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Parametric oscillations of dynamical systems with cubic term at the modulation depth under the influence of nonlinear frictions.

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This paper deals with the influence of nonlinear frictions to the parametric oscillations of dynamical systems described by the equation with the cubic term at the modulation depth

$$\ddot{x} + \omega^2 x + \varepsilon(cx + dx^3)\cos\gamma t + \varepsilon\alpha x^3 + \varepsilon R(x, \dot{x}) = 0, \qquad (0.1)$$

where ω, c, d, α are constants, ε is a small positive parameter, $R(x, \dot{x})$ is a nonlinear function of x, $\dot{x}$ characterized the frictions considered. Three forms of nonlinear frictions will be investigated here [1, 2]: the Coulomb friction, the turbulent one and their combination.

As will be seen later in the analysis, the sign and value of parameter d sharply change the motion picture and the stable regions.

It must be emphasized that the equation (0.1) describes the real physical systems more precisely than the one in which $d = 0$ [4, 5, 6]. The system of type (0.1) with linear friction was studied qualitatively by Minorsky [3] but no attempt has yet been made to investigate it with the Coulomb friction, turbulent one and their combination.

§ 1. STATIONARY OSCILLATIONS AND THEIR STABILITY

Let us consider the resonant case when there exists the following relation between the frequencies

$$\omega^2 = \frac{\gamma^2}{4} + \varepsilon\Delta \qquad (1.1)$$

where Δ is detuning. At first, we transform the equation (0.1) into standard form by means of the formulae[7]

$$x = a\sin\theta, \quad \dot{x} = \frac{\gamma}{2} a\cos\theta. \tag{1.2}$$

The transformed equations are

$$\begin{aligned} \frac{\gamma}{2} \dot{a} &= -\varepsilon\left[\Delta x + \alpha x^3 + (cx + dx^3)\cos\gamma t + R(x, \dot{x})\right]\cos\theta, \\ \frac{\gamma}{2} a\dot{\psi} &= \varepsilon\left[\Delta x + \alpha x^3 + (cx + dx^3)\cos\gamma t + R(x, \dot{x})\right]\sin\theta, \end{aligned} \tag{1.3}$$

where $\psi = \theta - \frac{\gamma}{2} t$ and a, ψ are slowly varying functions of t.

The asymptotic method of nonlinear oscillations gives in the first approximation the following equations

$$\begin{aligned} \frac{\gamma}{2} \dot{a} &= -\varepsilon\left[S(a, \gamma) + \left(\frac{ca}{4} + \frac{da^3}{8}\right)\sin 2\psi\right], \\ \frac{\gamma}{2} a\dot{\psi} &= \varepsilon\left[\frac{\Delta}{2} a + \frac{3}{8}\alpha a^3 + H(a, \gamma) - \left(\frac{ca}{4} + \frac{da^3}{4}\right)\cos 2\psi\right], \end{aligned} \tag{1.4}$$

received by averaging the right hand sides of (1.3) over the time, where we designate

$$\begin{aligned} S(a, \gamma) &= \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \,.\, R\left(a\sin\theta, \frac{\gamma}{2} a\cos\theta\right) d\theta, \\ H(a, \gamma) &= \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \,.\, R\left(a\sin\theta, \frac{\gamma}{2} a\cos\theta\right) d\theta. \end{aligned} \tag{1.5}$$

The steady state harmonic solution corresponding to $\dot{a} = 0$, $\dot{\psi} = 0$ are

$$\begin{aligned} \left(\frac{ca_o}{4} + \frac{da_o^3}{8}\right)\sin 2\psi_o &= -S(a_o, \gamma), \\ \left(\frac{ca_o}{4} + \frac{da_o^3}{4}\right)\cos 2\psi_o &= \frac{\Delta}{2} a_o + \frac{3}{8}\alpha a_o^3 + H(a_o, \gamma). \end{aligned} \tag{1.6}$$

Eliminating ψ_o gives

$$W(a_o, \gamma) = 0 \tag{1.7}$$

here

$$W(a_o, \gamma) = \frac{64 S^2}{a_o^2(2c + da_o^2)^2} + \frac{4\left(\Delta a_o + \frac{3}{4}\alpha a_o^2 + 2H\right)^2}{a_o^2(c + da_o^2)^2} - 1 \tag{1.8}$$

The subharmonic response given by equations (1.2) is obtainable only when it is physically stable. We study now the stability of stationary oscillations. Let δa and $\delta\psi$ be small perturbations and set $a = a_o + \delta a$, $\psi = \psi_o + \delta\psi$. Substituting these expressions into equations (1.4), neglecting powers of δa, $\delta\psi$ above the first and also making use of the relationships (1.6) yields

$$\frac{\Upsilon}{2} d \frac{\delta a}{dt} = -\varepsilon\left\{\left[S' + \left(\frac{c}{4} + \frac{3}{8} d a_o^2\right) \sin 2\psi_o\right]\delta a + \left(\frac{c}{2} + \frac{d}{4} a_o^2\right) a_o \cos 2\psi_o \delta\psi\right\},$$

$$\frac{\Upsilon a_o}{2} \frac{d\delta\psi}{dt} = \varepsilon\left\{\left[\frac{\Delta}{2} + \frac{9}{8}\alpha a_o^2 + H' - \left(\frac{c}{4} + \frac{3}{4} d a_o^2\right)\cos 2\psi_o\right]\delta a + \right.$$

$$\left. + \left(\frac{c}{2} + \frac{d}{2} a_o^2\right) a_o \sin 2\psi_o \delta\psi\right\}.$$

The characteristic equation of this system is given by

$$\left(\frac{\Upsilon}{2}\right)^2 a_o \lambda^2 + \frac{\varepsilon}{2}\Upsilon \frac{\delta}{\delta a_o}(a_o S)\lambda + \frac{a_o^2}{32}\omega^4 (\mathcal{C} + \mathcal{D} a_o^2)(2\mathcal{C} + \mathcal{D} a_o^2)\frac{\delta W}{\delta a_o} = 0, \quad (1.9)$$

where the following notation is introduced $\mathcal{C} = \frac{\varepsilon c}{\omega^2}$, $\mathcal{D} = \frac{\varepsilon d}{\omega^2}$.

