ACTA MATHEMATICA VIETNAMICA
TOM 4, №1 (1979)

ON SQUARE INTEGRABLE FACTOR REPRESENTATIONS OF LOCALLY COMPACT GROUPS

NGUYỄN HỮU ANH, VƯƠNG MẠNH SƠN

Institute of Polytechnics, Hanoi.

Let G be a separable locally compact unimodular group, Z a closed central subgroup of G. Let χ be a character of Z, a representation π of G is said to be a *χ-representation* if $\pi \mid Z$ is equivalent to a multiple of χ. A factor representation π of G (which must be a χ-representation for a certain χ) is said to be *square integrable mod Z* (abbrev. *SI mod Z*) if there exist non zero vectors φ, Ψ in the representation space $\mathcal{H}(\pi)$ such that:

$$\int_{G/Z} |(\pi(g)\varphi, \psi)|^2 d\bar{g} < \infty \qquad (1)$$

Note that the integration on G/Z makes sense since the integrand is constant on each Z-coset.

If π is a SI mod Z irreducible representation, (1) holds for all φ, ψ in $\mathcal{H}(\pi)$ and we have Schur's orthogonality relations. In this note we prove that a SI mod Z factor representations π is of type I if and only if (1) holds for all φ, ψ in $\mathcal{H}(\pi)$. In particular the unimodular Lie groups with factor representations SI mod Z in the latter sense are precisely those groups with SI mod Z irreducible representations which have been classified in [1]. In the second part of the note we give some necessary and sufficient conditions for a SI mod Z representation to be factor, and in particular to be irreducible.

1. A GENERALIZATION OF SCHUR'S ORTHOGONALITY RELATIONS

LEMMA 1. 1. If π is a χ-representation of G such that all matrix coefficients of π are SI mod z, then there exists a constant $C_\pi > O$ such that

$$\int_{G/z} |(\pi(g)\varphi, \psi)|^2 d\bar{g} \leqslant C_\pi \|\varphi\|^2 \|\psi\|^2, \ \varphi, \psi \in \mathcal{H}(\pi) \qquad (2)$$

Proof. For fixed φ and ψ the matrix coefficient $(\pi(g)\varphi, \psi)$ belongs to the hilbert space L^2_χ of all measurable functions f on G such that for all z in Z: $f(zx) = \chi(z) f(x)$ for almost all x, and $\int_{G/z} |f(x)|^2 d\bar{x} < \infty$. Let $\varphi_n \to \varphi$ in $\mathcal{H}(\pi)$ such that $(\pi(g)\varphi_n, \psi)$ is convergent in L^2_χ.

Since $(\pi(g)\varphi_n, \psi) \to (\pi(g)\varphi, \psi)$ for all g in G, $(\pi(g)\varphi_n, \varphi)$ also converges to $(\pi(g)\varphi, \psi)$ in L^2_χ. Thus by the closed graph theorem, the linear map $\varphi \to (\pi(.)\varphi, \psi)$ is continuous for every fixed ψ in $\mathcal{H}(\pi)$. Similarly, for every fixed φ the anti-linear map $\psi \to (\pi(.)\varphi, \psi)$ is also continuous. Therefore the sesquilinear map $(\varphi, \psi) \to (\pi(.)\varphi, \psi)$ is jointly continuous (cf [3], p. 83), and hence there exists $C_\pi > 0$ such that (2) holds.

PROPOSITION 1. 2. The assumptions beeing as in Lemma 1. 1, then (i) *forevery ψ, ψ' in $\mathcal{H}(\pi)$, there exists $d_\pi(\psi, \psi')$ in $\mathcal{L}(\mathcal{H}(\pi))$ such that*

$$\int_{G/Z} (\pi(g)\varphi, \psi)\overline{(\pi(g)\varphi', \psi')}\, d\bar{g} = (d_\pi(\psi, \psi')\varphi, \varphi') \tag{3}$$

(ii) $(\psi, \psi') \to d_\pi(\psi, \psi')$ *is a continuous sesquilinear mapping from*

$\mathcal{H}(\pi) \times \mathcal{H}(\pi)$ *into* $\mathcal{L}(\mathcal{H}(\pi))$.

(iii) *each* $d_\pi(\psi, \psi')$ *is an intertwining operator for* π.

