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STABILITY, SURJECTIVITY, — AND LOCAL INVERTIBILITY OF NON DIFFERENTIABLE MAPPINGS

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1. INTRODUCTION

Many questions in different fields of mathematics can be reduced to the solvability of an equation of the form

$$f(x) = y \tag{1}$$

where f is a mapping from an open subset of a linear topological space X into a linear topological space Y and y is an element of Y.

Suppose x^0 is a solution of the equation. It is often of interest to know whether this solution is *stable* in the following sense: for every neighbourhood W of x^0 one can find a neighbourhood V of y such that for every $v \in V$ the equation

$$f(x) = v$$

still has a solution in W. This condition amounts to requiring that $V \subset f(W)$, i. e. f maps every neighbourhood of x^0 onto a neighbourhood of $f(x^0)$.

We may say in this case that the mapping f is *surjective at* x^0. The problem of stability of x^0 may then be formulated as the question of under which conditions the mapping f will be surjective at this point.

In the simplest cases, when f is linear, along known answer to this question is furnished by the Banach classical open mapping theorem which asserts that if X, Y are Banach spaces and if a linear mapping $f : X \to Y$ is continuous and surjective then it is open (and hence, surjective) at every point x.

When f is nonlinear, a familiar method in classical analysis is to linearize f. i. e. to approximate it, in a certain sense, by means of a linear mapping

$g : X \to Y$. But then a second problem naturally arises: suppose that the linearization of f at x^0 is surjective at x^0; can one be sure that the mapping f itself is surjective at this point?

For the case of Banach spaces it can be shown that the answer will be positive, provided f is continuously Fréchet differentiable in a neighbourhood of x^0 and the linearization is $f(x^0) + f'(x - x^0)$, where f' is the derivative of f at x^0. This fact is related to a theorem of Ljusternik [5] asserting that, under the indicated conditions, the linear manifold $f'(x - x^0) = 0$ is tangent at x^0 to the manifold $f(x) = f(x^0)$

In Convex Analysis, motivated mainly by the applications to economics and optimization theory, we have often to consider equations of a more general form than (1). Namely we are frequently interested in the solvability of an equation of the type

$$f(x) \in y + M \tag{2}$$

where M is a closed convex cone given in the range spaee Y of f. Moreover, whereas in classical analysis the mapping f is generally assumed to enjoy very nice differentiability properties, the specific nature of many problems in economics and control theory make it necessary to abandon or at least to relax this assumption.

It is the aim of this paper to study the two problems formulated above for equations (2), or, more generally, for equations of the form $0 \in F(x)$, where F is a multivalued mapping from X into Y. The presence of the cone M and the lack of differentiability properties give rise to technical difficulties which have to be overcome. However, it turned out that several meaningful results can be obtained in this genneral setting which are reminiscent of familiar facts in classical analysis and can be fruitfully exploited for the purpose of applications.

The results that follow are related to ealier works of S.M. Robinson ([7], [8]) and Hoàng Tụy ([13] [14]) [15]). In particular, problems of the same kind as those we are considering here have been discussed in [15]; only, there, the mapping f was defined on a non necessarily open subset D of X, while the space Y was more specialized.

2. SURJECTIVE MULTIVALUED MAPPING THEOREM

We begin with the first problem.

Consider a multivalued mapping F from X into Y. We adopt the conventional notations.

dom $F = \{x \in X : F(x) \neq \varnothing\}$, graph $F = \{(x,y) : y \in F(x)\}$. It is convenient to recall the following definitions.

Definition 1. We say that F is *convex* if its graph is a convex set in $X \times Y$.

As can be easily seen, this amounts to requiring that for every pair $x, x' \in$ dom F and for every real number $\alpha \in [0, 1]$:

$$F(\alpha x + (1 - \alpha) x') \supset \alpha F(x) + (1 - \alpha) F(x') \tag{4}$$

Definition 2. We say that F is *closed* if its graph is a closed set in $X \times Y$.

The key for an answer to the question of when a solution x of the equation (2) is stable is furnished by the following result.

Theorem 1. *Assume that X, Y are Fréchet spaces and let F be a convex closed multivalued mapping from X into Y, whose range F (X) is a set of the second category in Y. Then for every $x \in$ int dom F and for every neighbourhood U of x contained in dom F we have*

$$F(x) \cap \operatorname{int} \overline{F(U)} \neq \emptyset \tag{5}$$

(As usually, int A and $\overline{A}$ denote the interior and the closure of the set A.

Proof. The proof is along the lines of the standard proof of Banach's open mapping theorem, although several technical details must be adjusted to the nonlinearity of F.

It suffices to prove (5) for the case where $x = 0$ and $F(0)$, since the general case can be reduced to this one by translating the origin in X to x and the origin in Y to an arbitrary point $y \in F(x)$. Also, without loss of generality, we may assume U to be a closed ball around 0.