The stability condition is given by the Routh-Hurwitz criterion that is

$$\frac{\delta}{\delta a_o}(a_o S) > 0, \quad (\mathcal{C} + \mathcal{D} a_o^2)(2\mathcal{C} + \mathcal{D} a_o^2)\frac{\delta W}{\delta a_o} > 0. \quad (1.10)$$

In the figures presented below the darkish areas correspond to the unstable regions where the conditions (1.10) violated and the undarkish ones — to the stable regions. Sometime the unstable branches of resonant curves are shown dotted to indicate that they are physically unobtainable.

As will be seen later the nonlinearity of the system under consideration/coefficient α/strongly influences to the maximum of amplitudes of stationary oscillations and their stability.

§ 2. THE INFLUENCE OF COULOMB FRICTION

Let us consider the Coulomb friction of type

$$R(x, \dot{x}) = h_o \operatorname{sign}\dot{x} \quad (2.1)$$

where

$$\operatorname{sign}\dot{x} = \begin{cases} 1 \text{ if } \dot{x} > 0, \\ -1 \text{ if } \dot{x} < 0, \\ 0 \text{ if } \dot{x} = 0. \end{cases}$$

In this case we have

$$S(a, \gamma) = \begin{cases} \dfrac{2h_o}{\pi} & \text{for } a \neq 0, \\ 0 & \text{for } a = 0, \end{cases}$$

and the equations (1.4) become

for $a \neq 0$:

$$\frac{\gamma}{2}\dot{a} = -\varepsilon\left[\frac{2}{\pi}h + \frac{a}{8}(2c + da^2)\sin 2\psi\right], \tag{2.2}$$

$$\frac{\gamma}{2}a\dot{\psi} = \varepsilon\left[\frac{\Delta}{2}a + \frac{3}{8}\alpha a^3 - \frac{a}{4}(c + da^2)\cos 2\psi\right],$$

and for $a = 0$:

$$\frac{\gamma}{2}\dot{a} = -\varepsilon\frac{a}{8}(2c + da^2)\sin 2\psi, \tag{2.3}$$

$$\frac{\gamma}{2}a\dot{\psi} = \frac{\varepsilon}{2}a\left[\Delta + \frac{3}{4}\alpha a^2 - \frac{1}{2}(c + da^2)\cos 2\psi\right].$$

The expression (1.8) now takes form

$$W(a_o, \gamma) = \frac{16\mathcal{H}_o^2}{a_o^2(2\mathcal{C} + \mathcal{D}a_o^2)^2} + \frac{4(1 - \eta^2 + \beta a_o^2)^2}{(\mathcal{C} + \mathcal{D}a_o^2)^2} - 1 \tag{2.4}$$

and the equation $W = 0$ gives

$$\eta^2 = 1 + \beta a_o^2 \pm \frac{1}{2}\left|\mathcal{C} + \mathcal{D}a_o^2\right|\sqrt{1 - \frac{16\mathcal{H}_o^2}{a_o^2(2\mathcal{C} + \mathcal{D}a_o^2)^2}} \tag{2.5}$$

$$\eta = \frac{\gamma}{2\omega}, \quad \beta = \frac{3}{4}\frac{\varepsilon}{\omega^2}\alpha, \quad \mathcal{H}_o = \frac{4}{\pi\omega^2}\varepsilon h_o, \quad \mathcal{C} = \frac{\varepsilon c}{\omega^2}, \quad \mathcal{D} = \frac{\varepsilon d}{\omega^2}.$$

Figs. 1 — 3 are obtained by plotting equation (2.5) for the positive $\beta = +0.1$/the resonant curves in the case of negative β are received by mirror reflection/. Figs. 1, 2 correspond to the negative value of d. For the fig. 1 we have $0 > \mathcal{D} > -2\mathcal{C}^3/27\mathcal{H}_o^2$, namely, $\mathcal{D} = -0.1$, $\mathcal{C} = 0.15$, and $\mathcal{H}_o = 0$/straight lines 1/, $\mathcal{H}_o^2 = 10^{-4}$/curve 2/, $\mathcal{H}_o^2 = 6{,}25 \cdot 10^{-4}$/curves 3/, $\mathcal{H}_o^2 = 12{,}5 \cdot 10^{-4}$/curves 4/. The parameters for the fig.2 are $\mathcal{D} < -2\mathcal{C}^3/27\,\mathcal{H}_o^2 < 0$: $\mathcal{D} = -0.1$, $\mathcal{C} = 0.1$, and $\mathcal{H}_o^2 = 9 \cdot 10^{-4}$/curve 2/, $\mathcal{H}_o^2 = 25 \cdot 10^{-4}$/curve 3/.