Proof : (i) follows from Lemma 1.1 and Riesz representation theorem. Moreover we have the estimate

$$\left|\int_{G/Z} (\pi(g)\varphi, \psi)\overline{(\pi(g)\varphi', \psi')}\, d\bar{g}\right|^2 \leqslant \int_{G/Z} |(\pi(g)\varphi, \psi)|^2\, d\bar{g} \times \int_{G/Z} |(\pi(g)\varphi', \psi')|^2\, d\bar{g}$$
$$\leqslant C_\pi^2 \|\varphi\|^2 \|\psi\|^2 \|\varphi'\|^2 \|\psi'\|^2$$

This proves (ii). Finally, for every ψ, ψ' in $\mathcal{H}(\pi)$ we have

$$\begin{aligned}(d_\pi(\psi, \psi')\pi(g)\varphi, \varphi') &= \int_{G/Z} (\pi(x)\pi(g)\varphi, \psi)\overline{(\pi(x)\varphi', \psi')}\, d\bar{x} \\ &= \int_{G/Z} (\pi(x)\varphi, \psi)\overline{(\pi(xg^{-1})\varphi', \psi')}\, d\bar{x} \\ &= (d_\pi(\psi, \psi')\varphi, \pi(g^{-1})\varphi') \\ &= (\pi(g)d_\pi(\psi, \psi')\varphi, \varphi')\end{aligned}$$

i. e. $\quad d_\pi(\psi, \psi')\pi(g) = \pi(g)d_\pi(\psi, \psi') \qquad \forall\, g \in G$

Q. E. D.

Remark : if π is irreducible, then $d_\pi(\psi, \psi')$ is a scalar operator and hence (3) is just the ordinary Schur's orthogonality relations.

LEMMA 1. 3. The assumptions beeing as above. Then each multiple of π has all of its matrix coefficents SI mod Z.

Proof: Let $n\pi$ be a multiple of π so that the space $\mathcal{H}(n\pi)$ may be realized as $\mathcal{H}(\pi) \otimes \mathcal{K}$ where $\mathcal{K}$ is a n-dimensional[1] hilbert space. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{K}$, then for every φ in $\mathcal{H}(n\pi)$ we have $\varphi = \sum_i \varphi_i \otimes e_i$ where $\varphi_i \in \mathcal{H}(\pi)$ and $\| \varphi \|^2 = \sum_i \| \varphi_i \|^2$. The action of $n\pi$ on φ is given by $n\pi(g)\varphi = \sum_i \pi(g)\varphi_i \otimes l_i$. We have $(n\pi(g)\varphi, \varphi) = \sum_i (\pi(g)\varphi_i, \varphi_i)$ for every g in G. On the other hand it follows from Lemma 1.1 that

$$\| (\pi(g)\varphi_i, \varphi_i) \|^2 = \int_{G/Z} | (\pi(g)\varphi_i, \varphi_i) |^2 d\bar{g} \leqslant C_\pi \| \varphi_i \|^4$$

Hence $\sum_i \| (\pi(g)\varphi_i, \varphi_i) \| \leqslant \sqrt{C_\pi} \sum_i \| \varphi_i \|^2 = \sqrt{C_\pi} \| \varphi \|^2$

Therefore $(n\pi(g)\varphi, \varphi) = \sum_i (\pi(g)\varphi_i, \varphi_i) \in L^2_\chi$. By polarization we see that all matrix coefficients of $n\pi$ belong to L^2_χ.

Q.E.D.

2. *THE PROOF OF THE MAIN RESULT.*

Recall that L^2_χ is the space of all measurable functions f on G such that for all z in Z we have $f(zx) = \chi(z) f(x)$ for almost all x, and $\int_{G/Z} | f(x) |^2 d\bar{x} < \infty$.

The representation ρ of G defined by $\rho(g) f(x) = f(xg)$ is called the induced representation of χ and denoted by $\mathrm{ind}_{Z\uparrow G}\chi$. Let λ be the representation of G in L^2_χ given by $\lambda(g) f(x) = f(g^{-1}x)$. Let $\mathcal{K}(G)$ be the algebra of continuous functions with compact support on G. For every f in $\mathcal{K}(G)$ we set $f_0(x) = \int_Z \chi(z)^{-1} f(zx)\, dz$, $\forall x \in G$. The set M_χ of all f_0 so obtained is dense in L^2_χ and each f_0 has compact support mod Z (cf [4], Lemma 3. 5). For f_0, g_0 in M_χ, put:

(i) $f_0^*(x) = \overline{f_0(x^{-1})} \qquad \forall x \in G$

(ii) $(f_0 * g_0)(x) = \int_{G/Z} f_0(y)\, g_0(y^{-1}x)\, dy \quad \forall x \in G$

(iii) $(f_0, g_0) = \int_{G/Z} f_0(y)\, \overline{g_0(y)}\, d\bar{y}$

The addition and multiplication are defined pointwise.

