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ON SYMMETRIC STABLE MEASURES ON SPACES l_p $(1 \leqslant p < +\infty)$

ĐẶNG HÙNG THẮNG
NGUYỄN DUY TIẾN
Hanoi University.

1. INTRODUCTION AND NOTATIONS.

Let E be a real separable Banach space, E' the topological dual space and $\mathcal{B}(E)$ the Borel σ-algebra on E. By E-valued random variable (E-valued r.v.), defined on the basic probability space $(\Omega, \mathcal{A}, P)$ we mean a measurable mapping $\xi: (\Omega, \mathcal{A}) \longrightarrow (E, \mathcal{B}(E))$. The distribution of ξ, denoted by $\mathcal{L}(\xi)$ is a probability measure on E given by

$$\mathcal{L}(\xi)(A) = P\{\omega : \xi(\omega) \in A\}, \ A \in \mathcal{B}(E)$$

Let us recall that a real random variable X is said to have a symmetric stable distribution of index α $(0 < \alpha \leqslant 2)$ and scale factor b if the characteristic function of X is of the form

$$\varphi_X(t) = \exp\{-b\,|t|^\alpha\} \qquad (b \geqslant 0)$$

A probability measure μ on E is said to be symmetric and stable of index α if every element $x' \in E'$ considered as a real random variable on the probability space $(E, \mathcal{B}(E))$, μ has a symmetric stable distribution of index α

Since the case $\alpha = 2$ corresponds to the Gaussian case, in what follows, for simplicity of writing a symmetric stable probability measure of index α will be called an α — Gaussian measure. A E-valued r. v. whose distribution is an α — Gaussian measure will be called an E-valued α — Gaussian r.v.

Through all the paper we shall consider only α — Gaussian measures with $1 < \alpha \leqslant 2$. In section 2 we investigate the almost sure (a.s.) convergence (in norm topology) of the series

$$\sum_{n=1}^{\infty} \xi_n , \tag{1}$$

where $(\xi_n)_{n=1}^{\infty}$ is a sequence of independent l_p-valued α — Gaussian r.v.s $(1 \leqslant p < +\infty)$. In the case $\alpha = 2$ this problem has been treated by Nguyên Zuy Tiên in [1]. For every α — Gaussian measure μ on l_p we define an element

$x_\mu \in l_p$. The condition for the a.s. convergence of the series (1) is given in terms of $\{x_{\mu_n}\}$ where $\mu_n = \mathcal{L}(\xi_n)$. Namely, for the series (1) to be convergent a.s. it is necessary that

$$\sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} |< x_{\mu_n}, e^*_i >|^\alpha\right)^{p/\alpha} < +\infty, \tag{2}$$

where $\mu_n = \mathcal{L}(\xi_n)$, $(e^*_i)_{i=1}^{\infty}$ is a sequence of coordinate functionals.

Moreover, if

$$P < \alpha^* =: \begin{cases} +\infty & \text{if } \alpha = 2, \\ \alpha & \text{if } \alpha < 2, \end{cases}$$

then the condition (2) is both necessary and sufficient for the series (1) to be convergent a. s.. But in the case $P \geqslant \alpha^*$ the condition (2) generally needs not be sufficient and this will be shown with the help of an example.

In section 3 we treat the problem of relative compactness of a family $\mathcal{M}$ of α — Gaussian measures on l_p. In the case $\alpha = 2$ this problem has recently been treated by Nguyên Zuy Tien, V.I. Tarieladze, S.A. Chobanjan in (2) where Gaussian measures were considered on more general Banach spaces. Here we shall show that the relative compactness of a family $(x_{\mu \in \mathcal{M}})$ in l_p is the necessary condition for the relative compactness of the family $\mathcal{M}$. This condition is also sufficient when $P<\alpha^*$ and not sufficient when $P\geqslant\alpha^*$. (It is interesting to note thathere we have an analogous situation to that in section 2). As a corollary, we obtain the necessary and sufficient condition for a sequence $(\mu_n)_{n=1}^{\infty}$ of α — Gaussian measures in l_p $(1 \leqslant P < \alpha^*)$ to converge weakly to μ.

