ACTA MATHEMATICA VIETNAMICA
TOM 3, N°2 (1978)

Classification of unimodular algebraic groups with square integrable representations

by
NGUYỄN HỮU ANH

Let G be a separable locally compact group, Z a central subgroup of G. An irreducible (unitary) representation π of G is said to be square integrable mod Z (or a member of the discrete series of G) if there exist non null vectors φ, ψ in the representation space $\mathcal{H}(\pi)$ of π such that

$$\int_{G/Z} |(\pi(g)\varphi, \psi)|^2 \, dg < \infty$$

where (φ, ψ) denotes the inner product in $\mathcal{H}(\pi)$ and dg is the right Haar measure of G/Z.

In a previous article we have classified those Lie groups G with discrete series in the case in which the radical of G is a connected simply connected nilpotent Lie group. In this article we shall treat the problem in the more general situation, namely when G is a unimodular algebraic group.

Thus let G be the neutral component of a real algebraic group. Then G is the almost semi direct product of its greatest connected normal nilpotent subgroup N and a connected reductive subgroup R. Assume also that G is a *unimodular* locally compact group with the Hausdorff topology, then aside from some technical requirements, we show that the discrete series of G exist if and only if:

(A) the center of N is the neutral component of the center of G.

(B) the discrete series of N and R exist, and the center of R is compact.

The important point here is the assumption that G is *unimodular*. Indeed in [1] and [2] we gave examples of unimodular Lie groups (infact algebraic groups) with discrete series in which the conditions (A) and (B) are not satisfied.

However it turns out that those examples are false for, in neither cases, the actions of $SL(3, \mathbb{R})$ on $\mathbb{R}^9$ as described have open orbits and hence Theorem 1.3 in [2] cannot be applied. The trouble is the groups in those examples are so chosen to be unimodular. However as we have seen above under this further assumption, the statements (A) and (B) hold. On the contrary simpler correct examples exist when G is not unimodular (cf § 2).

In the following we shall continue to use the notations and results of [2]. In particular underlined small letters $\underline{g}, \underline{h}, \underline{z}, \ldots$ are reserved for the Lie algebras of the Lie groups $G, H, Z, \ldots$ The subgroup of the dual $\hat{H}$ of an abelian locally compact group H consisting of those characters of H trivial on a closed subgroup Z of H is identified with $(H/Z)^{\wedge}$. Similarly the linear subspace of the dual V^* of a finite dimensional vector space V consisting of those linear functionals vanishing on a subspace W of V is identified with $(V/W)^*$.

1. PROOF OF THE MAIN RESULTS

Let V be a vector space over $\mathbb{R}$ with complexification V^c. A *real algebraic subgroup* of $GL(V)$ is the set of points rational over $\mathbb{R}$ of an algebraic subgroup of $GL(V^c)$ defined over $\mathbb{R}$. Recall that (cf [7]) a real algebraic subgroup of $GL(V)$ is compact in the Hausdorff topology if and only if the corresponding algebraic subgroup of $GL(V^c)$ is $\mathbb{R}$-compact

LEMMA 1. Let V be a vector space over $\mathbb{R}$, and G a *reductive* algebraic subgroup of $GL(V)$ with *compact center*. Assume that G has an open orbit in V. Then the isotropy subgroup corresponding to each open orbit is not reductive.

Proof. Let $\mathbf{G}$ be the algebraic subgroup of $GL(V^c)$ defined over $\mathbb{R}$ such that G is the set of $\mathbb{R}$-rational points of $\mathbf{G}$. Then $\mathbf{G}$ has an open orbit $\mathcal{O}$ such that $\mathcal{O} \cap V \neq \varnothing$ is the union of open G-orbits in V. Assume that the isotropy subgroup G_0 corresponding to $\mathcal{O}$ is reductive. Then it follows from [4] that $\mathcal{O}$ is an affine subvariety of V^c. Therefore $V^c - \mathcal{O} = Z(P)$ is the nullset of a polynomial $P \in \mathbb{C}[V^c]$. Since $\mathcal{O} \cap V \neq \varnothing, \mathcal{O}$ is defined over $\mathbb{R}$ and hence P may be chosen so that $P \in \mathbb{R}[V^c]$. Then it is clear that P is a semi invariant of $\mathbf{G}$ corresponding to a non trivial character of $\mathbf{G}$ defined over $\mathbb{R}$. This, however, contradicts the fact that the center of $\mathbf{G}$ is $\mathbb{R}$-compact.