Observe first that for every positive integer n:

$$F(nU) \subset n\overline{F(U)} \tag{6}$$

Indeed, if $v \in F(nU)$, then $v \in F(u)$ for some $u \in nU$ and from (4), where we set $x = u$, $x' = 0$, $\alpha = \frac{1}{n}$, it follows that

$$\frac{1}{n} F(u) + \left(1 - \frac{1}{n}\right) F(0) \subset F\left(\frac{1}{n} u\right),$$

and hence $\frac{1}{n} v + \left(1 - \frac{1}{n}\right) 0 \in F\left(\frac{1}{n} u\right) \subset F(U)$, i. e. $v \in n F(U)$

Further, since the set

$$F(X) = \bigcup_{n=1}^{\infty} F(nU)$$

is by hypothesis of the *second* category, there is a positive integer n for which int $\overline{F(nU)} \neq \emptyset$ Then, by (6)

$$\operatorname{int} \overline{F(U)} \neq \emptyset, \tag{7}$$

so that one can find a point $b \in \operatorname{int} \overline{F(U)}$ such that $b \in F(a)$ for some $a \in U$. Noting that $-a \in U \subset \operatorname{dom} F$, one can next take a point $b' \in F(-a)$, and using (4), one can write

$$\frac{1}{2} F(a) + \frac{1}{2} F(-a) \subset F\left(\frac{1}{2} a - \frac{1}{2} a\right) = F(0).$$

which implies that the point $c = \frac{1}{2}(b + b')$ belongs to $F(0)$. Furthermore, $c \in \operatorname{int} \overline{F(U)}$, because $b \in \operatorname{int} \overline{F(U)}$ and the set $\overline{F(U)}$ is convex. Thus we have found a point

$$c \in F(0) \cap \operatorname{int} \overline{F(U)}. \tag{8}$$

We now show that, actually,

$$c \in \operatorname{int} F(U) \tag{9}$$

which will conclude the proof

Let $\varepsilon > 0$ be the radius of U, and let U_i denote the closed ball of radius $\frac{\varepsilon}{2^i}$ around $0 \in X$ ($i = 0, 1, 2, \ldots$; so $U_0 = U$). By (8) there is a ball V_0 around $0 \in Y$ such that $c + V_0 \subset \overline{F(U_0)}$. Let $V_i = \frac{1}{2^i} V_0$. Then, by (6), $F(U_0) \subset 2^i F\left(\frac{1}{2^i} U_0\right)$, so that

$$V_i = \frac{1}{2^i} V_0 \subset \frac{1}{2^i} \overline{(F(U_0) - c)} \subset \overline{F\left(\frac{1}{2^i} U_0\right)} - c \tag{10}$$

Consider an arbitrary point $v \in V_1$. We shall construct by induction a sequence $z^1, z^2, \ldots$, such that

$$z^i \in F(U_0) - c \tag{11}$$

$$v - \left(\frac{z^1}{2} + \frac{z^2}{2^2} + \ldots + \frac{z^i}{2^i}\right) \in V_{i+1}. \tag{12}$$

Since $v \in V_1$, we have $2v \in V_0 \subset \overline{F(U_0)} - c$ and hence there is $z^1 \in F(U_0) - c$ satisfying $2v - z^1 \in V_1$, i. e. $v - \frac{z^1}{2} \in \frac{1}{2} V_1 = V_2$. Assume now that $z^1, z^2, \ldots z^i$ have been constructed. From (12) we have

$$2^{i+1}\left[v - \left(\frac{z^1}{2} + \frac{z^2}{2^2} + \ldots + \frac{z^i}{2^i}\right)\right] \in 2^{i+1} V_{i+1} = V_0 \subset \overline{F(U_0)} - c,$$

so we can find $z^{i+1} \in F(U_0) - c$ satisfying

$$2^{i+1}\left[v - \left(\frac{z^1}{2} + \frac{z^2}{2^2} + \ldots + \frac{z^i}{2^i}\right)\right] - z^{i+1} \in V_1.$$

Then

$$v - \left(\frac{z^1}{2} + \frac{z^2}{2^2} + \ldots + \frac{z^i}{2^{i+1}}\right) \in \frac{1}{2^{i+1}} V_1 = V_{i+2} \tag{13}$$

From (11) it follows that $z^i \in F(x^i) - c$ for some $x^i \in U_0$. By the convexity of F:

$$F\left(\sum_{i=1}^{k} \frac{x^i}{2^i}\right) = F\left(\sum_{i=1}^{k} \frac{x^i}{2^i} + \left(1 - \sum_{i=1}^{k} \frac{1}{2^i}\right) . 0\right) \supset$$

$$\supset \sum_{i=1}^{k} \frac{1}{2^i} F(x^i) + \left(1 - \sum_{i=1}^{k} \frac{1}{2^i}\right) F(0),$$

and since $0 \in F(0)$, we deduce

$$\sum_{i=1}^{k} \frac{1}{2^i} (z^i + c) + 0 \in F \left(\sum_{i=1}^{k} \frac{x^i}{2^i} \right). \tag{14}$$

Setting $s_k = \sum_{i=1}^{k} \frac{x^i}{2^i}$, we see that $\{s_k\}$ is a Cauchy sequence, because

$$| s_k - s_l | \leqslant \varepsilon \sum_{i=1}^{k} \frac{1}{2^i} \to 0 \text{ as } k, l \to \infty.$$

Therefore, since X is complete, $\{s_k\}$ converges to some $s^* \in U_0$. On the other hand, by (12),

$$\sum_{i=1}^{k} \frac{z^i}{2^i} \to v \ (k \to \infty).$$

Nothing that F is closed by hypothesis, we then conclude from (14):

$$v + c \in F(s^*) \subset F(U_0),$$

and hence, since this is true for any $v \in V_1$:

$$V_1 + c \subset F(U_0).$$

The proof of Theorem 1 is complete.

It is obvious that Theorem 1 yields as a corollary the classical Banach open mapping theorem when F is single — valued, since in that case F is convex in the sense of Definition 1 if and only if it is an affine mapping in the usual sense. Furthermore, one can derive from Theorem 1 the following proposition which provides a satisfactory answer to the first question put in the Introduction.

Definition 3. We say that a multivalued mapping F from X into Y, is *surjective* if $F(X) = Y$.