2. Convergence of sums of independent l_p — valued — Gaussian random variables.

At first let us note an important property of α — Gaussian measure on general Banach spaces (see [3]). Every α — Gaussian measure μ on E has a strong moment of p—th order for each $P < \alpha^*$ (i.e. $\int \| x \|^p_\mu \, (dx) < +\infty$ for all $P < \alpha^*$).

We begin by proving some needed lemmas:

2. 1. LEMMA. For every $0 < P<\alpha^*$ *there exists universal positive constants* $A_\alpha(p)$ *and* $B_\alpha(p)$ *such that for each* α *— Gaussian measure* μ *on* R^1 *with scale factor* b *we have.*

$$\int_{R^1} |x|^p \mu\,(dx) = b^{p/\alpha} A_\alpha(p) \tag{2—2}$$

$$\int_{R^1} |x|^p \mu(dx) = B_\alpha(p) \left(\int_{R^1} |x| \mu(dx) \right)^p \qquad (2\text{—}3)$$

Proof. Suppose that $\gamma^{(\alpha)}$ is a standard α — Gaussian r.v. (i.e. the ch. f. of $\gamma^{(\alpha)}$ is of the form $\exp\{-|t|^\alpha\}$). Then $\mu = \mathcal{L}(b^{1/\alpha}\gamma^{(\alpha)})$ and we have

$$\int_{R^1} |x|^p \mu(dx) = E|b^{1/\alpha}\gamma^{(\alpha)}|^p = b^{p/\alpha} E|\gamma^{(\alpha)}|^p = b^{p/\alpha} A_\alpha(p)$$

From (2—2), (2—3) follows.

2.2. *LEMMA. Let* $\left(\gamma_n^{(\alpha)}\right)_{n=1}$ *be a sequence of independent standard* α — *Gaussian r. v.'s. Then the series* $\sum_{n=1}^{\infty} C_n \gamma_n^{(\alpha)}$ ($C_n \in R^1$) *is convergent a.s. if and only if* $\sum_{n=1}^{\infty} |C_n|^\alpha < +\infty$.

Proof: The proof easily follows from the method of characteristic functions.

2. 3. *LEMMA.* (See [4]). Let $\left(\gamma_n^{(\alpha)}\right)_{n=1}^{\infty}$ *be a sequence of independent standard* α — *Gaussian r. v'.s.*

Then $\sum_{n=1}^{\infty} |C_n \gamma_n^{(\alpha)}|^p < +\infty$ *a.s. if and only if*

$$\sum_{n=1}^{\infty} |C_n|^p < +\infty \quad \text{in case } 1 \leqslant p < \alpha^*$$

$$\sum_{n=1}^{\infty} |C_n|^p \left(1 + \ln \frac{1}{|C_n|}\right) < +\infty \quad \text{in case } p = \alpha < 2$$

$$\sum_{n=1}^{\infty} |C_n|^\alpha < +\infty \qquad \text{in case } 2 \neq \alpha < p$$

2. 4. *LEMMA. Let* $(X_n)_{n=1}^{\infty}$ *be a sequence of real* α — *Gaussian r. v'. s such that for any integer* n *the vector* $(X_1, X_2, \ldots, X_n)$ *has an* α — *Gaussian distribution on* R^n.

a) *Suppose that* $\sum_{n=1}^{\infty} |X_n|^p < +\infty$ *a.s. Then we have*

$$\sum_{n=1}^{\infty} (E|X_n|)^p < +\infty \qquad (1 \leqslant p < +\infty)$$

b) *Conversely, if* $P < \alpha^*$ *then* $\sum_{n=1}^{\infty} (E|X_n|)^p < +\infty$

implies that $\sum_{n=1}^{\infty} |X_n|^p < +\infty$ *a.s.*

Proof.