For the positive value of d we have the resonant curves in fig.3 : $\mathcal{D} = 0.1$, $\beta = 0.1$, $\mathcal{C} = 0.15$ and $\mathcal{H}_o = 0$/straight lines 1/, $\mathcal{H}_o^2 = 10^{-4}$/curve 2/, $\mathcal{H}_o^2 = 16 \cdot 10^{-4}$/curve 3/.

Fig.4 represents the resonant curves in the case $d = 0$ for the parameters $\beta = 0.1$, $\mathcal{C} = 0.15$ and $\mathcal{H}_o = 0$/straight lines 1/, $\mathcal{H}_o^2 = 6{,}25 \cdot 10^{-4}$/curve 2/, $\mathcal{H}_o^2 = 12{,}5 \cdot 10^{-4}$/curve 3/, $\mathcal{H}_o^2 = 25 \cdot 10^{-4}$/curve 4/.

§ 3. TURBULENT FRICTION

$$R(x, \dot{x}) = h_2 \dot{x}^2 \operatorname{sign} \dot{x} \tag{3.1}$$

It is easily to show that in this case

$$S(a, \gamma) = \frac{1}{3\pi} h_2 \gamma^2 a^2$$

$$H(a, \gamma) = 0,$$

and therefore the averaging equations (1.4) take the form

$$\frac{\gamma}{2} \dot{a} = -\varepsilon \left[\frac{1}{3\pi} h_2 \gamma^2 a^2 + \left(\frac{c}{4} a + \frac{d}{8} a^3 \right) \sin 2\psi \right], \tag{3.2}$$

$$\frac{\gamma}{2} a \dot{\psi} = \varepsilon \left[\frac{\Delta}{2} a + \frac{3}{8} \alpha a^3 - \left(\frac{c}{4} a + \frac{d}{8} a^3 \right) \cos 2\psi \right].$$

Now the amplitude a_o of stationary oscillation and the frequency $\gamma/\nu = \frac{\gamma}{2\omega}$/ are related by the equation

$$\nu^2 = 1 + \beta a_o^2 \pm \frac{1}{2} | \mathcal{C} + \mathcal{D} a_o^2 | \sqrt{1 - \frac{4 \mathcal{H}_2^2 a_o^2}{(2\mathcal{C} + \mathcal{D} a_o^2)^2}} \tag{3.3}$$

where $\mathcal{H}_2 = \frac{16}{3\pi} \varepsilon h_2$. Such relationship for $d < 0$ are shown in fig.5/$\beta = 0.1$, $\mathcal{D} = -0.15$, $\mathcal{C} = 0.15$/. For $\mathcal{H}_2 = 0$ we have two crossing straight lines 1. With the small values of $\mathcal{H}_2$/$\mathcal{H}_2 = 10^{-2}$/ the resonant curve consists of three branches 2. The first branch lies above straight line $a_o^2 = -2\mathcal{C}/\mathcal{D}$, the second one — between $a_o^2 = -\mathcal{C}/\mathcal{D}$ and $a_o^2 = -2\mathcal{C}/\mathcal{D}$ and the thirdlower straight line $a_o^2 = -\mathcal{C}/\mathcal{D}$. The two last branches are tightened at the point $a_o^2 = -\mathcal{C}/\mathcal{D}$, $\nu^2 = 1 - \beta \frac{\mathcal{C}}{\mathcal{D}}$. With the growth of $\mathcal{H}_2$ the second branch becomes lower and lower, but the first moves up. For sufficiently large values of $\mathcal{H}_2$ the resonant curves consist of two branches/see curves 3 for $\mathcal{H}_2 = 0.1$/. One of which is above the straight line $a_o^2 = -2\mathcal{C}/\mathcal{D}$ and the other is lower $a_o^2 = -\mathcal{C}/\mathcal{D}$.

The resonant curves for the case $d > 0$ are shown in fig. 6. If $\mathcal{H}_2^2/2\mathcal{C} > \mathcal{D} > 0$ the resonant curve consists of two «parabolic» branches/see curves 3/. With the growth of $\mathcal{H}_2$ these branches move away. For $\mathcal{D} > \mathcal{H}_2^2/2\mathcal{C} > 0$ the resonant curve has form represented by branches 2. The parameters of the curves in fig. 6 are $\mathcal{D} = 0.15$, $\mathcal{C} = 0.15$, $\beta = 0.1$ and $\mathcal{H}_2 = 0$/straight lines 1/, $\mathcal{H}_2 = 0.15$/curves 2/, $\mathcal{H}_2 = 0.22$/curves 3/.

For comparison the resonant curves in the case $d = 0$ are given in fig. 7. The other parameters are: $\beta = 0.1$, $\mathcal{C} = 0.15$ and $\mathcal{H}_2 = 0$/straight lines 1/, $\mathcal{H}_2 = 0.05$/curve 2/, $\mathcal{H}_2 = 0.1$/curve 3/, $\mathcal{H}_2 = 0.15$/curve 4/.

§ 4. TURBULENT FRICTION TOGETHER WITH THE COULOMB ONE

In this section the nonlinear friction of form [2]

$$R(x, \dot{x}) = (h_o + h_2 \dot{x}^2)\, \mathrm{sign}\dot{x} \tag{4.1}$$

is investigated, where h_o, h_2 are the positive constants.