Q.E.D.

LEMMA 2. Let $\underline{n}$ be a nilpotent Lie algebra with center $\underline{z}$. Assume that there exists $l \in \underline{n}^*$ such that the antisymmetric bilinear form $B_l(\xi, \eta) = l([\xi, \eta])$ is non degenerate on $\underline{n}/\underline{z}$. Let τ be an automorphism of $\underline{n}$ such that τ is diagonalizable as an element of $gl(n)$ and $\tau\xi = \alpha\xi$, $\forall\xi \in z$. Then $\det_{n/z}(\tau) = \alpha^{\frac{q}{2}}$ where q is the dimension of n/z.

Proof. Factoring out $\ker l \cap z$ if necessary, we may assume that $\dim z = l$. First of all, let us prove by induction on q that there is a base ξ, ξ_i, η_i $\left(1 \leqslant i \leqslant \frac{q}{2}\right)$ of n consisting of eigen vectors of τ such that $[\xi_i, \eta_i] = \xi$ $\left(1 \leqslant i \leqslant \frac{q}{2}\right)$. Let indeed $\xi \in z$ be such that $l(\xi) = 1$. Let ξ_1 be an eigen vector of τ such that $\xi_1 + z$ belongs to the center of n/z. Then the centralizer m of ξ_1 in n has codimension 1 since B_l is non degenerate on n/z. It is clear that m is τ-invariant. Hence there exists an eigen vector η_1 of τ such that $l([\xi_1, \eta_1]) = 1$, i.e. $[\xi_1, \eta_1] = \xi$. Now the Lie algebra $m/\mathbb{R}\xi_1$ obviously satifies the conditions of Lemma 2 and hence the existence of the base ξ, ξ_i, η_i is proved by induction. Let α_i, β_i $\left(1 \leqslant i \leqslant \frac{q}{2}\right)$ be the eigen values of τ corresponding to ξ_i, η_i respectively. We have

$$\begin{aligned}\tau[\xi_i, \eta_i] &= [\tau\xi_i, \tau\eta_i]\\ &= \alpha_i\beta_i[\xi_i, \eta_i]\end{aligned}$$

i.e.

$$\alpha\xi = \alpha_i\beta_i\xi$$

Therefore

$$\det_{n/z}(\tau) = \prod_{i=1}^{\frac{q}{2}} \alpha_i\beta_i = \alpha^{\frac{q}{2}} \qquad \text{Q.E.D.}$$

Now let G be the neutral component of a real algebraic group. It follows from Proposition 5, §4.2 V of [5] that the Lie algebra g of G may be written as the direct sum $g = n + a + s$ where n is the greatest nilpotent ideal of g, s is a maximal semi simple subalgebra, and a is a commutative algebraic subalgebra such that each $ad_g(\xi)$, $\xi \in a$ is a diagonalizable linear transformation of g, and the elements of a commute with those of s. Let N, A, S be the analytic subgroups of G corresponding to n, a, s. Then G is the almost semi direct product of N and $R = AS$, i.e. $G = NR$ and $N \cap R$ is finite. On the other hand N is the direct product of its unipotent radical U and a central connected (algebraic) torus T since N is a nilpotent algebraic group. Moreover, since n consists of those elements ξ in g such that $ad_g(\xi)$ is nilpotent, we see that the neutral component of the center of G is the direct product of T with a connected subgroup Z lying in the center of U. Finally ZT has finite index in Cent (G) since a and s are algebraic Lie algebras.

THEOREM 1. The notations being as above, assume also that G is *unimodular*. Under these conditions if G has a square integrable mod ZT irreducible χ-representation π, where χ is a character of ZT, then

(i) Z coincides with the center of U and hence ZT is precisely the center of N.

(ii) N has a square integrable mod ZT irreducible χ-representation and thus U has a square integrable mod Z irreducible representation.

(iii) R has an atmost two sheeted covering with square integrable irreducible representation in the ordinary sense In particular the center of R is compact.