a) By the above assumption $\xi = (X_n)_{n=1}^{\infty}$ is an l_p — valued α — Gaussian r.v. Consider the random variable $\hat{\xi} = (|X_n|)_{n=1}^{\infty}$. We have $E\|\hat{\xi}\| = E\|\xi\| < +\infty$ (because of $\alpha > 1$). Hence $E\hat{\xi}$ exists and it is equal to $(E|X_n|)_{n=1}^{\infty} \in l_p$ i.e. $\sum_{n=1}^{\infty} (E|X_n|)^p < +\infty$.

b) If $p < \alpha^*$ by (2-3) we have $E|X_n|^p = B_\alpha(p)\,(E|X_n|)^p$. Hence $\sum_{n=1}^{\infty} (E|X_n|)^p < +\infty$ implies $E\left(\sum_{n=1}^{\infty} |X_n|^p\right) < +\infty$ and consequently

$$\sum_{n=1}^{\infty} |X_n|^p < +\infty \text{ a.s.}$$

Consider the mapping $\rho: l_p \to l_p$ defined by

$$\rho(x) = (|x_n|)_{n=1}^{\infty}, \quad x = (x_n)_{n=1}^{\infty}$$

If μ is an α — Gaussian measure on l_p. We define a vector $x_\mu \in l_p$ b

$$x_\mu = \int_{l_p} \rho(x)\,\mu(dx)$$

This last integral is a Bochner's integral (it exists since

$$\int_{l_p} \|\rho(x)\|\,\mu(dx) = \int_{l_p} \|x\|\,\mu(dx) < +\infty)$$

2.5. *THEOREM. Let* $(\xi_n)_{n=1}^{\infty}$ *be a sequence of independent* l_p *— valued* α *— Gaussian r.v.'s* $(1 < \alpha \leqslant 2)$, $(1 \leqslant p < +\infty)$.

For the series $\sum_{n=1}^{\infty} \zeta_n$ *to be convergent a.s. (in norm topology) it is necessary that*

$$\sum_{i=1}^{\infty} \Big(\sum_{n=1}^{\infty} | < x_{\mu_n}, e_i^* > |^{\alpha} \Big)^{p/\alpha} < + \infty \qquad (2-4)$$

where $\mu_n = L(\xi_n)$, $(e_n^*)_{n=1}^{\infty}$ *is a sequence of coordinate functionals.*

Moreover, in the case $p < \alpha^*$ *the condition* $(2-4)$ *is necessary as well as sufficient.*

Proof. Suppose that the series $\sum_{n=1}^{\infty} \zeta_n$ converges a.s. to S. Then for each e_i^* we have

$$\langle S, e_i^* \rangle = \sum_{n=1}^{\infty} \langle \xi_n, e_i^* \rangle$$

By Lemma 2 — 2 $\langle S, e_i^* \rangle$ is an α — Gaussian r.v. and

$$(E | \langle S, e_i^* \rangle |^{\alpha}) = \sum_{n=1}^{\infty} (E | \langle \xi_n, e_i^* \rangle |)^{\alpha} = \sum_{n=1}^{\infty} | \langle x_{\mu_n}, e_i^* \rangle |^{\alpha}$$

On the other hand

$$\sum_{i=1}^{\infty} | \langle S, e_i^* \rangle |^p = \| S \|^p < + \infty \text{ a.s.}$$

Applying Lemma 2 — 4 we get the condition (2 — 4).

Now suppose that the condition (2-4) is satisfied and $p < \alpha^*$. By Lemma 2-2, for each e_i^* $(i = 1,2,...)$ the series $\sum_{n=1}^{\infty} \langle \xi_n, e_i^* \rangle$ converges a.s. Let $S_i = \sum_{n=1}^{\infty} \langle \xi_n, e_i^* \rangle$. Then by Lemma 2-4:

$$\sum_{i=1}^{\infty} | S_i |^p < + \infty \qquad \text{a. s.}$$

This implies the existence of a l_p-valued r. v'. s such that for each e_i^* $(i = 1, 2, \dots)$

$$< S, e^*_i > = \sum_{i=1}^{\infty} < \xi_n , e^*_i > \quad \text{a.s.}$$

Hence, from Ito-Nisio's theorem [5] it follows that

$$S = \sum_{n=1}^{\infty} \xi_n \quad \text{a. s.}$$

Remark 1: In the case $P \geqslant \alpha^*$, the condition (2-4) generally needs not be sufficient. Indeed, consider the series

$$\sum_{n=1}^{\infty} c_n e_n \gamma_n^{(\alpha)} ,$$

where $(e_n)_{n=1}^{\infty}$ is the natural basis of l_p

$(c_n)_{n=1}^{\infty}$ is a sequence of real numbers.

