} \setminus \{0\}$ satisfying the condition

$$\lambda(1 - e^{-\pi} \dot{B}) < \infty \tag{3}$$

for every bounded Borel subset B of б. In what follows (α, λ) will be called canonical measures of ξ.

Let ξ be a stable random measure on δ with the Laplace transform given by (2). From (1) and from the uniqueness of the representation (2) we get the equations:

$$a_k \alpha = k\alpha \tag{4}$$

and

$$k\lambda = Ta_k \lambda \qquad (k = 1, 2, \ldots) \tag{5}$$

The symbol $Ta_k \lambda$ is obvious. Suppose now that $\xi \neq \sigma_0$ then either $\alpha \neq 0$, $\lambda = 0$ and $a_k = k$ $(k = 1, 2, \ldots)$ or $\alpha = 0$

$\lambda \neq 0$ and there is a constant $0 < c < 1$ such that $a_k = k^{\frac{1}{c}}$ $(k = 1, 2, \ldots)$.

Consider the case $\lambda \neq 0$ and suppose that the random measure ξ is supported by $\mathcal{M}_b$. By virtue of the proof of Theorem 6.1 ([1], p. 38) it follows that the measure λ'' on $\mathcal{M} \setminus \{o\}$ defined by the formula

$$\lambda'' (d\mu) = (1 - e^{-\mu\sigma}) \lambda(d\mu) \tag{6}$$

is finite. Consequently, by the proof of Theorem 6.1 ([1], p. 39) there is some continuous and strictly positive function g on δ such that the measure $Tg\lambda''$ is supported by $\mathcal{M}_b$. which together with (6) implies that the measure $T_g \lambda$ is supported by $\mathcal{M}_b$. Moreover, by (5) we get the equation:

$$k\, Tg\lambda = Ta_k\, Tg\lambda \quad (k = 1,2,\ldots) \tag{7}$$

Let $\mathcal{M}_1$ denote the class of all probability measures on δ. For every Borel subset W of $\mathcal{M}_1$ and $0 < r < \infty$ we put

$$J(r,W) = Tg\lambda\, (\{\mu \in \mathcal{M}_b : \mu\sigma \geqslant v;\ \mu/_{\mu\sigma} \in W\}). \tag{8}$$

It should be noted, by the condition (3), that the right hand side of (8) is finite.

Furtheremore, from the equation (7) with $a_k = k^{\frac{1}{c}}$ $(0 < c < 1;\ k = 1,2,\ldots)$ we have:

$$k\, J(r,W) = J(r\, k^{-\frac{1}{c}}, W)$$

wich by a simple computation implies that

$$J\left(\left(\frac{k}{n}\right)^{\frac{1}{c}}, W\right) = \frac{n}{k} J(1, W) \quad (k, n = 1,2\ldots)$$

Since for every W $J(r,W)$ is decreasing in r on $(0, \infty)$ the last equation implies:

$$J(r, W) = r^{-c} J(1, W) \tag{9}$$

for all r and W. Now putting $I = \{\mu \in \mathcal{M}_b : r_1 \leqslant \mu\sigma < r_2, \mu/_{\mu\sigma} \in W\}$, $\beta'_g(W)$ $= {}_c{}^{-1} J(1, W)$

and taking into account the formula (9) we get the formulas

$$T_g \lambda(I) = J(r_1, W) - J(r_2, W)$$

$$= \int_W \int_{r_1}^{r_2} \frac{dn}{n^{1+c}} \beta'(ds) \tag{10}$$

$$= \int_{\mathcal{M}_b} \chi_I(r,s) \frac{dr}{r^{1+c}} \beta'_g(ds)$$

where $\mu = rs$, $s = \mu/_{\mu\sigma} \in \mathcal{M}_1$.

Since all sets of type I form a semi—algebra of subsets of $\mathcal{M}_b \setminus \{o\}$ and they generate the Borel δ— algebra in $\mathcal{M}_b \setminus \{o\}$ it follws that for every Borel set A in $\mathcal{M}_b \setminus \{o\}$

$$(Tg(\lambda))(A) = \int_{\mathcal{M}_1} \chi_A (r,s \frac{dr}{r^{1+c}} \beta'_g(ds)$$

which together with (2) implies that

$$-\log L_\xi(f) = \lambda(1 - e^{-\pi} f)$$

$$= T_g \lambda(1 - e^{\pi f g^{-1}})$$

$$= \int_{\mathcal{M}_1} \int_0^\infty \left(1 - e^{-r\mu(fg^{-1})}\right) \frac{dr}{r^{1+c}} \beta'(d\mu)$$

$$= \int_{\mathcal{M}_1} [\mu(fg^{-1})]^c \beta(d\mu) \tag{11}$$

where $f \in \mathfrak{F}$ and $$\beta(d\mu) = \int_0^\infty \frac{1-e^{-x}}{x^{1+c}} dx\, \beta'g(d\mu)$$

$$= \frac{1}{c} \Gamma(1-c) \beta'(d\mu).$$

For general stable ξ (possibly ξ is not supported by $\mathcal{M}_b$) there exists some continuous and strictly positive function h on δ ([1], p. 39), such that $T_h \xi$ is supported by $\mathcal{M}_b$. Clearly, the random measure $T_h \xi$ is stable. Let