Definition 4. We say that a multivalued mapping F from X into Y is *surjective at a point* $x^0 \in$ int dom F if it carries every neighbourhood U of x^0 onto a neighbourhood $F(U)$ of $F(x^0)$. F is *locally surjective* if it is surjective at every point $x \in$ int dom F.

Clearly a mapping F is locally surjective if and only if it carries every open set contained in int dom F onto an open set.

Theorem 2. *Assume that X, Y are Fréchet spaces. Then every convex and closed multivalued mapping from X into Y which is surjective is locally surjective.*

Proof. Since $F(X) = Y$ is a Fréchet space, it is a set of the second category and by Theorem 1 property (5) holds, for every $x \in$ int dom F and every open set U containing x. To prove the local surjectivity of F it is enough to show that, actually,

$$F(x) \subset \text{int } F(U).$$

Let $c \in F(x) \cap$ int $F(U)$ and let $u \neq c$ be any element of $F(x)$. Consider any $v = c + \lambda (u - c)$ with $\lambda > 1$. Then $v \in F(z)$ for some z. For $\alpha > 0$ small enough, $x' = \alpha z + (1 - \alpha) x \in U$ and by the convexity of F we shall have $\alpha F(z) + (1 - \alpha) F(x) \subset F(x') \subset F(U)$, so that $w = \alpha v + (1 - \alpha) u \in F(U)$. But $u = \mu c + (1 - \mu) . w$ for some $\mu \in (0,1)$ and since $c \in \text{int } F(U)$, this shows, in view of the convexity of the set $F(U)$, that $u \in \text{int } F(U)$, which completes the proof.

Let M be a closed convex cone in Y.

Definition 5. We say that a (single — valued) mapping $f: X \to Y$ is *M — convex* (*M — closed*, *M — surjective*, resp) if the associated multivalued mapping $F(x) = f(x) - M$ is convex (closed, surjective, resp).

As an immediate consequence of Theorem 2 we obtain

Corollary 1. *Every mapping $f: X \to Y$ which is M — convex, M — closed and M — surjective is locally M — surjective.*

3. LOCAL INVERSION THEOREM

Turning to the second problem, we now consider an arbitrary mapping $f: U \to Y$ from an open subset U of a Banach space X into a Banach space Y. Let the space Y be equipped with a closed convex cone M and suppose that at some point x^0 the mapping f can be approximated (in a sense to be made precise) by a M — surjective mapping. What can be said about the M — surjectivity of f itself at the given point?

If the approximation is taken to be the ordinary linearization by means of the Fréchet derivative, then an answer to this question has been provided by a result of S.M. Robinson in [8]. Another answer, for the case where $M = \{0\}$, the approximation being understood in a more relaxed sense, has been furnished by a result established by A.E. Ioffe and V.M. Tikhomirov [3] in connection with a generalization of Ljusterniks' theorem.

In many applications, however, the mapping f is not differentiable, while $K \neq \{0\}$ (this situation is often encountered, for example, in optimal control theory). It seems therefore to be of interest to investigate the problem in the general case, as we are intended to do in this section.

Before we try to prove any precise answer to the question, we must, of course, specify two things:

1. the operation of «approximating»: what we mean precisely when we say that some mapping is an «approximation» of another one; in other words, what we propose to substitute for the usual notion of tangent mappings that is involved in the classical operation of differentiating?

2. the class of mappings to be used as «approximating mappings», i.e. what we propose to substitute for the ordinary linear transformations?

The results of the previous section suggest to use the class of M — convex mappings as a substitute for linear transformations. In fact, in order to obtain meaningful results, we shall use an intermediate class, namely the class of M— convex mappings that are positively homogeneous. On one hand this class is sufficiently large to cover virtually all cases of interest in practice. On the other hand, the mappings in this class have many properties near to those of linear transformations and, consequently, they are relatively easy to handle.

As a substitute for the classical notion of tangent mappings, we introduce the following

Definition 6. Let $f_1, f_2: U \to Y$ be two mappings from an open subset U of X into Y. We shall say that f_2 is in the (U, α) — *Lipschitz proximity* of f_1 (or f_1 is in the (U, α) — Lipschitz proximity of f_2) if the difference $f_1(x) - f_2(x)$ satisfies in U the α — Lipschitz condition, i. e.

$$(\forall\ x, x' \in U)\ |f_1(x) - f_2(x) - (f_1(x') - f_2(x'))| \leqslant \alpha\,|x-x'| \tag{15}$$

As will be seen later (Lemma 3), if f_1 is continuously Fréchet differentiable in U, then the linearization of f_1 at any point $x^0 \in U$, i.e. the mapping $f_2(x) = f_1(x^0) + f'_1\,(x-x^0)$, with f'_1 the derivative of f_1 at x^0, is in the (U, α) — — Lipschitz proximity of f_1 for some $\alpha > 0$. Thus the previous notion of approximation covers most usual cases of tangent mappings in the classical sense.

Remark. The terminology used above is motivated by the following fact. Let Y^M_{Lips} denote the linear space of all mappings from U into Y that are Lipschitzian in U. If for each $f \in Y^U_{\text{Lips}}$ we define the norm $\|f\|^U_{\text{Lips}}$ to be the smallest of all numbers $\alpha \geqslant 0$ such that

$$\forall\ x \in U \qquad |f\,(x)| \leqslant \alpha \tag{16}$$

$$\forall\ x, x' \in U\ |f\,(x) - f\,(x')| \leqslant \alpha\,|\,x - x'\,| \tag{17}$$

then Y^U_{Lips} becomes a normed space.