This series is convergent a. s. if and only if

$\sum_{n=1}^{\infty} | c_n \gamma_n^{(\alpha)} |^P < +\infty$ a. s. or, by Lemma 2-3, if and only if

$\sum_{n=1}^{\infty} | c_n |^{\alpha} < +\infty$ in case $P > \alpha \neq 2$

or $\sum_{n=1}^{\infty} | c_n |^{\alpha} \left(1 + \ln \frac{1}{| c_n |}\right) < +\infty$ in case $P = \alpha \neq 2$.

On the other hand

$$\sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} | < x_{\mu n} , e_i^* > |^{\alpha}\right)^{P/\alpha} = A_{\alpha}^{P} (1) \sum_{n=1}^{\infty} | c_n |^{p}$$

Therefore, if we choose a sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} | c_n |^p < +\infty$$

but $\sum_{n=1}^{\infty} | c_n |^p \left(1 + \ln \frac{1}{|c_n|}\right) = +\infty$ in case $p = \alpha \neq 2$;

or $\sum_{n=1}^{\infty} | c_n |^p < +\infty$

but $\sum_{n=1}^{\infty} |c_n|^{\alpha} = +\infty$ in case $p > \alpha \neq 2$

then the condition (2-4) is fulfilled but the series

$$\sum_{n=1}^{\infty} c_n e_n \gamma_n^{(\alpha)} \quad \text{is divergent a. s.}$$

Remark 2. Let us observe that in fact the « if » part in Lemma 2-3 remains valid without the assumption of independence of $\gamma_n^{(\alpha)}$. From this note, it follows that the condition

$$\sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} |\langle x_{\mu_n}, e_i^* \rangle|^{\alpha} \right) \left(1 + \ln \frac{1}{\sum_{n=1}^{\infty} |\langle x_{\mu_n}, e_i^* \rangle|^{\alpha}} \right) < +\infty$$

when $p > \alpha \neq 2$

or $$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle x_{\mu_n}, e_i^* \rangle|^{\alpha} < +\infty$$

when $p > \alpha \neq 2$

implies the a. s. finiteness of the series $\sum_{i=1}^{\infty} |S_i|^p$ i.e.

the a. s. convergence of the series $\sum_{n=1}^{\infty} \xi_n$. In general, however, this condition is not necessary.

3. *RELATIVE COMPACTNESS OF A FAMILY OF α — GAUSSIAN MEASURES ON L_p SPACES* $(1 \leqslant P < +\infty)$.

Before studying this problem it will be useful to investigate some general properties of α — Gaussian measures on general Banach spaces.

3. 1. THEOREM. Let $\mathcal{M}$ be a family of α — Gaussian measures on a Banach space E. If the family $\mathcal{M}$ is uniformly tight that is if $\forall \varepsilon > 0$ $\exists K$ compact $\subset E$ such that $\mu(K) \geqslant 1 - \varepsilon$ $\forall \mu \in \mathcal{M}$ then there exists a positive constant C such that

$$\forall \mu \in \mathcal{M} \quad \forall t > 0 \quad \mu\{ \|x\| > t \} \leqslant ct^{-\alpha}.$$

Proof. By the uniform tightness of $\mathcal{M}$ we can choose $T > 0$ such that

$$\mu\{ \|x\| > T \} \leqslant \frac{1}{4} \text{ for all } \mu \in \mathcal{M}. \tag{3 — 1}$$

