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Banach space valued brownian motions

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I. BANACH SPACE VALUED BROWNIAN MOTIONS

Abstract. It is known ([1], [3]) that every real Brownian motion $B(t)$, $t \in [0, 1]$, can be represented as

$$B(t) \doteq \sum_n Z_n \int_0^t g_n(s)\, ds$$

where $\{Z_n\}$ is a sequence of i.i.d symmetric Gaussian random variables, $\{g_n\}$ a CONS in $L^2[0, 1]$ and the series is convergent with probability one uniformly over $[0, 1]$. The aim of the present paper is to prove some complete analogues of this fact for Banach space valued Brownian motions.

This paper is concerned with random variables defined on a fixed probability system $(\Omega, \mathcal{F}, P)$. Let E denote a real separable Banach space with the topological dual space E^*. In the sequel, if Y is a normed space then its norm will be denoted by $\|\cdot\|_Y$. By an E-valued random variable we mean a measurable map $X: \Omega \to E$ (measurable in the weak sense). An E-valued stochastic process X_t, $t \in T$, is said to be Gaussian if for any $n = 1, 2, \ldots, y_1, y_2, \ldots, y_n \in E^*$ and $t_1, t_2, \ldots, t_n \in T$ the real random variable $\sum_{i=1}^{n} y_i(X_{t_i})$ is Gaussian. In particular, an E-valued random variable X is Gaussian if $y(X)$ is Gaussian for each $y \in E^*$. Further, for the characteristic functional of an E-valued symmetric Gaussian random variable X we have the formula

$$E\exp i\langle y, X\rangle = \exp\left(-\frac{1}{2}\langle y, Ry\rangle\right) \quad (y \in E^*).$$

R being a covariance operator i.e. a compact operator from E^* into E such that $\langle y, Ry\rangle \geqslant 0$ and $\langle y_1, Ry_2\rangle = \langle y_2, Ry_1\rangle$ for all $y, y_1, y_2 \in E^*$.

The following consequence of Jain-Kallianpur Theorem ([2], Theorem 3) will be needed.

PROPOSITION 1. Let R be a covariance operator. Then there exists a separable Hilbert space H with the inner product $\langle \cdot, \cdot \rangle_H$ such that

$$R(E^*) \subset H \subset E$$

and for any $y \in E^*$ and $h \in H$

$$y(h) = \langle Ry, h\rangle_H.$$

Moreover, for every E-valued symmetric Gaussian random variable X with the covariance R and for every CONS $\{e_n\}$, $n = 1, 2, \ldots$, in H there exists a sequence $\{U_n\}$ of independent real random variables with distribution $N(0, 1)$ such that

$$X = \sum_n U_n e_n$$

where the series is convergent in the norm $\|\cdot\|_E$ with probability one.

Let ξ_t, $t \in [0, 1]$, be an E-valued stochastic process. Then it is called a Brownian motion if

(i) $\xi_o = 0$ (P.1),

(ii) $\{\xi_t\}$ is a symmetric homogeneous process with independent increments,

(iii) the realizations of $\{\xi_t\}$ are continuous (in the norm topology of E) with probability one.

An equivalent definition of Banach-space-valued Brownian motions is given by the following theorem:

THEOREM 1. An E-valued process ξ_t, $t \in [0, 1]$, is a Brownian motion if and only if it is Gaussian and for any $t, s \in [0, 1]$ and $x, y \in E^*$

/1/ $$Ex(\xi_t)\, y(\xi_s) = \langle y, Rx\rangle\, (t \wedge s)$$

R being the covariance operator of ξ_1.

To prove this Theorem we need the following lemma:

LEMMA 1. Let X_t, $t \in [0, 1]$, be an E-valued symmetric process with independent increments. If for every $y \in E^*$ the realizations of the process $y(X_t)$, $t \in [0, 1]$, are continuous with probability one then the realizations of X_t, $t \in [0, 1]$, are continuous (in the norm topology of E) with probability one.