Now, before stating our main theorem in this section, we recall some pertinent notions and results about *convex processes.* Those are by definition ([10]) multivalued mappings from X into Y, whose graphs are convex cones containing the origin in $X \times Y$.

In what follows, given a mapping $f: X \to Y$ we shall try to approximate it (in the sense of Definition 6) by a mapping $f\,(x^0) + g\,(x - x^0)$, where g is a M — convex and positively homogeneous mapping. Then the associated multivalued mapping $G\,(x) = g\,(x) - M$ is obviously a convex process.

The *norm* of a convex process G, written $|\,G\,|$, is defined to be the smallest of all numbers $\gamma \geqslant 0$ such that for every $x \in \text{dom}\ G$ there is an $y \in G\,(x)$ satisfying

$$|\,y\,| \leqslant \gamma\,|\,x\,|$$

(if no such γ exists we set $|\,G\,| = +\infty$)

The inverse of a convex process G is the convex process G^{-1} that carries every y into $G^{-1}\,(y) = \{\,x:\ y \in G\,(x)\,\}$. Clearly $|\,G^{-1}\,|$ is the smallest of all numbers $\gamma \geqslant$ o such that for every y in the range of G there is an $x \in G^{-1}\,(y)$ satisfying

$$|\,x\,| \leqslant \gamma\,|\,y\,|$$

Lemma 1. *If a convex proccess G is surjective, then $|G^{-1}|$ is finite and for every pair $y, y' \in Y$ we have.*

$$h(G^{-1}(y), G^{-1}(y')) \leqslant |G^{-1}| \cdot |y - y'| \tag{18}$$

where $h(A,B)$ denotes the Hausdorff distance between sets A and B.

Proof. This proposition follows easily from the results in [7]. For the sake of completeness, we give here a direct proof, which is very simple.

By Theorem 2 G is locally surjective, and so the unit ball $U = \{x : |x| \leqslant 1\}$ is mapped by G onto a set $G(U)$ containing a ball V around $0 \in Y$. Let γ denote the radius of V. Then for every $y \in Y$ there is $u \in V$ such that $\gamma \frac{y}{|y|} \in G(u)$. Setting $x = \frac{|y|}{\gamma} u$, we have $y \in G(x)$, and $|x| = \frac{|y|}{\gamma} |u| \leqslant \frac{|y|}{\gamma}$, which shows that $|G^{-1}| \leqslant \gamma < \infty$. Let now $y, y' \in Y$ and $x \in G^{-1}(y)$. By the definition of $|G^{-1}|$ there is an $u \in G^{-1}(y' - y)$ satisfying $|u| \leqslant |G^{-1}| \cdot |y' - y|$. Setting $x' = x + u$ we get $y' \in y + G(u) \subset G(x) + G(u) \subset G(x')$, i. e. $x' \in G^{-1}(y')$. Since $|x' - x| = |u| \leqslant |G^{-1}| \cdot |y' - y|$, (18) follows.

Let us recall also a lemma on multivalued contraction mappings, which is essentially due to S. B. Nadler [60] (see also [8], or [3]; the following variant is taken from [8]. We shall need this lemma in the proof of our Theorem 3.

Lemma 2. *Let $S = S(x^0, r)$ denote the closed ball of radius r around a point $x^0 \in X$, and let P be a multivalued mapping from S into X such that for every x the set $P(x)$ is nonempty and closed. If there exists a number θ such that $0 < \theta < 1$ and.*

$$h(P(x), P(x')) < \theta |x - x'| \quad \forall x, x' \in S;$$

$$\rho(x^0, P(x^0)) < (1 - \theta) r,$$

then for every $\delta > 0$ there exists a point $x \in P(x)$ such that $|x - x^0| < \frac{1+\delta}{1-\theta} \rho(x^0, P(x^0))$

$$\text{(here } \rho(x^{\bullet}, B) = \inf_{x \in B} |x - x^0| \text{)}.$$

We are now in a position to state the main result of this section.

Theorem 3. *Let $f : U \to Y$ be a mapping defined in an open subset U of X, and let $x^0 \in U$. Assume there is in a (U, α) — Lipschitz proximity of f a M—surjective mapping $x \to f(x^0) + g(x - x^0)$ such that $g : X \to Y$ is continuous, M — convex, positively homogeneous, and $\alpha |G^{-1}| = \theta < 1$, where G denotes the convex process $G(x) = g(x) - M$.*

Then f is M — surjective at x^0

More precisely, if $r = \rho(x^0, X \setminus U)$, then for every v such that $|v - f(x^0)| < \frac{r(1-\theta)}{|G^{-1}|}$ and for every number $\delta > o$ the equation.

$$f(x) \in v + M \tag{19}$$

has at least one solution x satisfying

$$| x - x^0 | \leqslant c \; | v - f(x^0) | \tag{20}$$

with $c = \dfrac{1+\delta}{1-\theta} \; | G^{-1} |$

Proof. Denote by V the open ball of radius $\dfrac{r(1-\theta}{|G^{-1}|}$ around $f(x^0)$. Let v be an arbitrary element of V and define a multivalued mapping P from S into X such that $P(x) = x^0 + G^{-1}(v - f(x) + g(x - x^0))$.
Since g is M — surjective and continuous by hypothesis, the set $P(x)$ is nonempty and closed. Further, since $f(x^0) + g(x - x^0)$ is in the $(\mathring{U}, \alpha)$—Lipschitz proximity of f, we have for all $x, x' \in$ U :

$$| f(x) - f(x') - (g(x - x^0) - g(x' - x^0)) | \leqslant \alpha \, | x - x' | \tag{21}$$