---

(1) Here n can be an arbitrary cardinal, finite or infinite.

LEMMA 2.1. M_χ is a hilbert algebra.

Proof: Note that the multiplication $f_{o*}g_o$ in M_χ is well defined. Now it is clear that:

$$(f_o, g_o) = (g_o^*, f_o^*) \text{ and}$$

$$(f_{o*}g_o, h_o) = (g_o, f_{o*}^* h_o)$$

for all f_o, g_o, h_o in M_χ. Moreover

$$\begin{aligned}(f_{o*}g_o)(x) &= \int_{G/Z} f_o(y)\, g_o(y^{-1}x)\, d\bar{y} \\ &= \int_{G/Z}\int_Z f(zy)\chi(z)^{-1} g_o(y^{-1}x)\, dz\, d\bar{y} \\ &= \int_G f(y)\, g_o(y^{-1}x)\, dy = \lambda(f)\, g_o(x)\end{aligned}$$

Hence the mapping $g_o \to f_o * g_o$ is continuous in the prehilbert space M_χ. Finally we have

$$(f_o * g_o)_x(y) = (f_o * (g_o)_x)(y), \text{ and}$$

$$(\alpha f_o)(x) = (\alpha f)_o(x)$$

for all f, g in $\mathcal{K}(G)$, and α is a bounded continuous function on G which is constant on the Z—cosets, where f_x is the right translation of f by $x: f_x(y) = f(yx)$. Thus by Lemma 3.3 of [4] $M_\chi * M_\chi$ is dense in M_χ, and M_χ is a hilbert algebra (cf. [2], A 54)

Q.E.D.

Let A_χ be the perfect hilbert algebra of M_χ consisting of all bounded elements in L^2_χ. By the same computation as above we see that for all f in $\mathcal{K}(G)$ and all ξ in L^2_χ: $(f_o * \xi)(x) = \lambda(f)\xi(x)$. Similarly: $(\xi * f_o)(x) = \rho(\check{f})\xi(x)$ where $\check{f}(x) = f(x^{-1})$. Hence $\mathcal{U}(A_\chi) = \lambda(G)'$ and $\mathcal{V}(A_\chi) = \rho(G)'$. But $\mathcal{U}(A_\chi) = \mathcal{V}(A_\chi)'$ (cf. [2], A54), therefore $\lambda(G)' = \rho(G)''$.

Recall that if ρ is a representation of G, and π is a subrepresentation of ρ defined by a projection $E \in \rho(G)'$, then there exists a unique projection F in the center of $\rho(G)'$ such that $E \leqslant F$ and F is minimal amongs the projections lying in the center of $\rho(G)'$ majorizing E; the projection F is called the central support of π. The representation π is a factor if and only if F is minimal in the center of $\rho(G)'$.

Now assume that π is a *SI* mod Z factor representation of G, than π is quasi equivalent to a subrepresentation π' of $\rho = \mathrm{ind}_{G\uparrow Z}\chi$ (cf.[5]). Let π'' be the subrepresentation of ρ corresponding to the central support F of π'. Since F is minimal in the center of $\rho(G)'$, π'' is a factor and $\pi \sim \pi' \sim \pi''$. Assume in addition that all matrix coefficients of π are *SI* mod *Z*. By Lemma 1.3 all matrix coefficients of $\infty\pi$ are *SI* mod *Z* and hence so are those of π''. Thus for all ξ and η in $\mathcal{H}(\pi'') = FL^2_\chi$ we have: $(\eta^* * \xi)(g) = (\eta^*, \lambda(g)\xi^*) = (\eta^*, (\rho(g)\xi)^*) = (\pi''(g)\xi, \eta) \in L^2_\chi$. Since $(\eta^* * \xi)(g) = (\rho(g)\xi, \eta) = 0$ if ξ is perpendicular to FL^2_χ we see that for every fixed η in FL^2_χ, $\xi \to \eta^* * \xi$ is an every where defined linear operator from L^2_χ into itself. Therefore by the same argument as in the proof of Lemma 1.1 we see that this operator is continuous, i.e. $\eta^* \in A_\chi$. Finally, since $F\ L^2_\chi$ is clearly self conjugate, $FL^2_\chi = FA_\chi$ is a complete hilbert algebra. Hence the Von Neumann algebra $\mathcal{U}(A_\chi)_F = \mathcal{U}(F, A_\chi)$ is of type I, i. e. π is of type I (cf. [2], A 65). Thus we have proved the first part of.