Now the equations (1.4) become

$$\begin{aligned} \frac{\gamma}{2}\dot{a} &= -\varepsilon\left[\frac{2h_o}{\pi} + \frac{h_2}{3\pi}\gamma^2 a^2 + \left(\frac{c}{4}a + \frac{d}{8}a^3\right)\sin 2\psi\right], \\ \frac{\gamma}{2}a\dot{\psi} &= \varepsilon\left[\frac{\Delta}{2}a + \frac{3}{8}\alpha a^3 - \left(\frac{c}{4}a + \frac{d}{4}a^3\right)\cos 2\psi\right]. \end{aligned} \tag{4.2}$$

and the equation (1.7) takes form

$$\frac{4(2\mathcal{H}_o + \mathcal{H}_2\eta^2 a_o^2)^2}{a_o^2(2\mathcal{C} + \mathcal{D}a_o^2)^2} + \frac{4(1 - \eta^2 + \beta a_o^2)^2}{(\mathcal{C} + \mathcal{D}a_o^2)^2} - 1 = 0 \tag{4.3}$$

For $d > 0$ the resonant curves have form presented in fig.8/$\beta = 0.1$, $\mathcal{C} = \mathcal{D} = 0.15$/. Straight lines 1 correspond $\mathcal{H}_o = \mathcal{H}_2 = 0$ and curves 2,3 correspond to $\mathcal{H}_o^2 + \mathcal{H}_2^2 \neq 0$: $\mathcal{H}_o = \mathcal{H}_2 = 5.10^{-2}$/curve 2/, $\mathcal{H}_o = \mathcal{H}_2 = 7{,}5.10^{-2}$/curve 3/.

If $d < 0$, then depending on the disposition of the curves

$$y = 4(2\mathcal{H}_o + \mathcal{H}_2 A)^2, \quad z = A(2\mathcal{C} + \mathcal{D}A)^2 \tag{4.4}$$

the resonant curves have forms shown in fig.9. The curve 2 corresponds to the case when there exists only a point of intersection of the curves (4.4). Curve 3 and point 3 correspond to the points of intersection A_1, A_2, A_3 of the curves (4,4): $A_1 > -2\mathcal{C}/\mathcal{D}$, $A_2 = A_3 = -\mathcal{C}/\mathcal{D}$. If the curves (4.4) have three separated points of intersection then the resonant curves have form «4» in fig.9 if $A_1 > -2\mathcal{C}/\mathcal{D}$, $A_2 < -\mathcal{C}/\mathcal{D}$, $A_3 < -\mathcal{C}/\mathcal{D}$, and form «5» if $A_1 > -2\mathcal{C}/\mathcal{D}$, $-2\mathcal{C}/\mathcal{D} > A_2 > -\mathcal{C}/\mathcal{D}$, $A_3 < -\mathcal{C}/\mathcal{D}$.

The resonant curves in the case $d = 0$ are represented in fig.10 for $\beta = 0.25$, $\mathcal{C} = 0.16$, $\mathcal{H}_o = 2.10^{-2}$ and $\mathcal{H}_2 = 0.16$/point 2/, $\mathcal{H}_2 = \frac{46}{3}\cdot 10^{-2}$ /curve 3/, $\mathcal{H}_2 = \frac{40}{3}\cdot 10^{-2}$/curve 4/, $\mathcal{H}_2 = 12.10^{-2}$/curve 5/.

To compare with the linear friction in figs. 11, 12 and 13 the amplitude — frequency responses in the system (0.1) with linear friction $R(x, \dot{x}) = h\dot{x}$ are plotted for the case $d < 0$/fig.11/, $d > 0$/fig.12/and $d = 0$/fig.13/. The curves in these figures are presented for the case $\beta = 0.1$, $\mathcal{C} = 0.15$. The other parameters for fig.11 are $\mathcal{D} = -0.15$ and $\mathcal{H} = 0$/curve 1/$\mathcal{H} = 0.03$/curve 2/, $\mathcal{H} = 0.21$ /curve 3/, $\mathcal{H} = 0.3$/curve 4/. For fig.12 we have $\mathcal{D} = 0.15$ and $\mathcal{H} = 0$/curve 1/, $\mathcal{H} = 0.2$/curve 2/, $\mathcal{H} = 0.3$/curve 3/, $\mathcal{H} = 0.45$/curve 4/and for fig.13 : $d = 0$, $\mathcal{H} = 0$/curve 1/, $\mathcal{H} = 0.27$/curve 2/, $\mathcal{H} = 0.28$/curve 3/, $\mathcal{H} = 0.297$/curve 4/, where $\mathcal{H} = 4\varepsilon h/\omega$.