Proof of the Lemma. Given $t \in [0, 1]$ and a sequence $\{t_n\} \subset [0, 1]$ such that $t_n \to t$. Without loss of generality we may assume that $t_1 < t_2 < \ldots < t_n \to t$ or $t_1 > t_2 > \ldots > t_n \to t$. Consider the first case. Put $Z_1 = X_{t_1}$, $Z_n = X_{t_n} - X_{t_{n-1}} (n > 1)$ and $U_n = X_t - X_{t_n}$. Then for every $n = 1, 2, \ldots$ the random variables $Z_1, Z_2, \ldots, Z_n, U_n$ are independent. Moreover, $X_t = \sum_{i=1}^{n} Z_i + U_n (n=1, 2, \ldots)$ By Theorem 2.2 [4] and Theorem 4.1 [1] the series $\sum_{i=1}^{\infty} Z_i$ converges with probability one. Hence $\lim_{n\to\infty} U_n = U$ (P.1) exists. Further, for every $y \in E^*$ we have $y(X_t) = \sum_{i=1}^{\infty} y(Z_i) + y(U)$ and $y(U) = 0$ (P.1) by the continuity of the process $y(X_t)$, $t \in [0, 1]$. Consequently, $U = 0$ (P.1) which shows that

$$\lim_{n\to\infty} X_{t_n} = X_t \quad (\text{P.}1)$$

The proof of this equality for $t_1 > t_2 > \ldots > t_n \to t$ is the same. The Lemma is thus proved.

Proof of Theorem 1. Suppose that ξ_t, $t \in [0, 1]$, is an E-valued Brownian motion. Then for every $y \in E^*$ $y(\xi_t)$ is a real Brownian motion. Consequently, every ξ_t, $t \in [0, 1]$, is an E-valued Gaussian random variable with $E\xi_t = 0$. Hence and by the assumption that $\{\xi_t\}$ is a process with independent inorements it follows that it is a Gaussian process. Further, for any $0 \leqslant s < t \leqslant 1$ and $x, y \in E^*$ we have

$$\begin{aligned} Ex(\xi_s)y(\xi_t) &= Ex(\xi_s)y(\xi_s) \\ &= \frac{1}{4}\Big[E\big((x+y)\xi_s\big)^2 - E\big((x-y)\xi_s\big)^2\Big] \\ &= s\langle y, Rx\rangle \end{aligned}$$

where R is the covariance operator of ξ_1.

Conversely, suppose that ξ_t, $t \in [0, 1]$, is an E-valued Symmetric Gaussian process satisfying the condition /1/. It is easy to check that $\{\xi_t\}$ is a homogeneous process with independent increments. Moreover, for every $y \in E^*$ $\{y(\xi_t)\}$ is a real Brownian motion. Consequently, the realizations of $\{y(\xi_t)\}$ are continuous with probability one. By Lemma 1 it follows that the realizations of $\{\xi_t\}$ are continuous in the norm topology of E with probability one. Consequently, $\{\xi_t\}$ is a Brownian motion which completes the proof of the Theorem.

In the sequel we fix an E-valued Brownian motion ξ_t, $t \in [0, 1]$, and call the covariance operator R of ξ_1 its associated covariance operator. Let H be a real separable Hilbert space. By $L^2([0, 1], H)$ we shall denote the Hilbert space of measurable functions $f: [0, 1] \to H$ such that

$$\int_0^1 \|f(s)\|_H^2 \, ds < \infty .$$

Its inner product is defined in a natural way. In particular, if H is the real line then we denote $L^2([0, 1], H)$ by the usual symbol L^2.