It then follows from Lemma 1 that for all $x, x' \in S$:

$$h(P(x), P(x')) \leqslant | G^{-1} | \; | f(x) - f(x') - (g(x - x^0) - g(x' - x^0)) |$$
$$\leqslant \alpha \, | G^{-1} | \, . \, | x - x' | = \theta \, | x - x' | \tag{22}$$

On the other hand, using the definition of $| G^{-1} |$, we can write

$$\rho(x^0, P(x^0)) = \inf \{ | x - x^0 | : x \in x^0 + G^{-1}(v - f(x^0)) \}$$
$$\leqslant | G^{-1} | \; | v - f(x^0) | \tag{23}$$

and, therefore, since $v \in V$:

$$\rho(x^0, P(x^0)) < (1 - \theta) \, r \tag{24}$$

This, together with (22), shows that Lemma 2 applies to the mapping P. Hence there exists a point $x \in S$ such that

1/ $x \in P(x)$, i. e. $v - f(x) + g(x - x^0) \in g(x - x^0) - M$ or $f(x) \in v + M$;

2/ $| x - x^0 | \leqslant \dfrac{1+\delta}{1-\theta} \rho(x^0, P(x^0)) < \dfrac{1+\delta}{1-\theta} \; | G^{-1} | \; | v - f(x^0) |$.

Thus for every $v \in V$ the equation (19) has a solution x satisfying (20). This implies that, given an arbitrary neighbourhood U' of x^0, one can choose a neighbourhood V of $f(x^0)$ so that $V \subset$ F (U') for F $(x) = f(x) - M$. Therefore, the mapping f is M — surjective at x^0

The proof is complete.

Corollary 2. *Assume, in addition to all the conditions specified in Theorem 3, that g is linear. Then for every* $u \in U$, *for every* v *such that* $| v - f(u) | < \dfrac{1-\theta}{|G^{-1}|} \rho(u, X \setminus U)$ *and for every* $\delta > o$ *the equation*

$$f(x) \in v + M$$

has at least one solution x satisfying

$$| x - u | \leqslant c \, | v - f(u) | ,$$

where c is the same constant as in Theorem 3. Hence f is M — surjective at every point of U and the mapping F $(x) = f(x) - M$ *carries every open subset of U onto an open subset of F (U).*

Proof. Let u be an arbitrary element of U. From (21), using the linearity of g, we have.

$$\forall\ x, x' \in U \mid f(x) - f(x') - (g(x-u) - g(x'-u)) \mid \leqslant \alpha \mid x - x' \mid$$
which shows that $x \to f(u) + g(x-u)$ — is also in the (U, α) — Lipschitz proximity of f. Therefore, all the conclusions of Theorem 3 apply for $x^0 = u$. This proves the first assertion of the Corollary. The second assertion follows easily.

4. *Applications*

The first application of Theorem 3 that suggests itself is a *generalization of Ljusternik's tangent space theorem.*

Before stating this generalization let us point out the basis for the application of Theorem 3 to situations where the mapping f is differentiable in the classical sense.

Lemma 3. *Let f be a mapping from an open subset of X into Y. If f is Fréchet differentiable in a neighbourhood U of x^0 and if*

$$\sup \{ \| f'_x - f'_{x'} \| : x, x' \in U \} \leqslant \alpha \tag{25}$$

then the mapping $x \to f(x^0) + f'_{x^0}(x - x^0)$ is in the (U, α) Lipschitz proximity of f.

(here f'_x denotes the derivative of f at point x; $\| \cdot \|$ is the norm in the space of linear operators from X into Y).

Proof. By the mean value theorem (see for ex. [4]) we have for all $x, x' \in U$;

$$\mid f(x) - f(x') - (f'_x(x - x^0) - f'_{x^0}(x' - x^0) \mid = \mid f(x) - f(x') - f'_{x^0}(x - x') \leqslant$$
$$\leqslant \sup_{0 \leqslant t \leqslant 1} \| f'_{x+t(x'-x)} - f'_{x'} \| \cdot \mid x - x' \mid \leqslant \alpha \mid x - x' \mid.$$

Remark. If the derivative f'_x is continuous with respect to x in a neighbourhood of x^0, then (25) necessarily holds, provided U is small enough.

This Lemma shows that Theorem 3 contains as special cases both the mentioned results of S. M. Robinson [8] and of A. D. Ioffe and V. M. Tikhomirov [3].

Let us recall the following

Definition 7. Let K be a subset of X. We say that a vector $z \in X$ is *tangent* to K at a point $x^0 \in K$ if for some $\partial > 0$ there exists a mapping $x : [0, \partial] \to K$ such that $x(t) = x^0 + tz + o(t)$.

The set of all tangent vectors to K at x^0, is a closed nonempty cone written $T_{x^0}(K)$ and called the *tangent cone* to K at x^0.