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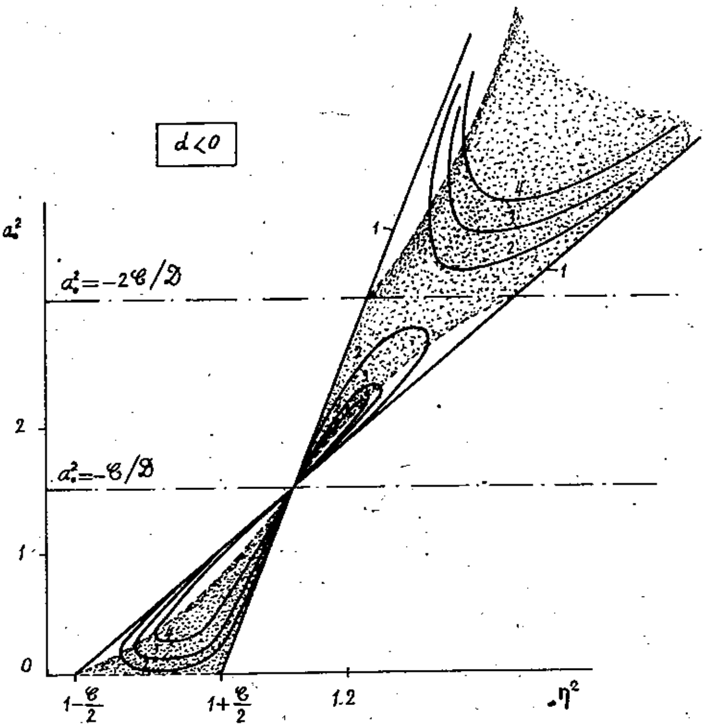
d<0
a_0^2
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$a_0^2=-\mathcal{C}/\mathcal{D}$
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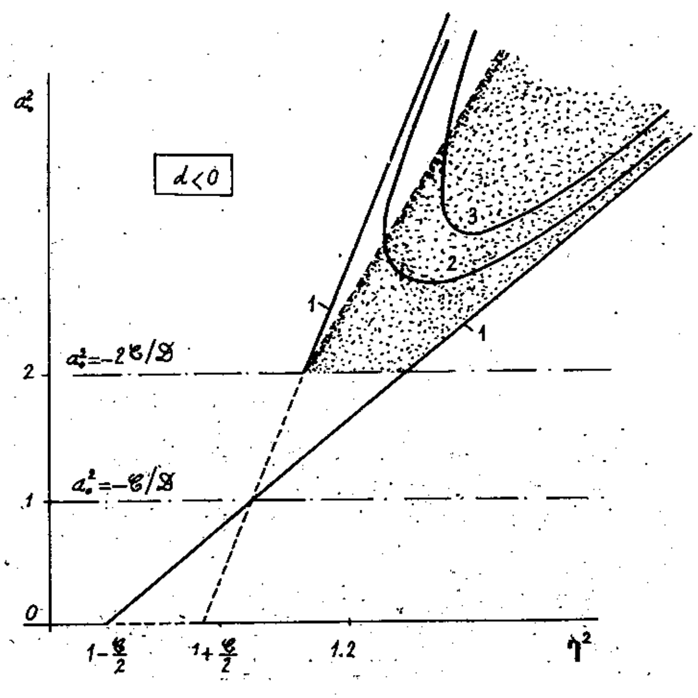
1
0
$1-\frac{\mathcal{C}}{2}$
$1+\frac{\mathcal{C}}{2}$
1.2
η^2

Fig. 1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a_0^2
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$a_0^2=-\mathcal{C}/\mathcal{D}$
2
1
0
$1-\frac{\mathcal{C}}{2}$
$1+\frac{\mathcal{C}}{2}$
1.2
η^2