PROPOSITION 2. For every function f in L^2 the stochastic integral

$$I(f) := \int_0^1 f(s) \, d\xi_s$$

is defined such that for any $x, y \in E^*$ and $f, g \in L^2$

$$/2/ \qquad Ex\big(I(f)\big)\, y\big(I(g)\big) = \langle y, Rx\rangle \int_0^1 f(s) g(s) \, ds$$

Proof. Let L denote the set of all simple functions of the form

$$f = \sum_{i=1}^{n} r_i \chi_{(t_{i-1}, t_i)}$$

where $r_1, r_2, \dots, r_n$ are some real numbers and $0 = t_0 < t_1 < \dots < t_n = 1$. Define an E-valued stochastic integral for such functions f as follows

$$I(f) := \int_0^1 f(s) \, d\xi_s = \sum_{i=1}^{n} r_i \left(\xi_{t_i} - \xi_{t_{i-1}}\right)$$

By Proposition 1 it follows that for any $x, y \in E^*$ and $f, g \in L$ we have

$$/3/ \qquad Ex\,(I(f))\; y(I(g)) = \langle y, Rx\rangle \int_0^1 f(s)\, g(s) ds$$

Consequently, if the Brownian motion $\{\xi_t\}$ is non-zero then for any $f_1, f_2, \dots, f_n \in L$ the random variables $I(f_i)$ are independent (res. identically distributed) if and only if the functions f_i, $i = 1, 2, \dots, n$, are orthogonal (res. have the same norm in L^2).

Our further aim is to define the stochastic integral $I(f) := \int_0^1 f(s) d\xi_s$ for every $f \in L^2$.

Let f be an arbitrary function in L^2 and $\{e_n\}$ be a CONS in L^2 such that $\{e_n\} \subset L$. By the Parseval identity we have

$$/4/ \qquad \|f\|_{L^2}^2 = \sum_n \langle f, e_n\rangle_{L^2}^2$$

Put $S_n = \sum_{i=1}^{n} \langle f, e_1 \rangle_{L^2} I(e_i)$, $n = 1, 2, \ldots$ By the above remark the ramdom variables $I(e_i)$, $i = 1, 2, \ldots$, are independent and identically distributed. Further, for every $y \in E^*$ we have

$$Ey^2(S_n) = \langle y, Ry \rangle \sum_{i=1}^{n} \langle f, e_i \rangle_{L^2}^2 \to \langle y, Ry \rangle \|f\|_{L^2}^2$$

as $n \to \infty$. Therefore,

/5/ $$\lim_{n \to \infty} E \exp i\, y(S_n) = \exp \left\{ -\frac{1}{2} \langle y, Ry \rangle \|f\|_{L^2}^2 \right\}$$

Since the last limit is a characteristic functional of an E-valued random variable it follows, by Ito-Nisio theorems ([1], Theorems 3.1 and 4.1), that there exists an E-valued Gaussian random variable S such that $S_n \to S$ with probability one. It is easy to prove that the limit S does not depend on any choice of the CONS $\{e_n\} \subset L$. Thus we can define

/6/ $$I(f) = \int_0^1 f(s)\, d\xi_s = S = \sum_{n=1}^{\infty} \langle f, e_n \rangle_{L^2} I(e_n)$$

From this definition it follows that for any $f, g \in L^2$ and $x, y \in E^*$

/7/ $$Ex(I(f))y(I(g)) = \langle y, Rx \rangle \int_0^1 f(s)\, g(s) ds.$$

which complets the proof of the Proposition.

COROLLARY 1. If the Brownian motion $\{\xi_t\}$ is non-zero then for any $f_1, f_2, \ldots, f_n \in L^2$ the random variables $I(f_i)$, $i = 1, 2, \ldots, n$, are independent (res. identically distributed) if and only if the functions f_i, $i = 1, 2, \ldots, n$, are orthogonal (res. have the same norm) in L^2.

Proof. It is an easy consequence of the equality /2/.

Now we shall formulate the main results of this paper.

THEOREM 2. Let ξ_t, $t \in [0, 1]$, be an E-valued Brownian motion. Then to every CONS $\{e_n\}$ in L^2 there corresponds a sequence of i.i.d E-valued symmetric Gaussian random variables $\{Z_n\}$ such that

/7/ $$\xi_t = \sum_{n=1}^{\infty} Z_n \int_0^t g_n(s) ds \qquad (t \in [0, 1]).$$

where the series is convergent in the norm of E with probability one uniformly over $[0, 1]$.