THEOREM 2. 2. Let π be a SI mod Z factor representation of G, then π is of type I if and only if all of its matrix coefficients are SI mod Z, i. e. for all φ, Ψ in $\mathcal{H}(\pi)$

$$\int_{G/Z} |(\pi(g)\varphi, \Psi)|^2 \, d\dot{g} < \infty$$

Proof: It remains to prove the necessary condition. Assume that π is of type I so that π is equivalent to a multiple of some irreducible representation π_0 of type I so that π is equivalent to a multiple of some irreducille representation π_0 of *G*. Since π is quasi equivalent to a subrepresentation of $\mathrm{ind}_{Z\uparrow G}\chi$, so is π_0, i.e. π_0 is *SI* mod *Z*. Therefore by Schur's orthogonality relations, all matrix coefficients of π_0 are *SI* mod *Z*. Thus by Lemma 1. 3 all matrix coefficients of π are also *SI* mod *Z*.

Q. E. D.

3. In view of Theorem 2. 2, it would be interesting if we could find conditions for a representation with all matrix coefficients *SI* mod *Z* to be a factor, and in particular to be irreducible. First note that if π is the sum of two non-equivalent *SI* mod *Z* irreducible representations then all matrix coefficients of π are *SI* mod *Z* but π is not a factor. Now let π be a χ-representation such that all of its matrix coefficients are *SI* mod *Z*. For all ψ and ψ' in $\mathcal{H}(\pi)$, let $d_\pi(\psi, \psi')$ be the intertwining operator for π determined by Lemma 1. 3.

Put $\mathcal{D} = \{ d_\pi(\psi, \psi') \mid \psi, \psi' \in \mathcal{H}(\pi) \}$. It is easy to see that $\mathcal{D}$ is self conjugate.

Hence $\mathcal{A} = \mathcal{D}''$ is a Von Neumann algebra. We have

PROPOSITION 3. 1. π is a factor if and only if $\mathcal{A}$ is a factor. Moreover if it is the case then $\mathcal{A} = \pi(G)'$.

Proof: Assume that π is a factor, then π is of type I by Theorem 2. 2, hence we may assume that $\pi = \pi_0 \otimes \mathrm{Id}$, $\mathcal{H}(\pi) = \mathcal{H}(\pi_0) \otimes \mathcal{K}$, where π_0 is an irreducible representation and $\mathcal{K}$ is some hilbert space. Thus $\pi(G)' = C \otimes \mathcal{L}(\mathcal{K})$. In virtue of Proposition 1.2. $\mathcal{D} \subset \pi(G)'$, hence for every ψ, ψ' in $\mathcal{H}(\pi)$, there exists an operator $d'_\pi(\psi, \psi') \in \mathcal{L}(\mathcal{K})$ such that $d_\pi(\psi, \psi') = \mathrm{Id} \otimes d'_\pi(\psi, \psi')$.

Set $\varphi = \varphi_o \otimes \varphi_1$, $\psi = \psi_o \otimes \psi_1$, $\varphi' = \varphi'_o \otimes \varphi'_1$, $\psi' = \psi'_o \otimes \psi'_1$ in (3) we obtain

$$C_\pi \overline{(\psi_o, \psi'_o)} (\varphi_1, \psi_1)(\psi'_1, \varphi_1') = (d'_\pi(\psi, \psi') \varphi_1, \varphi'_1),$$

where $\frac{1}{C_\pi}$ is the formal degree (or dimension) of π_o, i. e.

$$d'_\pi(\psi, \psi') \varphi_1 = C_\pi \overline{(\psi_o, \psi'_o)} (\varphi_1, \psi_1) \psi'_1 \qquad \text{for all } \varphi_1, \psi_1, \psi'_1 \in \mathcal{K}$$

From this it follows that $\mathcal{A} = \pi(G)' \simeq \mathcal{L}(\mathcal{K})$.