Proof. First note that according to Lemma 2.3 of [3], each irreducible representation of G has the form $\lambda(t)\,\pi\,(\text{nas})$ where λ is a character of T and π is an irreducible representation of the subgroup NAS such that $\pi|_{T\cap NAS} \cong$ mult $\lambda|_{T\cap NAS}$. On the other hand it is clear that such a representation of G is square integrable mod ZT if and only if π is square integrable mod Z. Therefore we may assume $T=\{1\}$ without loss of generality. We now prove the Theorem by a double induction on dim R and dim N. Thus suppose that it holds for those groups G such that dim $R<p$, dim N arbitrary, or dim $R=p$ and dim $N<q$. Let us prove it for the case dim $R=p$ and dim $N=q$. Let l_1 be the unique element of z^* such that $\chi=\exp\sqrt{-1}\,l_1$. By factoring out ker l_1 if necessary we may assume that dim $z\leqslant 1$. Assume that i) is false. Since $Ad_n(R)$ is the neutral component of a reductive real algebraic group, n contains an R-invariant subspace $\bar{n}$ complementing z. Let $\bar{h}$ be a minimal R-invariant subspace of $\bar{n}$ lying in the center of n and $\bar{H}$ be the corresponding analytic subgroup of G. Put $H=Z\bar{H}$. According to Theorem 1.3 in [2], there exists l_2 in $\bar{h}^*$ such that the G-orbit of $\lambda=\chi\exp\sqrt{-1}\,l_2$ in $\hat{H}$ has the form $\chi\mathcal{O}$ where $\mathcal{O}$ is the (open) G-orbit of $\exp\sqrt{-1}\,l_2$ in $\hat{\bar{H}}$ (here l_1 and l_2 are extended in a natural manner to the whole h so that $l_1(\bar{h})=\{0\}$ and $l_2(z)=\{0\}$). Moreover π is induced from a square integrable mod H irreducible λ-representation σ of the subgroup

$$
\begin{aligned}
G_0 &= \{g\in G/\lambda(ghg^{-1})=\lambda(h),\ \forall\ h\in H\}\\
&= \{g\in G/\lambda(ghg^{-1})=\lambda(h),\ \forall\ h\in \bar{H}\}\\
&= \{g\in G/\ Ad^*_{\bar{h}^*}(g)\ l_2=l_2\}\\
&= NR_0
\end{aligned}
$$

where $R_0=\{r\in R/\ Ad^*_{\bar{h}^*}(r)\ l_2=l_2\}$

Note that G_0/H is unimodular since G/Z is (cf Corollary 1.5 in [2]). Put ${}^0h=\ker(l_1+l_2)$ and let ${}^0H$ be the corresponding analytic subgroup of G_0. Then $\tilde{Z}=H/{}^0H$ is a central subgroup of $\tilde{G}=G_0/{}^0H=(N/{}^0H)\,R_0$. Moreover let $\tilde{\lambda}$ and $\tilde{\sigma}$ be the character by $\tilde{Z}$ and the representation of $\tilde{G}$ determined by λ and σ respectively by passing to quotient, then σ is square integrable mod $\widetilde{Z}$. Therefore by the induction hypothesis, the unipotent radical of $\widetilde{G}$ has a square integrable mod $\widetilde{Z}$ irreducible representation. Since the unipotent radical of $\widetilde{G}$ is clearly the semi direct product of $\widetilde{N} = N/{}^0H$ and the unipotent radical of R_0, it follows from Proposition 4.2 of [2] that the latter reduces to the identity, and hence $\widetilde{N}$ has a square integrable mod $\widetilde{Z}$ irreducible representation and R_0 is reductive. Therefore the center of $Ad^*_{\bar{h}*}(R)$ and hence the center of $Ad_{\bar{h}}(R)$ is non compact as indicated by Lemma 1. Let a be an element of the maximal $\mathbb{R}$-spilit torus of A such that $Ad_{\bar{h}}(a) = \alpha$ Id, $\alpha \neq 1$ (recall that $\bar{h}$ is minimal). By Theorem 1 of [8] there is a linear form $\widetilde{l}$ on $\widetilde{n}$ such that the antisymmetric bilinear form $B_{\widetilde{l}}(.,.) = \widetilde{l}([.,.])$ is non degenerate on $\widetilde{n}/\widetilde{z}$ since $\widetilde{N}$ has square integrable mod $\widetilde{Z}$ irreducible representations. Therefore it follows from Lemma 2 that

$$\det_{\widetilde{n}/\widetilde{z}}(Ad\ a) = \det_{n/h}(Ad\ a) = \alpha^k$$

where $k = 0$ if $z \neq \{0\}$ and $k = \dim n/h$ if $z = \{0\}$. Hence $\det_n(Ad\ a) = = \alpha^k \cdot \alpha^{\dim \bar{h}} \neq 1$: this contradicts the fact that G is unimodular. Thus i) is proved. In particular $\dim z = 1$.