Proof. For a trivial Brownian motion $\xi_t = 0(P.1)$, $t \in [0, 1]$, the expansion /7/ holds for $Z_n = 0(P.1)$, $n = 1, 2, \ldots$. Suppose that $\{\xi_t\}$ is non-zero. Let $\{g_n\}$ be a CONS in L^2. Putting $Z_n = I(g_n)$, $n = 1, 2, \ldots$, and taking into account Proposition 2 we infer that the E-valued symmetric Gaussian random variables Z_n, $n = 1, 2, \ldots$, are independent and identically destributed.

We shall prove that the series

$$/8/ \qquad \sum_{n=1}^{\infty} Z_n \int_0^t g_n(s)\,ds \qquad (t \in [0, 1])$$

is convergent in the norm of E with probability one uniformly over $[0, 1]$ to ξ_t.

Let U denote the unit ball in E^*. It is known that if U were endowed with the E-topology then the product $K = [0, 1] \times U$ is a compact metric space. Further, define

$$S(t, y) = y(\xi_t)$$

and

$$S_n(t, y) = y(S_n(t))$$

where $S_n(t)$ is the n-th sum of the series /8/ and $(t, y) \in K$. It is clear that the real Gaussian processes $S(t, y)$ and $S_n(t, y)$ on K have continuous realizations with probability one. Our aim is to prove that $S_n(t, y)$ converges to $S(t, y)$ with probability one uniformly over K.

By $C(K)$ we shall denote the Banach space of all continuous real-valued functions defined on the compact metric space K with the norm supremum. Let τ be a signed measure on Borel subsets of K with the variation $\|\tau\|$. Then we have

$$E \left| \int_K (S(t, y) - S_n(t, y)\, d\tau(t, y) \right| \leq$$

$$\int_K E |S(t, y) - S_n(t, y)\, d|\tau|(t, y)$$

$$\leq \int_K (E|S(t, y) - S_n(t, y)|^2)^{\frac{1}{2}}\, d|\tau|(t, y)$$

$$\int_K \left[\langle y, Ry \rangle \right]^{\frac{1}{2}} \left[\sum_{i=n+1}^{\infty} (\int_0^t g_n(s) ds^2) \right]^{\frac{1}{2}} d|\tau|(t, y)$$

$$\to 0 \text{ as } n \to \infty.$$

because $\sum_{i=n+1}^{\infty} (\int_0^1 g_n(s)\,ds)^2 \to 0$ and also is bounded by t.

Consequently, if we consider $S(t, y)$ and $S_n(t, y)$ as $C(K)$ —valued random variables then by Ito-Nisio Theorem ([1], Theorem 4.1), $S_n(t,y) \to S(t,y)$ in the norm of $C(K)$ with probability one. Hence it follows that the series /8/ is convergent to ξ_t in the norm of E with probability one uniformly over $[0, 1]$. The Theorem is thus proved.

THEOREM 3. Let ξ_t, $t \in [0,1]$, be an E-valued Brownian motion and R be its associated covariance operator. Let H be the Hilbert space corresponding to R as described in Proposition 1. Then for every CONS $\{e_n\}$, $n = 1, 2, \ldots$, in H there exists a sequence $\{B_n(t)\}$, $t \in [0, 1]$ and $n = 1, 2, \ldots$, of independent identically distributed real Brownian motions such that

/9/ $$\xi_t = \sum_{m} B_m(t)\, e_m \qquad (\mathrm{P.1}) \qquad (t \in [0, 1])$$

where the series is convergent in the norm of E with probability one uniformly over $[0, 1]$.