Conversely assume that $\mathcal{A}$ is a factor. Let E be any projection belonging to the center of $\pi(G)'$, we will prolve that $E = 0$ or 1.

Note that $E \in \mathcal{A}'$ since $\pi(G)'' \subset \mathcal{D}' \subset \mathcal{A}'$. Now let $A \in \mathcal{A}'$, then:

$$\begin{aligned}
(d_\pi(\psi, \psi') \varphi, E A \varphi') &= (d_\pi(\psi, \psi') E \varphi, A\varphi') \\
&= \int_{G/Z} (\pi(g) E\varphi, \psi) \overline{(\pi(g) A \varphi', \psi')}\, d\bar{g} \\
&= \int_{G/Z} (\pi(g) \varphi, E\psi) \overline{(\pi(g) A \varphi', \psi')}\, d\bar{g} \\
&= (d_\pi(E\psi, \psi') \varphi, A\varphi') \\
&= (d_\pi(E\psi, \psi') A^* \varphi, \varphi') \\
&= \int_{G/Z} (\pi(g) A^* \varphi, E\psi) \overline{(\pi(g) \varphi', \psi')}\, d\bar{g} \\
&= \int_{G/Z} (\pi(g) EA^* \varphi, \psi) \overline{(\pi(g) \varphi, \psi')}\, d\bar{g} \\
&= (d_\pi(\psi, \psi') AE^* \varphi, \varphi') \\
&= (d_\pi(\psi, \psi') \varphi, AE\varphi')
\end{aligned}$$

Since the vectors $d_\pi(\psi, \psi')\varphi$ where $\psi, \psi', \varphi \in \mathcal{H}(\pi)$ span $\mathcal{H}(\pi)$, we see that $AE = EA$, i. e. $E \in \mathcal{A}'' = \mathcal{A}$. Therefore $E \in \mathcal{A} \cap \mathcal{A}' = \Phi.\mathrm{Id}$, and hence $E = 0$ or 1.

Q.E.D.

COROLLARY 3. 2. π is irreducible if and only if for all φ, ψ in $\mathcal{H}(\pi)$ we have

$$\int_{G/Z} |(\pi(g)\varphi,\psi)|^2 \, d\bar{g} = C_\pi \, \|\varphi\|^2 \|\psi\|^2, \tag{4}$$

where C_π is a positive constant.

Proof: the necessary condition follows immediately from Schur's orthogonality relations. To prove the sufficient condition we observe that it follows from (3) and (4):

$$(d_\pi(\psi,\psi)\varphi,\varphi) = (C_\pi(\psi,\psi)\varphi,\varphi)$$

Thus by polarization:

$$d_\pi(\psi,\psi) = C_\pi \, \|\psi\|^2 \, \mathrm{Id}$$

Hence $\mathcal{A} \simeq \mathbf{C}$ and $\pi(G)' = \mathcal{A} \simeq \mathbf{C}$, i. e. π is irreducible.

Q.E.D

Remark: the sufficient condition may also be proved directly by observing that if π is not irreducible then we can select φ and ψ in two non zero mutually orthogonal invariant subspaces of $\mathcal{H}(\pi)$ so that

$\int_{G/Z} |(\pi(g)\varphi,\psi)|^2 \, d\bar{g} = 0$, while (4) implies that

$\int_{G/Z} |(\pi(g)\varphi,\psi)|^2 \, d\bar{g} = C_\pi \|\varphi\|^2 \|\psi\|^2 \neq 0$: contradiction

PROPOSITION 3. 3. Let π be a χ — representation such that

$$\int_{G/Z} |(\pi(g)\varphi,\varphi)|^2 \, d\bar{g} = C \|\varphi\|^4 \tag{5}$$

for all φ in $\mathcal{H}(\pi)$, where C is some positive constant.

Under this condition, if π is not irreducible then it is equivalent to a multiple of some one-dimensional representation of G. In particular G/Z must be compact.