Theorem 4. *The notations being the same as in Theorem 3, assume that $f : U \to Y$ is continuously differentiable in U and that the derivative f'_{x^0} at a point $x^0 \in U$ is M — surjective. Then the tangent cone at x^0 to the set*

$$K = \{x \in U : f(x) \in f(x^0) + M\} \tag{26}$$

is the solution set of the equation $f'_{x^0} z \in M$, i.e.

$$T_{x^0}(K) = \{z : f'_{x^0} \cdot z \in M\}, \tag{27}$$

Proof. We have only to prove $T_{x^0}(K) \supset \{z : f'_{x^0}. z \in M\}$ since the converse inclusion is trivial. Let z satisfy $f'_{x^0}. z \in M$. Using the continuity of f'_x, we may assume U to be small enough to ensure that

$$\forall\ x, x' \in U \ \| f'_x - f' \ \| \leqslant \alpha, \alpha \mid G^{-1} \mid < 1,$$

where G is the convex process $x \to f'_0 x(x - x^0) - M$. Then by Lemma 3, $f(x^0) + f'_{x^0}(x - x^0)$ is in the (U,α) — Lipschitz proximity of f, so Corollary 2 applies with $g = f_{x^0}$ For $t \in [0, \partial]$, with $\partial > 0$ sufficiently small, we have

$$x^0 + tz \in U, \text{ and } \mid f(x^0 + tz) - f(x^0) - f'_{x^0}(tz) \mid < \frac{1-\theta}{\mid G^{-1} \mid} \rho(x^0 + t\,z, X \setminus U);$$

Then by Corollary 2, where $u = x^0 + tz$, $v = f(x^0) + f'_{x_0}(tz)$, there exists a vector $x(t)$ satisfying

$$f(x(t)) \in v + M = f(x^0) + f'_{x^0}(t\ z) + M \tag{27}$$

$$\mid x(t) - (x^0 + tz) \mid \leqslant c \mid f(x^0 + tz) - f(x^0) - f'_{x^0}(tz) \mid \tag{28}$$

Since by hypothesis $f'_{x^0}(t\,z) \in M$, it follows from (27) that $f(x(t)) \in f(x^0) + M$, i.e. $x(t) \in K$. Furthermore (28) implies, obviously, $\mid x(t) - (x^0 + tz) \mid = o(t)$, and hence $z \in T_{x^0}(K)$, as was to be shown.

Remark. Ljusternik's theorem corresponds to the special case $M = \{0\}$ of the previous proposition. In an earlier paper [13] one of us has proved a proposition (Theorem 4) which was essentially the same as the present result, except that, instead of assuming the M — surjectivity of the derivative f'_{x^0} we assumed there a condition which in the present context may be formulated as: $(\exists\, \gamma > 0)$ $(\forall\, y, y' \in Y)$

$$h\,(f'^{-1}_{x^0}(y), f'^{-1}_{x^0}(y')) \leqslant \gamma \mid y - y' \mid.$$

From Lemma 1 we know that the M — surjectivity of f'_{x^0} always implies the just formualed condition. In fact the two conditions are equivalent, as can be easily shown. Thus the present Theorem 4 is essentially equivalent to Theorem 4 in [13]. The two methods of proof are different, however.

As the second application of Theorem 3 we shall prove some *stability properties* of equations (2).

Theorem 5. *Under the same assumptions as in Theorem 3, let $f(x^0) \in M$. Then for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every mapping A : $U \to Y$ satisfying.*

$$\forall\, x \in U \ \mid A(x) \mid \leqslant \eta \tag{29}$$

$$\forall x, x \in U \ \mid A(x) - A(x') \mid \leqslant \eta \mid x - x' \mid \tag{30}$$

the equation

$$f(x) + A(x) \in M \tag{31}$$

has at least one solution x such that $\mid x - x^0 \mid < \varepsilon$.

More precisely, if

$$\eta < \frac{r}{1+r} \cdot \frac{1-\alpha \mid G^{-1} \mid}{\mid G^{-1} \mid} \tag{32}$$

with $r = \rho(x^0, X \setminus U)$, *then for each mapping A satisfying (29), (30) and for each* $\delta > 0$ *the equation (31) has at least one solution x such that*

$$\mid x - x^0 \mid \leqslant \frac{(1+\delta) \mid G^{-1} \mid}{1-(\eta+\alpha) \mid G^{-1} \mid} \cdot \mid A(x^0) \mid \tag{33}$$

Remark. The set of mappings A satisfying (29), (30) is nothing else than the η — ball around O in the space Y^U_{Lips} introduced in the Remark following Definition 6 Thus one can restate. Theorem 5 as follows:

If $f(x^0) \in M$, if $f(x) - f(x^0)$ is in the — α — neighbourhood of some M — convex, positively homogeneous and M — surjective mapping g such that $\alpha \mid G^{-1} \mid < 1$, then for every $\varepsilon > 0$ thereexists an $\eta >$ o such that for every $\tilde{f}$ in the η — neighbourhood of f the equation $\tilde{f}(x) \in M$ has at least one solution in the ε — neighbourhood of x^0.

Therefore the solution x^0 *of the equation* $f(x) \in M$ *is stable under small perturbations in* Y^U_{Lips} (i.e. small perturbations preserving the local Lipschitz-property of f).

Proof of Theorem 5. Let α satisfy (32). Then upon simple computations, we deduce $(\alpha+\eta) \mid G^{-1} \mid < 1$ (34)

$$\eta < \frac{r(1-(\alpha+\eta) \mid G^{-1} \mid)}{\mid G^{-1} \mid}. \tag{35}$$