Proof. By virtue of Theorem 2 the E-valued Brownian motion $\{\xi_t\}$ can be represented by the random series /7/ where Z_n, $n = 1, 2, \ldots$, are some i.i.d symmetric Gaussian E-valued random variables with a common covariance operator R. Let H be the Hilbert space corresponding to R as described in Proposition 1. Let $\{e_n\}$ be an arbitrary CONS in H. From Proposition 1 it follows that for every $n = 1, 2, \ldots$ there exists a sequence $U_{n,m}$, $m = 1, 2\ldots$ of independent real random variables with distribution $N(0, 1)$ such that

/10/ $$Z_n = \sum_{m} U_{n,m}\, e_m \qquad (n = 1, 2, \ldots)$$

where the series is convergent in the norm of E with probability one. It should be noted that the family $\{U_{n,m}\}$ is consisted of i.i.d real Gaussian random variables. Putting

/11/ $$B_m(t) = \sum_{n} U_{n,m} \int_0^t g_n(s)\, ds \qquad (t \in [0, 1])$$

where $\{g_n\}$ is a CONS in L^2, we get, by Theorem 5.2[1], a sequence of i.i.d real Brownian motions $B_m(t)$, $m = 1, 2, \ldots$ Further, from /7/ and /10/ it is easy seen that

$$\xi_t = \sum_{m} B_m(t)\, e_m$$

where the series is convergent in the norm of E with probability one for every $t \in [0, 1]$. Moreover, by the same technique as in the proof of Theorem 2 one can prove that this series is convergent in the norm of E with probability one uniformly over $[0, 1]$. Thus the Theorem is proved.

THEOREM 4. Let ξ_t, $t \in [0, 1]$, be an E-valued Brownian motion and

R be its associated covariance operator. Let H be the Hilbert space corresponding to R as described in Proposition 1.

Then for every CONS $\{f_n\}$, $n = 1, 2, \ldots$, in $L^2([0, 1], H)$ there exists a sequence of independent real random variables $\{U_n\}$, $n = 1, 2, \ldots$, with distribution $N(0, 1)$ such that

/12/ $$\xi_t = \sum_n U_n \int_0^t f_n(s)\, ds \qquad (t \in [0, 1])$$

where the series is convergent in the norm of E with probability one uniformly over $[0, 1]$.

Proof. Given an E-valued Brownian motion let G denote the Hilbert space spanned by all real Gaussian random variables and closed under the square convergence. Then for any $x, y \in E^*$ and $t, s \in [0, 1]$ we have the equations

$$\begin{aligned} Ex(\xi_t)\, y(\xi_s) &= \langle y, Rx\rangle\, (t \wedge s) \\ &= \langle Ry, Rx\rangle_H (t \wedge s) \\ &= \langle Rx\, \chi_{(0, t)}, Ry\, \chi_{(0, s)}\rangle_{L^2([0, 1], H)} \end{aligned}$$

which, by the fact that the random variables $x(\xi_t)$, $x \in E^*$ and $t \in [0, 1]$, are linearly dense in G and the simple functions $Rx\, \chi_{(0, t)}$, $x \in E^*$ and $t \in [0, 1]$, are linearly dense in $L^2([0, 1], H)$, imply that there exists an isometric isomorfism φ from $L^2([0, 1], H)$ into G such that $\varphi(Rx\, \chi_{(0, t)}) = x(\xi_t)$ for all $x \in E^*$ and $t \in [0, 1]$. Let $\{f_n\}$ be an arbitrary CONS in $L^2([0, 1], H)$. Then using the isomorfism φ we put $U_n = \varphi(f_n)$, $n = 1, 2, \ldots$ It is clear that $\{U_n\}$ is a CONS in G. In particular, it is a sequence of i.i.d symmetric real Gaussian random variables. As $Rx\, \chi_{(0, t)}$ has an orthogonal expansion

$$\begin{aligned} Rx\, \chi_{(0, t)} &= \sum_n f_n \int_0^t \langle Rx, f_n(s)\rangle_H\, ds \\ &= \sum_n f_n \int_0^t x(f_n(s))\, ds \qquad \text{(Proposition 1)} \\ &= \sum_n f_n\, x\left(\int_0^t f_n(s) ds\right) \end{aligned}$$

where the integral is taken in the Bochner sense, we have an orthogonal expansion