Proof: it is easy to see that (5) implies that all matrix coefficients of π are *SI* mod Z. Let $\mathcal{A}$ be as above then $\mathcal{A} \subset \pi(G)'$. Let E be any projection belonging to the center off $\mathcal{A}$. Put $F = 1 - E$. It follows from (3) and (5) that:

$$(d_\pi(\varphi,\varphi)\varphi,\varphi) = C \|\varphi\|^4 \tag{6}$$

On the other hand:

$$\begin{aligned} C\|\varphi\|^4 &= C(\|E\varphi\|^2 + \|F\varphi\|^2)^2 \\ &= C\|E\varphi\|^4 + C\|F\varphi\|^4 + 2C\|E\varphi\|^2\|F\varphi\|^2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} (d_\pi(\varphi,\varphi)\varphi,\varphi) &= (d_\pi(\varphi,\varphi)E\varphi,E\varphi) + (d_\pi(\varphi,\varphi)F\varphi,F\varphi) \\ &= \int_{G/Z} |(\pi(g)E\varphi,\varphi)|^2 dg + \int_{G/Z} |(\pi(g)F\varphi,\varphi)|^2 d\bar{g} \\ &= \int_{G/Z} |(\pi(g)E\varphi,E\varphi)|^2 d\bar{g} + \int_{G/Z} |(\pi(g)F\varphi,F\varphi)^2 d\bar{g} \\ &= C\|E\varphi\|^4 + C\|F\varphi\|^4 \end{aligned} \tag{8}$$

Now it follows from (6), (7) and (8) that

$$\| E \varphi \|^2 \| F\varphi \|^2 = 0 \qquad \forall \varphi \in \mathcal{H}(\pi)$$

Therefore $E = 0$ or 1, i. e. $\mathcal{A}$ is a factor and hence π is a factor as indicated by Proposition 3.1. Moreover π is of type I by Theorem 2.2. Assume that π is not irreducible. Then we can select two mutually orthogonal minimal invariant subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}(\pi)$. Let T be a unitary intertwining operator between two equivalent irreducible subrepresentations of π corresponding to $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively so that $\pi(g)T\varphi = T\pi(g)\varphi$, $\forall \varphi \in \mathcal{H}^1$ By Schur's orthogonality relations we have

$$\int_{G/Z} (\pi(g)\varphi_1, \varphi_1)\overline{(\pi(g)\varphi_2, \varphi_2)}\, d\bar{g} =$$

$$\int_{G/Z} (\pi(g)\varphi_1', \varphi_1)\overline{(\pi(g)T^{-1}\varphi_2, T^{-1}\varphi_2)}\, d\bar{g} = C\, |(\varphi_1, T^{-1}\varphi_2)|^2 \qquad (9)$$

Note that the formal degree of $\pi \mid \mathcal{H}_1$ is just C. On the other hand it follows from (5) that

$$C \| \varphi_1 + \varphi_2 \|^4 = \int_{G/Z} |(\pi(g)\varphi_1 + \varphi_2, \varphi_1 + \varphi_2)|^2\, d\bar{g}$$

$$= C \|\varphi_1\|^4 + C \|\varphi_2\|^4 + 2\,\mathrm{Re}\int_{G/Z} (\pi(g)\varphi_1, \varphi_1)\ \overline{(\pi(g)\varphi_2, \varphi_2)}\, d\bar{g}$$

i. e.

$$\mathrm{Re}\int_{G/Z} (\pi(g)\varphi_1, \varphi_1)\overline{(\pi(g)\varphi_2, \varphi_2)}\, d\bar{g} = C \|\varphi_1\|^2 \|\varphi_2\|^2 \qquad (10)$$

Now (9) and (10) imply:

$$|(\varphi_1, T^{-1}\varphi_2)|^2 = \|\varphi_1\|^2 \|T^{-1}\varphi_2\|^2 \qquad \forall \varphi_1 \in \mathcal{H}_1,\ \forall \varphi_2 \in \mathcal{H}_2$$

Thus by Schwartz — Cauchy inequality we must have $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 1$. Moreover in this case G/Z must be compact or, otherwise π can not be *SI* mod Z.

Q.E.D.

Received April 15th, 1978.

REFERENCES

1. N. H. Anh, *Classification of unimodular Lie groups with square integrable representations.* Preprint, the Institute of Polytechnics, Hanoi, 1978.

2. J. Dixmier, *Les C^*—algèbres et leurs représentations.* Gauthier Villars, Paris, 1964.

3. N. Dunford and J. T. Schwarz, *Lineear operators,* vol 1 Interscience Publishers, New York, 1958.

4. G. W. Mackey, *Induced representations of locally compact groups I,* Ann. of Math. Vol 55 (1952), pp. 101 — 139.

5. J. Rosenberg, *Square integrable factor representations of locally compact groups.* Preprint, University of California, Berkeley, USA.