Consider now an arbitrary mapping $A : U \to Y$ satisfying (29), (30), We have, for all $x, x' \in U$:

$$\mid f(x) + A(x) - f(x') - A(x') - (g(x-x^0) - g(x'-x^0)) \mid \leqslant$$
$$\leqslant \mid f(x) - f(x') - (g(x-x^0) - g(x'-x^0)) \mid + \mid A(x) - A(x') \mid$$
$$\leqslant (\alpha+\eta) \mid x - x' \mid .$$

since f is in the (U, α) — proximity of $f(x^0) + g(x - x^0)$. Thus $f(x) + A(x)$ is in the $(U, \alpha+\eta)$ — Lipschitz proximity of $f(x^0) + A(x^0) + g(x - x^0)$. Since (34) holds, it follows from Theorem 3 that for every $\delta > 0$ and for every v such that

$$\mid v - f(x^0) - A(x^0) \mid < \frac{r(1-(\alpha+\eta) \mid G^{-1} \mid)}{\mid G^{-1} \mid} \tag{36}$$

the equation

$$f(x) + A(x) \in v + M$$

has at least one solution x satisfying

$$\mid x - x^0 \mid \leqslant \frac{(1+\delta) \mid G^{-1} \mid}{1-(\alpha+\eta) \mid G^{-1} \mid} \mid v - f(x^0) - A(x^0) \mid . \tag{37}$$

But from (29) and (35) we see that (36) holds for $v = f(x^0)$. Hence the equation

$$f(x) + A(x) \in f(x^0) + M$$

is solved by at least one x satisfying (37). Since $f(x^0) \in M$ by hypothesis, we have $f(x) + A(x) \in M$, which completes the second assertion in the Theorem. The first assertion follows readily, because one can ensure $|x - x^0| < \varepsilon$ by choosing η so as to satisfy (32) and

$$\frac{2\,|G^{-1}|\,\alpha}{1-(\eta+\alpha)\,|G^{-1}|} < \varepsilon$$

An immediate consequence of Theorem 5 is the following proposition which reduces to a known result on operator equations when $M = \{0\}$ (see [5]).

Denote by Y^U_{cb1} the space of continuously differentiable mappings $f : U \to Y$ defined on an open set $U \subset X$ and bounded together with their derivatives. As it is known ([12], chapter III), this is a Banach space with the norm

$$|||f||| = \sup_{x \in U} (\,|f(x)|\,,\,\|f'_x\|\,).$$

Corollary 3 *Let x^0 be a solution of the equation*

$$f(x) \in M$$

where $f \in Y^U_{cb,1}$, and M si a closed convex cone in Y If f'_{x^0} is M — surjective, then for every $\varepsilon > 0$ there exists $\eta > 0$ such that, for every $\tilde{f} \in Y^U_{cb,1}$ satisfying $|||\tilde{f} - f||| \leqslant \eta$ the equation $\tilde{f}(x) \in M$ has at least one solution x with $|x - x^0| < \varepsilon$

Proof. It suffices to observe that $|||\psi||| \leqslant \eta$ implies $\sup_{x \in U} \|\psi'_x\| \leqslant \eta$, so that for all $x, x' \in U$, $|\psi(x) - \psi(x')| \leqslant \eta\,|x - x'|$, and therefore $|||\psi||| \leqslant \|\psi\|^U_{\text{Lips}}$.

Some times it may be convenient to formulate Theorem 5 in the form of an « implicit function theorem ».

Let X, Y be Banach spaces and let U be an open set in X. Recall that Y^U_{Lips} is a normed space, the norm $\|f\|^U_{\text{Lips}}$ being the smallest number $\alpha \geqslant 0$ satisfying

$$\forall\, x \in U \;\; |f(x)| \leqslant \alpha \tag{38}$$

$$\forall x, x' \in U \;\; |f(x) - f(x')| \leqslant \alpha\,|x - x'| \tag{39}$$

Theorem 6. *Let W be an arbitrary set and tet $f : X \times W \to Y$ be a mapping such that, for some $(x^0, w^0) \in U \times W : f(x^0, w^0) \in M$, where M is, as before, a closed convex cone in Y. Assume that $f(.,w) \in Y^U_{\text{Lips}}$ for every $w \in W$ and there exists a mapping $g \in Y^U_{\text{Lips}}$ which is M-convex, positively homogeneous, M — surjective and such that :*

1/ $\| f(.\,, w^0) - f^0(.) \|^U_{\text{Lips}} \leqslant \alpha$, where $f^0(x) = f(x^0, w^0) + g(x - x^0)$,

2/ $\| f(.\,, w^0) - f(.\,, w) \|^U_{\text{Lips}} \leqslant \eta$ for all $w \in W$

$$3/\ (\alpha + \eta) \mid G^{-1} \mid < 1,\ \eta < \frac{r}{1+r} \frac{(1-\alpha \mid G^{-1} \mid)}{\mid G^{-1} \mid} \tag{40}$$

where $r = \rho(x^0, X \setminus U)$, $G(x) = g(x) - M$.

Then for every $w \in W$ and every $\delta > 0$ the equation

$$f(x, w) \in M \tag{41}$$

has at least one solution $x \in U$ such that

$$\mid x - x^0 \mid \leqslant \frac{(1+\delta) \mid G^{-1} \mid}{1-(\alpha+\eta) |G^{-1}|} |f(x^0, w) - f(x^0, w^0)| \tag{42}$$

Proof. The mapping $f(., w^0)$ satisfies all the assumptions of Theorem 3, whereas for every $w \in W$ the mapping $A_{(w)}(.) = f(., w) - f(., w^0)$ satisfies conditions (29) and (30) of Theorem 5. Therefore Theorem 5 applies.

Remark. Assume, in addition to all the conditions already specified in Theorem 6, that g is linear and that for every $(x, w) \in U \times W$

$$\rho(f(x, w), M) \leqslant \alpha \tag{43}$$

Then the conclusion can be sharpened in the following way.

For every $u \in U$, for every $w \in W$ and for every $\delta > 0$ the equation

$$f(x, w) \in M \tag{44}$$

has at least one solution x such that

$$\mid x - u \mid \leqslant \frac{(1+\delta). \mid G^{-1} \mid}{1-(\alpha+\eta) \mid G^{-1} \mid} \rho(f(u,z), M) \tag{45}$$

(Obviously, this implies (42) when $u = x^0$; because $f(x^0, w^0) \in M$).