/13/ $$x(\xi_t) = \sum_n U_n\, x\left(\int_0^t f_n(s)\right) ds.$$

By Ito-Nisio Theorem ([1], Theorem 4.1) it follows that

/14/ $$\xi_t = \sum_n U_n \int_0^t f_n(s) ds$$

where the series is convergent in the norm of E with probability one for every $t \in [0, 1]$. Moreover, by (13), (14) and by the same technique as in the proof of Theorem 2, one can prove that the series (14) is convergent in the norm of E with probability one uniformly over $[0, 1]$. The Theorem is thus proved.

Theorems 2, 3 and 4 suggest that we can construct an E-valued Brownian motion as follows. Let R be a covariance operator of an E-valued symmetric Gaussian random variable. Let H be a Hilbert space corresponding to R as described in Proposition 1. Let $\{U_n\}$, $n = 1, 2, \ldots$, be a sequence of independent real random variables with distribution $N(0, 1)$ and $\{Z_n\}$, $n = 1,2, \ldots$, a sequence of independent symmetric Gaussian E-valued random variables with the common covariance operator R. Further, let $\{e_n\}$ be a CONS in H, $\{f_n\}$ be a CONS in $L^2([0, 1], H)$ and $\{g_n\}$ be a CONS in L^2. Finally, let $\{B_n(t)\}$ be a sequence of independent real Brownian motions with $E\ B_n^2(1) = 1$ for every $n = 1, 2, \ldots$

THEOREM 5. The series

$$\sum_n Z_n \int_0^t g_n(s)\, ds \qquad (t \in [0, 1]) \tag{15}$$

$$\sum_n B_n(t)\, e_n \qquad (t \in [0, 1]) \tag{16}$$

$$\sum_n U_n \int_0^t f_n(s)\, ds \qquad (t \in [0, 1]) \tag{17}$$

converge in the norm of E with probability one uniformly over $[0, 1]$ to E-valued Brownian motions whose the associated covariance operator is R.

Proof. Consider the series /15/. Put

$$S_n(t) = \sum_{i=1}^{n} Z_i \int_0^t g_i(s)\, ds \qquad (n = 1, 2, \ldots \text{ and } t \in [0, 1]).$$

Then for every $y \in E^*$ we have

$$Ey\left(S_n(t)\right)^2 = \langle y, Ry\rangle \sum_{i=1}^{n} \left(\int_0^t g_i(s)\, ds\right)^2$$

$$\to \langle y, Ry\rangle t \text{ as } n \to \infty$$

which, by virtue of Ito-Nisio Theorem ([1], Theorem 4.1), implies that the series /15/ converges in the norm of E with probability one for every $t \in [0, 1]$. Let ξ_t, $t \in [0, 1]$, denote the limit process. Then for any $t, s \in [0, 1]$ and $x, y \in E^*$ we have

$$Ex(\xi_t)\, y(\xi_s) = \langle y, Rx\rangle\, (t \wedge s)$$

Consequently, by Theorem 1, it follows that $\{\xi_t\}$ is a Brownian motion. Moreover, its associated covariance operator is R. Now, by the same technique as in the proof of Theorem 2, one can prove that the series /15/ converges to $\{\xi_t\}$ in the norm of E with probability one uniformly over $[0, 1]$.

The proof of the remainder parts of the Theorem is the same which completes the proof of the Theorem.

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II. A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTION

Abstract. In this paper we define a stochastic integral for some class of operator-valued functions based on a Banach space valued Brownian motion.