Next by applying Theorem 4.5 of [2] one see that iii) is a consequence of i) and ii). Therefore it remains to prove ii). The case in which N is isomorphic to some Heisenberg group is taken care by Lemma 3.1 of [2]. Thus assume that N is not isomorphic to any Heisenberg group. Then it follows from Propposition 2.3 of [2] that there is an R-invariant subspace $\bar{k}$ of $\bar{n}$ such that $(\bar{k} + z)/z$ is contained in the center of n/z. Put $k = \bar{k} + z$. Let us use the same notation to indicate the extension of l_1 to the whole k such that $l_1(\bar{k}) = \{0\}$. Since $Ad(N)$ reduces to the identity on z and k/z respectively, the N-orbit of l_1 in k^* has the form $l_1 + V$ where V is a linear subspace of $(k/z)^*$. Let us prove that $V = (k/z)^*$. Assume indeed the contrary, then there exists $\xi \in k$ such that $l_1(Ad(n)\xi) = \{0\}$. $\forall n \in N$, i.e. $Ad(n)\xi \in \ker l_1$, $\forall n \in N$. Hence $Ad(n)\xi - - \xi \in \ker l_1 \cap z = \{0\}$, $\forall n \in N$, i.e. $Ad(n)\xi = \xi$, $\forall n \in N$. This contradicts the fact that z is the center of n. Thus $V = (k/z)^*$. This implies in particular that the N-orbit and hence the G-orbit of $\chi = \exp\sqrt{-1}\, l_1$ in $\widehat{K}$ har the form $\chi(K/Z)^{\wedge}$. Therefore the quasi invariant measure on $\widehat{K}$ determined by $\pi|_K$ is concentrated in this orbit. By Theorem 1.3 of [2] π is induced from a square integrable mod K irreducible χ-representation ρ of the subgroup

$$G' = \{ g \in G / Ad^*_{\bar{k}^*}(g)\, l_1 = l_1 \}$$
$$= \{ g \in G / Ad(g)\, \bar{k} \subset \bar{k} \}$$
$$= N'R$$

where $N' = \{ n \in N / Ad(n) \bar{k} \subset \bar{k} \}$

Let χ' be the character of $Z' = K/\exp\bar{k}$ and ρ' the representation of $G'/\exp\bar{k} = (N' \exp\bar{k}) R$ uniquely determined by χ and ρ respectively by passing to quotient. Then it follows from the induction hypothesis that $N'/\exp\bar{k}$ has a square integrable mod Z' irreducible χ'-representation. Composing it with the canonical homomorphism $N' \to N'/\exp\bar{k}$ we obtain a square integrable mod K irreducible χ-representation of N'. Finally this representation together with the N-orbit $\chi\widehat{(K/Z)}$ define a square integrable mod Z irreducible χ-representation of N as indicated by Theorem 1.3 of [2].

Q.E.D.

Remark. Theorem 4.5 of [2] may be viewed again as a converse of Theorem 1. These two theorems together with the results in [6] and [8] give the following algebraic characterization of unimodular algebraic groups with discrete series

THEOREM 2. Let $G = NR$ being as above, then there exists a finite covering of G with discrete series if and only if the following conditions hold

(A) the center of N has finite index in the center of G.

(B) the center of the universal envelopping algebra of n coincides with the symmetric algebra $S(z)$ of z.

(C) R has a compact Cartan subgroup.

2. AN EXAMPLE

We conclude with the example of an algebraic group with square integrable irreducible representation in the ordinary sense in which the conditions (A) and (C) are not satisfied. Of course such a group can not be unimodular.

Let $R = GL(n, \mathbf{R})$ and $N = \mathcal{M}(n, \mathbf{R})$ — the additive group of $n \times n$-matrices with coefficiefts in $\mathbf{R}$. Then R acts naturally on N by left multiplication, and the semi direct product $G = NR$ is a real algebraic group in which the condition (A) does not hold since the center of G reduces to the identity while N is abelian. On the other hand (C) does not hold. Finally the action of R on $\widehat{N}$ has a unique open orbit, namely the set of non singular matrices such that the corresponding isotropy subgroup reduces to the identity. Therefore an arbitrary character of N in this orbit will induce to an irreducible representation of G which is square integrable in the ordinary sense according to Theorem 1.3 of [2].