Fix $\mu \in \mathcal{M}$. Suppose that X is an E-valued r. v. with distribution μ and $X_1, X_2, \ldots, X_n$ are independent copies of X. We have

$$P\{\|X\|\leqslant t\}^n = P\left\{\max_{1\leqslant j\leqslant n}\|X_j\|\leqslant t\right\} =$$

$$=1-P\left\{\max_{1\leqslant j\leqslant n}\|X_j\|>t\right\}\geqslant 1-P\left\{\max_{1\leqslant j\leqslant n}\|S_j\|>\frac{1}{2}t\right\},$$

where $S_j = X_1 + X_2 + \ldots + X_j$.

By Levy's inequality

$$P\left\{\max_{1\leqslant j\leqslant n}\|S_j\|\geqslant\frac{1}{2}t\right\}\leqslant 2P\left\{\|S_n\|\geqslant\frac{1}{2}t\right\}$$

we get

$$P\{\|X\|\leqslant t\}^n\geqslant 1=2P\left\{\|S_n\|>\frac{1}{2}t\right\} \qquad (3-2)$$

By choosing $t = 2n^{1/\alpha}T$ (3-2) gives

$$P\{\|X\|\geqslant 2n^{1/\alpha}T\}\leqslant 1-[1-2P\{\|S_n\|>n^{1/\alpha}T\}]^{1/n}.$$

Noting that $\frac{S_n}{n^{1/\alpha}}$ is distributed like X we have by (3-1)

$$P\{\|X\|\geqslant 2n^{1/\alpha}T\}\leqslant 1-[1-2P\{\|X\|>T\}]^{1/n}\leqslant$$

$$\leqslant 1-\left(\frac{1}{2}\right)^{1/n}\leqslant\frac{\ln 4}{n+1}. \qquad (3-3)$$

If $t\geqslant 0$ then we can find $n\geqslant 1$ such that

$2(n-1)^{1/\alpha}T\leqslant t\leqslant 2n^{1/\alpha}T$ and so

$$P\{\|X\|>t\}\leqslant P\{\|X\|>2(n-1)^{1/\alpha}T\}\leqslant\frac{\ln 4}{n}\leqslant\frac{2^\alpha T\ln 4}{t^\alpha}.$$

So we find

$$\mu\{\|x\|>t\}\leqslant C\,t^{-\alpha} \qquad \forall t\geqslant 0 \qquad \forall\mu\in\mathcal{M},$$

where $C = 2^\alpha T\ln 4$

The theorem is proved.

Let $\mathcal{M}$ be an arbitrary family of measures on E. A measurable mapping $f: E\rightarrow R^1$ is said to be equi-integrable with respect to $\mathcal{M}$ if

$$\sup_{\mu\in\mathcal{M}}\int_{\{|f|>a\}}|f(x)|\,\mu(dx)\rightarrow 0 \qquad \text{as } a\rightarrow\infty$$

It is not hard to prove

3. 2. *LEMMA:*

a) *Let f be a measurable mapping: $E \to R^1$ If f is equi-integrable with respect to $\mathcal{M}$ then*

$$\sup_{\mu \in \mathcal{M}} \int_E |f(x)| \, \mu(dx) < +\infty.$$

b) *Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of measures on E such that $\mathcal{M}_n \to \mu$. If f is continuous and equi-integrable with respect to $(\mu_n)_{n=1}^{\infty}$ then we have*

$$\lim_{n \to \infty} \int_E f(x) \, \mu_n(dx) = \int_E f(x) \, \mu(dx)$$

The next theorem is an application of theorem 3-1.