Throughout this paper we shall preserve the terminology and notation in [2]. In particular, by E we shall denote a real separable Banach space with the norm $\|\cdot\|_E$ and the dual space E^*. Let ξ_t, $t \in [0, 1]$, be an E-valued Brownian motion with the associated operator R. Further, let $B(E)$ denote the space of all bounded linear operators on E. Mordifying the technique developed in [3] by Vahaniya and Kandelaki for the Hilbert space case we introduce a norm in $B(E)$ as follows: For every $A \in B(E)$ and for every number $1 \leqslant p < \infty$ we put

/1/ $$\|A\|_p = \left(E\|A\ \xi_1\|_E^p\right)\frac{1}{p}\cdot$$

In the sequel we shall identify the operators $A, B \in B(E)$ for which

$$\|A - B\|_p = 0.$$

In such a way we get a normed linear space $(B(E), \|\cdot\|_p)$. Let M denote the completion of $B(E)$ in the norm $\|\cdot\|_p$. It is evident that M is a separable Banach space. Let us denote by $L^1([0, 1], M)$ the Banach space of all measurable M-valued functions f defined on $[0, 1]$ such that

$$\|f\|_{L^1([0,1],M)} := \int_0^1 \|f(s)\|_M\, ds < \infty$$

where $\|\cdot\|_M$ denotes the norm in M. It should be noted [1] that the set of all simple functions of the form

/2/ $$f = \sum_{i=1}^{n} A_i \chi_{(t_{i-1}, t_i]}$$

where $A_1, A_2, \ldots, A_n \in B(E)$ and $0 = t_0 < t_1 < \ldots < t_n \leqslant 1$, is dense in $L^1([0,1], M)$.

We now proceed to define a stochastic integral for functions $f \in L^1([0, 1], M)$. First for a simple function f of the form (2) we put

/3/ $$J(f) := \int_0^1 f(s)\, d\xi_s = \sum_{i=1}^{n} A_i(\xi_{t_i} - \xi_{t_{i-1}}).$$

Then we have

/4/ $$E\|J(f)\|_E^p \frac{1}{p} \leqslant \sum_{i=1}^{n} \left(E\|A_i(\xi_{t_i} - \xi_{t_{i-1}})\|_E^p\right) \frac{1}{p} = \sum_{i=1}^{n} (t_i - t_{i-1})\|A_i\|_p$$

$$= \int_0^1 \|f(s)\|_M ds = \|f\|_{L^1([0,1], M)}.$$

Let f be an arbitrary function in $L^1([0, 1], M)$. Choose a sequence $\{f_n\}$ of simple functions of the form /2/ such that $f_n \to f$ in the norm of $L^1([0,1], M)$. By /4/ it follows that the sequence of E-valued Gaussian random variable $\{J(f_n)\}$ is fundamental in the $L^p(\Omega, \mathcal{G}, P; E)$ norm. Since the last space is complete it follows that there exists a limit

$$\lim_{n \to \infty} J(fn)$$

in the $L^p(\Omega, \mathcal{G}, P; E)$ norm. Define

/5/ $$J(f) := \int_0^1 f(s)\, d\xi_s$$

$$= \lim_{n \to \infty} J(f_n)$$

It is easy to check that $J(f)$ does not depend on any choice of $\{f_n\}$. Thus the stochastic integral $J(f)$ is defined for every function $f \in L^1([0, 1], M)$.

We remark that if f is a function in $L^1([0, 1], M)$ with the property that $f([0, 1]) \subset B(E)$ then by /3/ and /5/ it follows that the covariance operator of the E-valued Gaussian random variable $J(f)$ is given by the formula

/6/ $$\int_0^1 f(s) R\, f(s)^* ds$$

where the integral is taken in such a way that for any $x, y \in E^*$

$$Ex\big(J(f)\big)\ y\big(J(f)\big) = \int_0^1 < f^*(s)y, \quad Rf^*(s)x > ds.$$

Finally, if E is a Hilbert space and $\{\xi_t\}$ is a Hilbert space valued Brownian motion then our definition of the stochastic integral coincides with that given in [3] by Vahaniya and Kandelaki.

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