(α, λ) and (α_h, λ_h) be canonical measures of ξ and $T_h \xi$, respectively. Then if $\alpha = 0$ and $\lambda \neq 0$ it follows that $\alpha_h = 0$ and $\lambda_h \neq 0$. In this case, by virtue of (11) we get the formula

$$- \log L_\xi (f) = - \log L_{T_h \xi} (fh^{-1})$$

$$= \int_{\mathcal{M}_1} [\mu (fg^{-1} h^{-1})]^c \beta_h (d\mu)$$

$$= \int_{\mathcal{A}} [\mu (f)]^c T_{gh} \beta_h (d\mu) \tag{12}$$

for every $f \in \mathcal{F}$. Here $\mathcal{A}$ denotes the support of a finite measure $\beta =: T_{gh} \beta_h$.

It is clear that:

$$\mathcal{A} \subseteq \{ gh\mu : \mu \in \mathcal{M}_1 \} \subset \mathcal{M} \setminus \{ o \}.$$

Let $\mathcal{K}$ be a compact subset of б. Since the functions g and h are continuous and strictly positive we infer that

$$\sup_{\mu \in \mathcal{A}} \mu (\mathcal{K}) < \infty \tag{13}$$

Conversely, given a Borel subset $\mathcal{A}$ of $\mathcal{M} \setminus \{ o \}$ such that for every compact $\mathcal{K}$ of б the condition (14) holds and given a finite measure β on $\mathcal{M} \setminus \{ o \}$ supported by $\mathcal{A}$ the formula:

$$- \log L_\xi (f) = \int_{\mathcal{A}} [\mu (f)]^c \beta (d\mu) \qquad (o < c < 1) \tag{14}$$

defines a random measure ξ on б. In particular, this random measure must be stable. Thus we have proved the following theorem.

THEOREM 1. Let ξ be a stable random measure on δ. *Then there is a measure $\alpha \in \mathcal{M}$ such that*

$$- \log L_\xi (f) = \alpha f \qquad (f \in \mathcal{F}) \tag{15}$$

or there is a number $o < c < 1$, a Borel subset $\mathcal{A}$ of $\mathcal{M} \setminus \{o\}$ with the property (13), a finite Borel measure β on $\mathcal{A}$ and the formula (14) holds.

Conversely, for any α, β, $\mathcal{A}$, c mentioned above the formulas (14) and (15) define some stable random measures on б.

We now consider a particular case when the stable random measure ξ has independent increments. From Theorem 7. 2. ([1], p. 46) it follows that:

$$- \log L_\xi (f) = \alpha f + \int_{(o, \infty) \times б} (1 - e^{-xf(t)}) \gamma (dx, dt) \tag{16}$$

($f \in \mathfrak{F}$), where $\alpha \in \mathcal{M}$, γ is a Radon measure on the product $(0, \infty) \times \sigma$ such that for every Borel bounded subset B of б

$$\int_0^\infty (1 - e^{-x}) \, \gamma (dx, B) < \infty. \tag{17}$$

Moreover, if $\xi \neq \delta_0$ then either $\alpha \neq 0$ and $\gamma = 0$ or $\alpha = 0$ and $\gamma \neq 0$. Assume that $\gamma \neq 0$. For every number a > 0 we define a measure $T_a \gamma$ by

$$T_a \gamma (dx, dt) = \gamma (a^{-1} dx, dt)$$

Then, by virtue of (1), it follows that there is a number $o < c < 1$ such that for every k $=$ 1,2,... we have the equation

$$k \gamma = T_{a_k} \gamma \tag{18}$$

where $a_k = k^{\frac{1}{c}}$ (k = 1, 2,...). It is exactly the same as in the case of stable probability measures on $[o, \infty)$ it follows by the conditions (17) and (18), that for every bounded Borel subset B of б

$$\gamma (dx, B) = \frac{1}{x^{1+c}} dx \; \gamma (1, B) \tag{19}$$

Now putting $\mu (B) = \frac{\Gamma(1-c)}{c} \gamma (1, B)$ and taking into account the formulas (16) and (19) we have:

$$- \log L_\xi (f) = \int_б \int_0^\infty \left(1 - e^{-xf(t)} \right) \frac{dx}{x^{1+c}} \mu (dt)$$

$$= \mu (f^c) \qquad (f \in \mathfrak{F}).$$

Thus we have proved the following theorem:

THEOREM 2. Let ξ be a stable random measure on б with independent increments. Then there exists a Radon measure μ on б and a number $o < c \leqslant 1$ such that

$$- \log L_\xi (f) = \mu (f^c) \qquad (f \in \mathfrak{F}) \tag{20}$$

If ξ is non-degenerate then (μ, c) is uniquely determined by ξ.

Received April 20th, 1978.

REFERENCES

1. O. Kallenberg, *Random measures*, Akademie — Verlag. Berlin, 1976.
2. J. Kuelbs, *A representation theorem for symmetric stable processes stable and measures on H*, Z. Wahrscheinlichkeits theorie verw. Geb. 26, (259 — 271), 1973.