Fig. 2

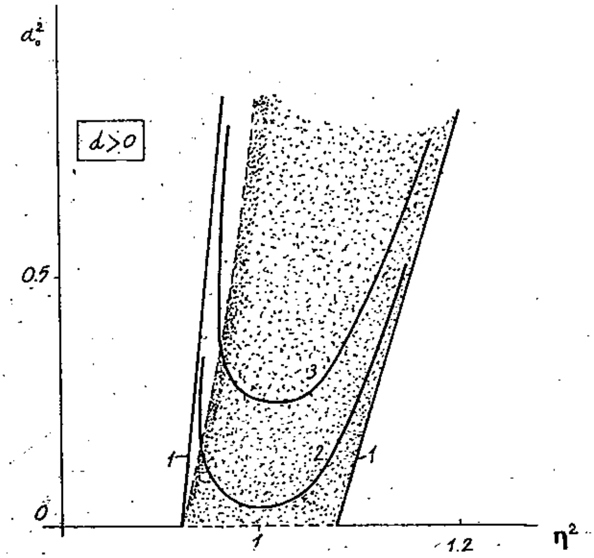


Fig. 3

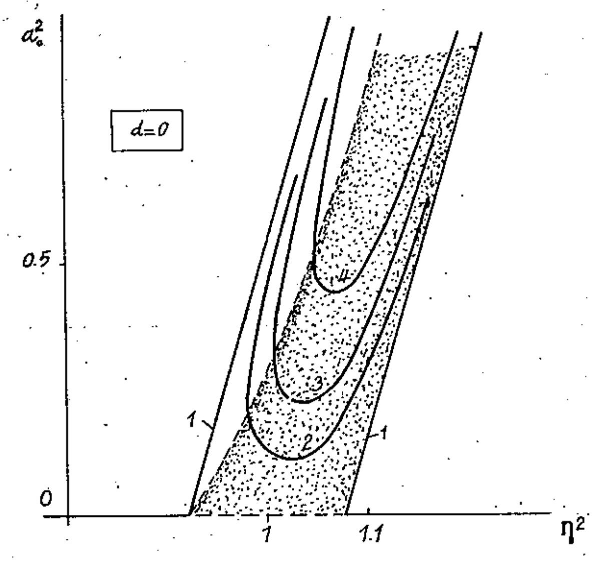
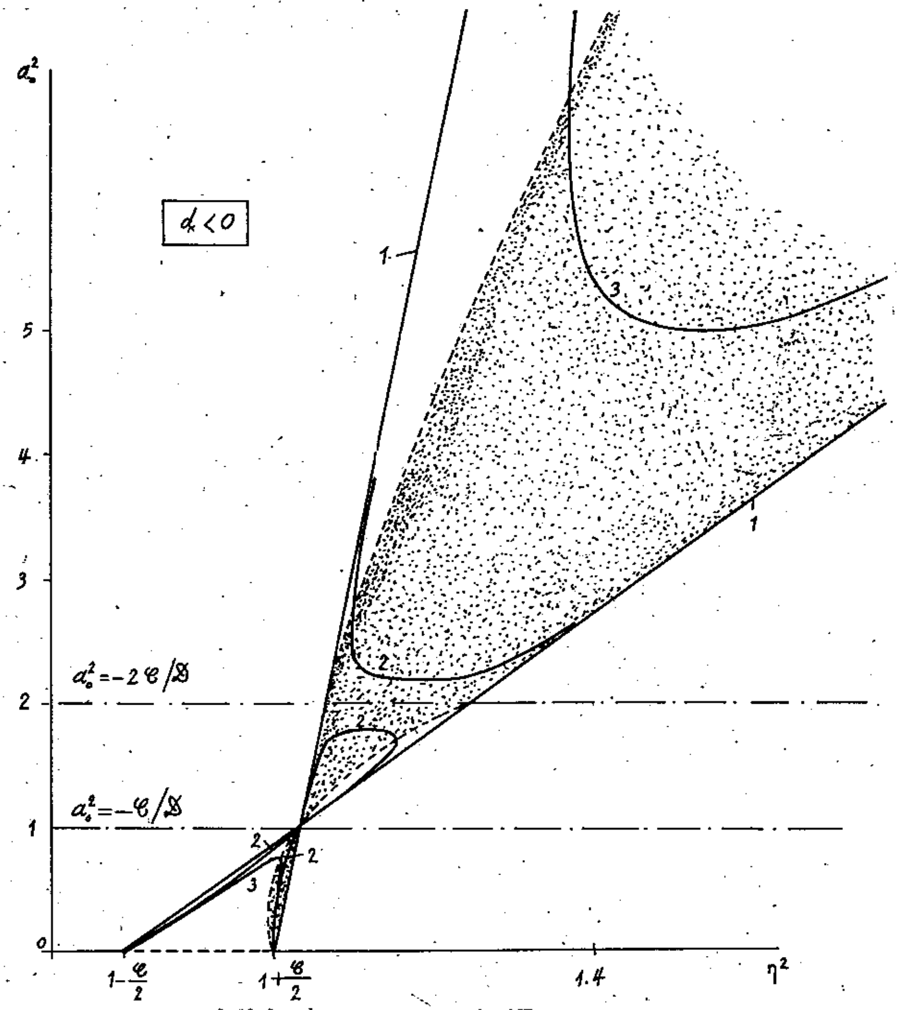
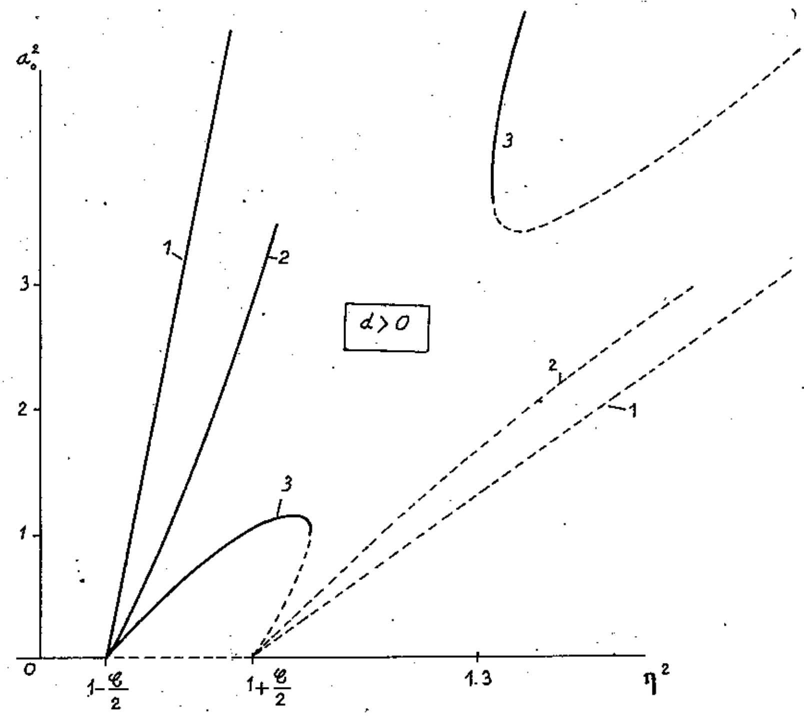