REFERENCES

[1] N.H. Anh, *Algebraic groups with square integrable representations*, Bull. A.M.S. 80 (1974), 539—542

[2] N.H. Anh, *Lie groups with square integrable representations*, Annals of Mathematics, 104 (1976), 431—458

[3] N.H. Anh, *Restriction of the Principal series of SL(n, C) to some Reductive subgroups*, Pacific Journal of Mathematics, 38 (1971), 295—314

[4] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Annals of Mathematics 75 (1962), 485—535

[5] C. Chevalley, *Théorie des groupes de Lie*, Vol 3, Act. Sci. Ind. n° 1226, Hermann, Paris 1955

[6] Harish-Chandra, *The discrete series for semi simple Lie groups II, Explicit determination of the characters*, Acta Math. 116 (1966), 1—111

[7] I. Satake, *Classification theory for semi simple algebraic groups*, Lecture notes in pure and applied math., Marcel Dekker Inc., New York 1971

[8] J.A. Wolf and C.C. Moore, *Square integrable representations of nilpotent groups*, Proc. Sympos. Pure Math., A.M.S. 26 (1973), 239—244

Department of Mathematics and Physics
The Institute of Polytechnics, Hà Nội

Reçu le 15-1-1978

ERRATA

CLASSIFICATION OF UNIMODULAR ALGEBRAIC GROUPS WITH SQUARE INTEGRABLE REPRESENTATIONS

by
NGUYÊN HỮU ANH

In the proof of Theorem 1, in addition to Lemma 2 we need also the following Lemma 3. Moreover we include a shorter proof of Lemma 2 which is based on the same principle as that of Lemma 3.

LEMMA 2. Let n be a Lie algebra with center z. Assume that there exists l in n^* such that the antisymmetric bilinear form $B_l(\xi, \eta) = l([\xi, \eta])$ is non degenerate on n/z. Let τ be an automorphism of n such that τ is diagonalizable as an element of $gl(n)$ and that there is a τ-invariant subspace $\bar{n}$ of n complementing z such that $l(\bar{n}) = 0$. Finally assume that $\tau | z$ has just one eigenvalue α. Then det $\tau = \alpha^q$, where $q = \frac{1}{2}$ dim h/z + dim z.

Proof. Let $\{\xi_i\}$ be a basis of $\bar{n}$ consisting of eigen vectors of τ corresponding to the eigenvalues $\{\alpha_i\}$. We have $\tau[\xi_i, \xi_j] = [\tau\xi_i, \tau\xi_j] = \alpha_i\alpha_j[\xi_i, \xi_j]$. Hence $\alpha_i\alpha_j = \alpha$ if $B_l(\xi_i, \xi_j) \neq 0$, i.e. $\tau B_l = \alpha B_l$. Here τB_l is the result of the induced action of τ on the 2-form B_l. Thus det $\tau | \bar{n} = Pf(\tau B_l)/Pf(B_l) = \alpha^{\frac{1}{2}\dim n/z}$, where Pf is the Pfaffian polynomial on 2-forms with respect to the determinant on $\bar{n}$.

Q.E.D.

LEMMA 3. Let N, Z, τ be as above. Assume now $\tau | z$ has exactly two eigenvalues 1 and $\alpha \neq \pm 1$. Then $|\det \tau| \neq 1$.

Proof. Let $|\tau|$ be the linear transformation on $\bar{n}$ such that $|\tau|\xi_i = |\alpha_i|\xi_i, \forall i$. It is sufficient to prove that det$|\tau|$ is not equal to any negative integral power of $|\alpha|$. Let $|\tau|^t = e^{t \log|\tau|}$, then as in the proof of Lemma 2 we have

$$|\alpha_i|^t |\alpha_j|^t = |\alpha|^{tk_{ij}} \text{ if } B_l(\xi_i, \xi_j) \neq 0, \text{ where}$$

$k_{ij} = 0$ or 1.

Hence det $|\tau|^t = Pf(|\tau|^t B_l)/Pf(B_l)$ is a polynomial in $|\alpha|^t$. Therefore by the unique factorization of the polynomial ring in one variable with coefficients in $\mathbb{R}$, det $|\tau|^t$ is not equal to any negative integral power of $|\alpha|^t$.

Q.E.D.