3. 3. *THEOREM. Let $\mathcal{M}$ be a family of uniformly tight α — Gaussian measures on E. Then for each $r < \alpha$ the mapping $x \longrightarrow \| x \|^r$ is equi-integrable with respect to $\mathcal{M}$.*

Proof. By integrating by parts an applying theorem 3-1 we have

$$\int_{\{\|x\|^r > a\}} \|x\|^r \, \mu(dx) = -\int_{a^{1/r}}^{\infty} t^r \, d_\mu \{\|x\| > t\} =$$

$$= r \int_{a^{1/r}}^{\infty} t^{r-1} \mu\{\|x\| > t\} \, dt + a\mu\{\|x\| > a^{1/r}\}$$

$$< C r \int_{a^{1/r}}^{\infty} t^{r-1-\alpha} \, dt + C a^{1-\alpha/r}$$

Since

$$\int_{a^{1/r}}^{\infty} t^{r-1-\alpha} \, dt \longrightarrow 0 \qquad \text{as } a \to \infty$$

$$a^{1-\frac{\alpha}{r}} \longrightarrow 0 \qquad \text{as } a \to \infty$$

that is

$$\int_{\{\|x\|^r > a\}} \|x\|^r \, \mu(dx) \to 0 \qquad \text{uniformly as } a \to \infty$$

3-4. COROLLARY. *Let $\mathcal{M}$ be a sequence of α — Gaussian measures such that $\mu_n \to \mu$. Then for each $r < \alpha$ we have*

$$\lim_{n \to \infty} \int_E \|x\|^r \, \mu_n(dx) = \int_E \|x\|^r \, \mu(dx).$$

In order to investigate the problem of compactness of α —Gaussian measures on l_p we prove the following theorem giving a general necessary and sufficient condition for uniform tightness of a given family of α — Gaussian measures on Banach spaces with Schauder basis.

3. 5. THEOREM. Let $\mathcal{M}$ be a family of α—Gaussian measures on the Banach space with Schauder basis $(e_k)_{k=1}^{\infty}$.

i) *If $\mathcal{M}$ is uniformly tight then for each $r < \alpha$*

$$1) \sup_{\mu\in\mathcal{M}} \int_E \| x \|^r \mu(dx) < +\infty , \tag{3-4}$$

$$2) \lim_{N\to\infty} \sup_{\mu\in\mathcal{M}} \int_E \| V_N x \|^r \mu(dx) = 0, \tag{3-5}$$

where

$$V_N x = \sum_{n=N}^{\infty} \langle x, e_n^* \rangle e_n$$

$(e_n^*)_{n=1}^{\infty}$ is a sequence of coordinate functionals.

ii) Conversely, if these conditions (3-4) and (3-5) are satisfied for some $r \geqslant 1$ then $\mathcal{M}$ is uniformly tight.

Proof: we use the following result from [6] valid for a family of arbitrary measures on a Banach space with Schauder basis. For a given family $\mathcal{M}$ of measures on E to be uniformly tight it is necessary and sufficient that

$$\lim_{R\to\infty} \sup_{\mu\in\mathcal{M}} \mu\{\|x\| > R\} = 0, \tag{3-6}$$

$$\lim_{N\to\infty} \sup_{\mu\in\mathcal{M}} \mu\{\|V_N x\| > \sigma\} = 0 \text{ for each } \sigma > 0. \tag{3-7}$$

i) The condition (3-4) follows directly from Theorem 3-3 and Lemma 3-2. We must only show (3-5). According to (3-7)

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \text{ such that } \sup_{\mu\in\mathcal{M}} \mu\{\|V_N x\| > 1\} < \frac{\varepsilon}{2}$$

for all $N \geqslant N(\varepsilon)$.

Fix $N \geqslant N(\varepsilon)$, $\mu\in\mathcal{M}$. Since $V_N : E \to E$ is a linear continuous operator, V_N presents itself as an E-valued α — Gaussian r. v. on the probability space $(E, \mathcal{B}(E), \mu)$. The same argument which led us to (3—3) in the proving theorem 3—1 now shows that

$$\mu\left\{\|V_N x\| > 2n^{1/\alpha}\right\} \leqslant 1 - (1-\varepsilon)^{1/n} \leqslant \frac{2\varepsilon}{n+1}.$$