Fig. 4



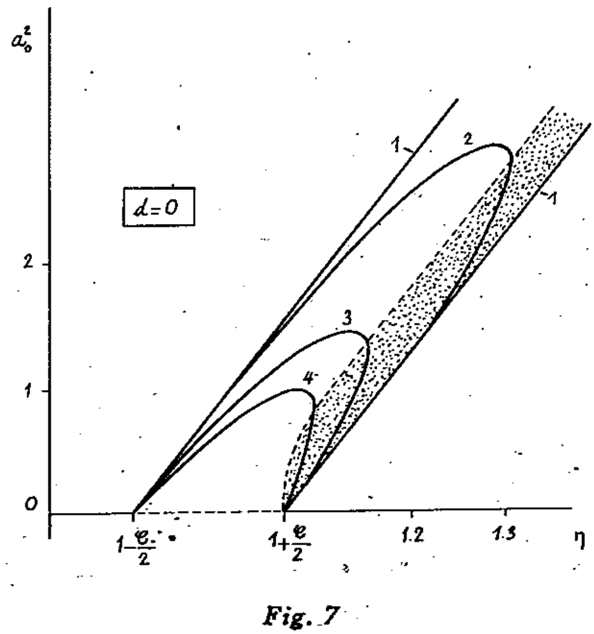
a_0^2
$d < 0$
5
4
3
2
1
0
$a_0^2 = -2\mathscr{C}/\mathscr{D}$
$a_0^2 = -\mathscr{C}/\mathscr{D}$
1
2
3
$1-\frac{\mathscr{C}}{2}$
$1+\frac{\mathscr{C}}{2}$
1.4
η^2

Fig. 5



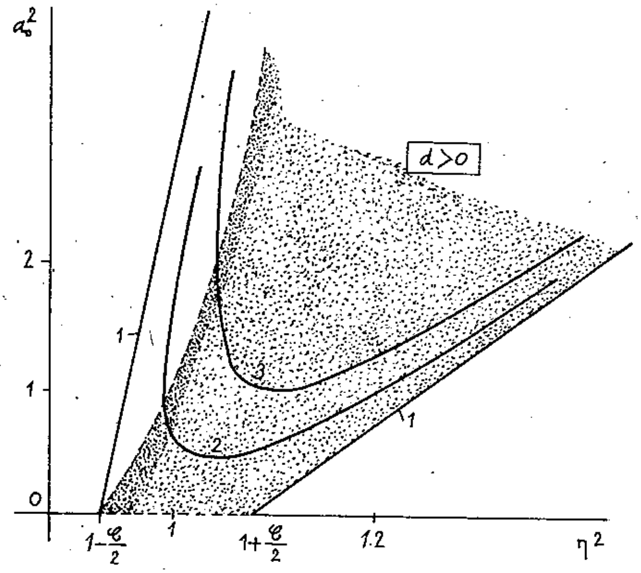
a_0^2
3
2
1
0
1
2
3
2
1
$d>0$
$1-\frac{\mathscr{C}}{2}$
$1+\frac{\mathscr{C}}{2}$
1.3
η^2

Fig. 6



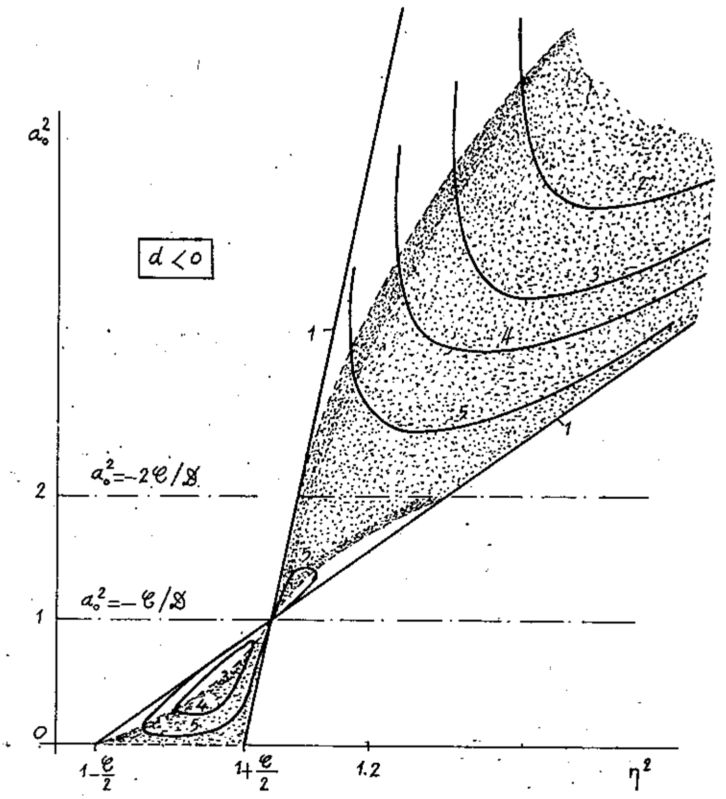
a_0^2
$d=0$
2
1
0
1
2
3
4
1
$1-\frac{\mathscr{C}}{2}$
$1+\frac{\mathscr{C}}{2}$
1.2
1.3
η

Fig. 7



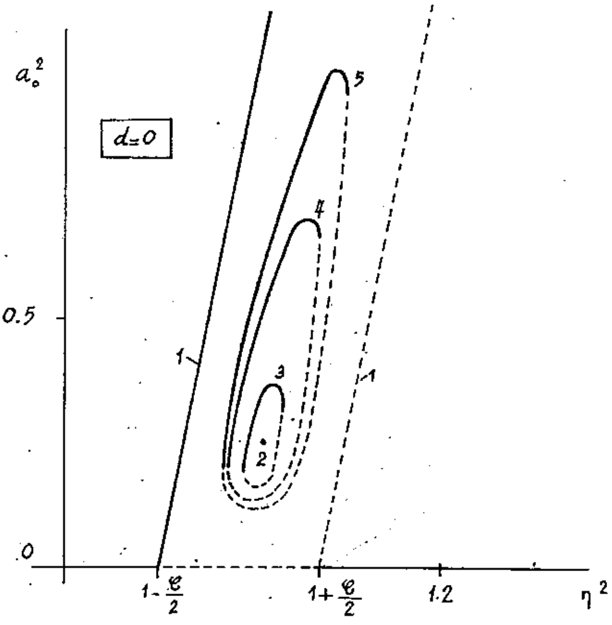
a_o^2
d > 0
2
1
0
$1-\frac{C}{2}$
1
$1+\frac{C}{2}$
1.2
η^2