If $t > 0$ then we can find $n > 1$ such that

$$2(n-1)^{1/\alpha} \leqslant t \leqslant 2\,n^{1/\alpha}.$$

So

$$\mu\left\{\| V_N x \| > t\right\} \leqslant \mu\left\{\| V_N x\| > 2(n-1)^{1/\alpha}\right\} \leqslant \frac{2\varepsilon}{n} \leqslant \frac{C\varepsilon}{t^{\alpha}}$$

From here, it follows that

$$\int_E \| V_N \; x\|^r \mu\,(dx) = \int_0^\infty r t^{r-1} \mu\,\{\| V_N x\| > t\}\,dt \leqslant \varepsilon C r \int_0^\infty t^{\,r-1-\alpha} dt = A\varepsilon,$$

where

$$A = 2^{\alpha} r \int_0^\infty t^{\,r-1-\alpha} dt$$

Thus we have shown that
$\forall \varepsilon > 0 \quad \exists N(\varepsilon)$ such that for all $\mu \in \mathcal{M}$, $N > N(\varepsilon)$

$$\int_E \| V_N x \|^r \mu\,(dx) \leqslant \varepsilon,$$

that is

$$\lim_{N\to\infty} \sup_{\mu\in\mathcal{M}} \int_E \| V_N x \|^r \mu\,(dx) = 0.$$

ii) This assertion is an immediate consequence of (3—6) and (3—7) (using Chebysev inequality).

Now applying the above theorem to l_p spaces, we obtain the condition for relative compactness of a given family of α — Gaussian measures on l_p. The condition given here may often be easy to verify.

3—6. THEOREM. Let $\mathcal{M}$ be a given family of α—Gaussian measures on l_p.

i) Assume that $1 \leqslant p < \alpha^*$. Then $\mathcal{M}$ is relatively compact if and only if the family $(x_\mu)_{\mu\in\mathcal{M}}$ is relatively compact in l_p.

ii) In all cases, the relative compactness of the family $(x_\mu)_{\mu\in\mathcal{M}}$ in l_p is always necessary for the relative compactness of the family $\mathcal{M}$.

Proof. Suppose that $\mathcal{M}$ is relatively compact that is $\mathcal{M}$ is uniformly tight by Prohorov's theorem. By theorem 3—5 we have

$$\sup_{\mu\in\mathcal{M}} \int_{l_p} \| x \| \mu(dx) < +\infty,$$

$$\lim_{N\to\infty} \sup_{\mu\in\mathcal{M}} \int_{l_p} \| V_N x \| \mu(dx) = 0.$$

On the other hand

$$\| x_\mu \| = \| \int_{l_p} \rho(x)\, \mu(dx) \| \leqslant \int_{l_p} \| \rho(x) \| \mu(dx) = \int_{l_p} \| x \| \mu(dx),$$

so

$$\sup_{\mu \in \mathcal{M}} \| x_\mu \| < +\infty. \tag{3—8}$$

Since $V_N x_\mu = V_N \int_{l_p} \rho(x)\, \mu(dx) = \int_{l_p} V_N\, \rho(x)\, \mu(dx)$,

hence

$$\| V_N x_\mu \| = \| \int_{l_p} V_N\, \rho(x)\, \mu(dx) \| \leqslant$$

$$\leqslant \int_{l_p} \| V_N\, \rho(x) \| \mu(dx) = \int_{l_p} \| V_N x \| \mu(dx),$$

so

$$\lim_{N\to\infty} \sup_{\mu \in \mathcal{M}} \| V_N x_\mu \| = 0. \tag{3-9}$$

According to the compact criterion in l_p (see [7] page 248), which states that for a set $K \subset l_p$ to be relatively compact it is necessary and sufficient that

1. $$\operatorname{Sup}_{x \in K} \| x \| < +\infty, \tag{3-10}$$

2. $$\lim_{N\to\infty} \sup_{x \in K} \| V_N x \| = 0, \tag{3-11}$$

we conclude that the family $(x_\mu)_{\mu \in \mathcal{M}}$ is relatively compact in l_p.