Fig. 8



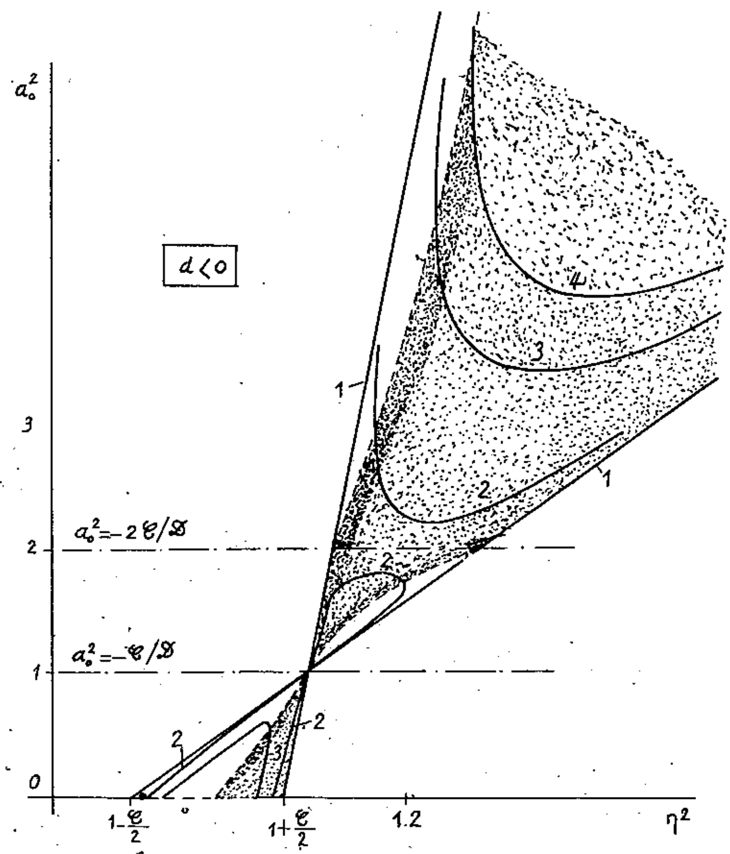
a_o^2
d < 0
$a_o^2 = -2C/D$
$a_o^2 = -C/D$
2
1
0
$1-\frac{C}{2}$
$1+\frac{C}{2}$
1.2
η^2

Fig. 9



a_o^2
d=0
0.5
0
1
2
3
4
5
$1-\frac{\mathscr{C}}{2}$
$1+\frac{\mathscr{C}}{2}$
1.2
η^2

Fig. 10



a_o^2
d<0
3
2
1
0
$a_o^2=-2\mathscr{C}/\mathscr{D}$
$a_o^2=-\mathscr{C}/\mathscr{D}$
1
2
3
4
$1-\frac{\mathscr{C}}{2}$
$1+\frac{\mathscr{C}}{2}$
1.2
η^2

Fig. 11

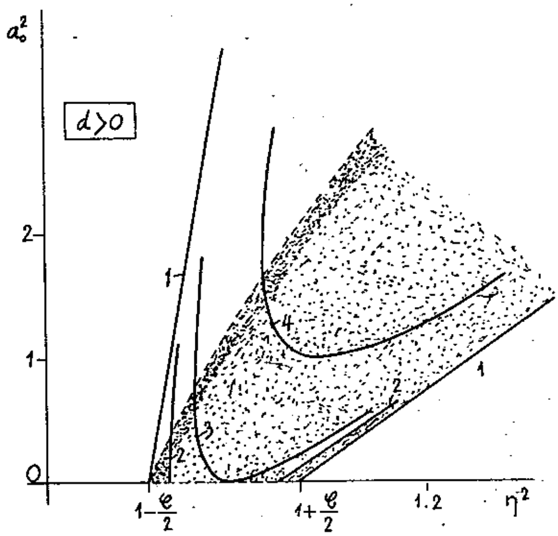

a_0^2
$d>0$
2
1
0
$1-\frac{\varphi}{2}$
$1+\frac{\varphi}{2}$
1.2
η^2
1
2
3
4

Fig. 12

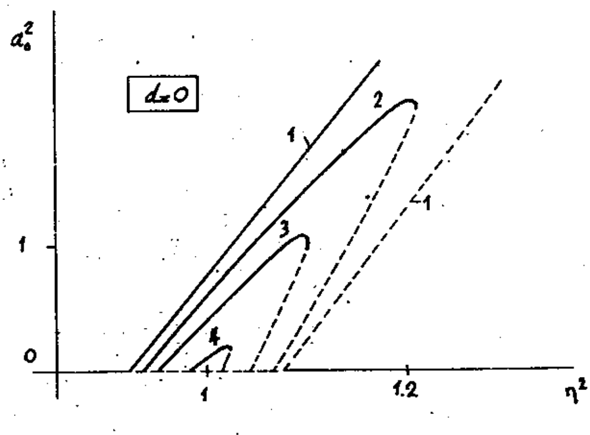

a_0^2
$d=0$
1
0
1
1.2
η^2
1
2
3
4

Fig. 13