If $(x_\mu)_{\mu \in \mathcal{M}}$ is relatively compact in l_p and $1 \leqslant p < \alpha^*$ then we have:

$$\int_{l_p} \| x \|^p \mu(dx) = \sum_{n=1}^{\infty} \int_{l_p} |\langle x, e_n^* \rangle|^p \mu(dx) =$$

$$= B_\alpha(p) \sum_{n=1}^{\infty} \left(\int_{l_p} |\langle x, e_n^* \rangle| \mu(dx)\right)^p = B_\alpha(p) \sum_{n=1}^{\infty} | \langle x_\mu, e_n \rangle |^p$$

$$= B_\alpha(p) \| x_\mu \|^p,$$

$$\int_{l_p} \| V_N x \|^{p} \mu(dx) = \sum_{n=N}^{\infty} \int_{l_p} |\langle x, e_n^* \rangle|^p \mu(dx)$$

$$= B_\alpha(p) \sum_{n=N}^{\infty} \left(\int_{l_p} |\langle x, e_n^* \rangle| \mu(dx)\right)^p = B_\alpha(p) \sum_{n=N}^{\infty} |\langle x_\mu, e_n^* \rangle|^p$$

$$= B_\alpha(p) \| V_N x_\mu \|^p.$$

So it follows by (3-10), (3-11) and theorem 3-5 that $\mathcal{M}$ is relatively compact Finally, we give an example to show that in case $P \geqslant \alpha$ the relative compactness of $(x_\mu)_{\mu \in \mathcal{M}}$ does not, in general, imply the relative compactness of $\mathcal{M}$.

Example. Let $(c_n)_{n=1}^{\infty}$ be a sequence of real numbers. Consider

$\mu_n = L(S_n)(n = 1, 2, \ldots)$ where $S_n = \sum_{i=1}^{\infty} c_i e_i \gamma_i^{(\alpha)}$. The relative compactness of $(\mu_n)_{n=1}^{\infty}$ is equivalent to the convergence a.s. of the series $\sum_{n=1}^{\infty} c_n e_n \gamma_n^{(\alpha)}$ by Ito-Nisio's theorem [5]. Therefore $(\mu_n)_{n=1}^{\infty}$ is relatively compact if and only if

$$\sum_{n=1}^{\infty} |c_n|^\alpha \left(1 + \ln \frac{1}{|c_n|}\right) < +\infty \text{ in case } P = \alpha \neq 2$$

or

$$\sum_{n=1}^{\infty} |c_n|^\alpha < +\infty \qquad \text{in case } P > \alpha \neq 2$$

It is easily seen that $x_{\mu_n} = (A_\alpha(1)c_1, \ldots, A_\alpha(1)c_n, 0, 0, \ldots)$

Hence, if we choose a sequence of real numbers $(c_n)_{n=1}^{\infty}$

so that $$\sum_{n=1}^{\infty} |c_n|^\alpha < +\infty$$

but $$\sum_{n=1}^{\infty} |c_n|^\alpha \left(1 + \ln \frac{1}{|c_n|}\right) = +\infty \qquad \text{in case } P = \alpha \neq 2$$

or $$\sum_{n=1}^{\infty} |c_n|^P < +\infty \quad \text{but} \quad \sum_{n=1}^{\infty} |c_n|^\alpha = +\infty \qquad \text{in case } P > \alpha \neq 2$$

then the family $(x_{\mu_n})_{n=1}^{\infty}$ is relatively compact but the family $(\mu_n)_{n=1}^{\infty}$ is not so.

3.7. Corollary. For a sequence $(\mu_n)_{n=1}^{\infty}$ of α — Gaussian measures on l_p $(1 \leqslant P < +\infty)$ to converge weakly to an α — Gaussian measure μ it is necessary that

1. $\lim_{n\to\infty} x_{\mu_n} = x_\mu$ *(in topology norm)*

2. $\lim_{n\to\infty} \int_{l_p} e_i \langle x, x^* \rangle \mu_n(dx) = \int_{l_p} e_i \langle x, x^* \rangle \mu(dx)$ for each $x^* \in l_p^*$

In the case $P < \alpha^*$ *this condition is also sufficient.*

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