Proof. As seen in the proof of Theorem 5, each $f(., w)$ is in the $(\alpha+\eta)$ — Lipschitz proximity of $f(x^0, w) + g(x + x^0)$. If now g is linear, then, by Corollary 2, for every $u \in U$, every $w \in W$, every $\delta > 0$ and every v such that

$$\mid v - f(u, w) \mid < \frac{r(1-(\alpha+\eta) \mid G^{-1} \mid)}{\mid G^{-1} \mid} \tag{46}$$

the equation

$$f(x, w) \in v + M \tag{47}$$

has at least one solution x satisfying

$$\mid x - u \mid \leqslant \frac{(1+\delta) \mid G^{-1} \mid}{1-(\alpha+\eta). \mid G^{-1} \mid} \mid v - f(u, w \mid \tag{48}$$

Since η satisfies (40), it follows from (43) and the closedness of M that an element $v \in M$ may be found such that

$$\mid v - f(u, w) \mid = \rho(f(u, w), M) \leqslant \alpha < \frac{r(1-(\alpha+\eta). \mid G^{-1} \mid)}{\mid G^{-1} \mid}.$$

Therefore, with this v the equation (47) (which implies (44) since $v \in M$) has at least one solution x satisfying (48), and hence, (45).

For the case of classical differentiability we have the following

Corollary 4. *Let X, Y, Z be Banach spaces, let M be a closed cone in Y. Consider a mapping $f: U \times W \to Y$ from an open subset $U \times W$ of $X \times Z$ into Y, such that $f(x^0, w^0) \in M$ for some point $(x^0, w^0) \in U \times W$. Assume that f has at every point $(x, w) \in U \times W$ a partial Fréchet derevative $f'_x(x, w)$, that f and f'_x are continuous in some neighbourhood of (x^0, w^0) and finally, that $f'_x(x^0, w^0)$ is M — surjective.*

Then for every $\varepsilon > 0$ there exists a neighbourhood $U' \times W'$ of (x^0, w^0) such that for every $(u, w) \in U' \times W'$ the equation

$$f(x, w) \in M \tag{43}$$

has at least one solution x satisfying

$$| x - u | \leqslant (1 + \varepsilon) \, | G^{-1} | \, \rho(f(u, w), M) \tag{44}$$

where G is the convex process $x \to G(x) = f'_x(x^0, w^0).x - M$

Proof. We apply Theorem 6, and the remark following it, with $g = f'_x(x^0, w^0)$. For $\varepsilon >$ given, we can choose $\alpha > 0$, $\eta > 0$, $\delta > 0$ so that

$$(\alpha + \eta) \, | G^{-1} | < 1, \quad \frac{1 + \delta}{1 - (\alpha + \eta) \, | G^{-1} |} \leqslant 1 + \varepsilon \tag{45}$$

Next, using the continuity of f and f'_x at (x^0, w^0), we can find a neighbourhood U' of x^0 such that

$$\forall x \in U' \; | f(x, w^0) - f(x^0, w^0) - g(x - x^0) | \leqslant \alpha,$$

$$\forall x, x' \in U' \; \| f'_x(x, w^0) - f'_x(x', w^0) \| \leqslant \alpha$$

which, by Lemma 3, will ensure condition 1) in Theorem 6 (with U' replacing U). Further, since

$$| (f(x, w) - f(x, w^0)) - (f(x' w) - f(x', w^0)) | \leqslant$$

$$\leqslant \sup_{0 < \theta < 1} \| f'_x(x + \theta(x' - x), w) - f'_x(x + \theta(x' - x), w^0) \| \, . \, | x - x' | ,$$

it follows from the continuity of f and f'_x that condition 2) in Theorem 6 (with W' replacing W) is fulfilled, provided $U' \times W'$ is small enough. Since $\rho(f(x^0, w^0), M) = 0$ we may also assume (43) to hold for all $(x, w) \in U' \times W'$.

Finally, we can choose η so as to have (40), with $r = \rho(x^0, X \setminus U')$.

Thus, all the conditions required in Theorem 6 are satisfied. This concludes the proof.

Addendum. All the essential results in this paper have been presented on the seminars on Convex Analysis and the Scientific sessions of the Institute of Mathematics in Hanoi at the end of 1975 and the beginning of 1976. At that time the authors were unaware of the paper [7] wich contained, among other things, a result intermediate between the Banach open mapping theorem and our Corollary 1.

Some times after the present work had been achieved, we learned from professor S.M. Robinson (August 1976) that many results close to ours had just been published in his two recent papers [9] and [10]. It turned out that our Theorem 2 here is included in Theoem 1 of [9]. On the other hand, Lemma 1 and Theorem 1 of [10] overlap with our Corollary 4 and Theorem 6 respectively: in the case of differentiable mappings our above mentioned results are properly contained in the corresponding results of [10], whereas in the case where the mappings are considered in an open set ($L = Z$ in Lemma 1 of [10] or $C = X$ in Theorem 1 of [10]), the latter results are properly included in ours. Apparently, a slight extension of our Theorem 3 would allow it to include all these results together.

In any case, it should be noted once more that all the results established in the present paper concern mappings which are assumed only to be *locally Lipschitz*. In a subsequent paper we shall demonstrate how they can be applied to the theory of *Lipschitzian optimization* that has been recently initiated by F.H. Clarke ([1]), [2]). This will provide one more motivation for the approach taken here.

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REFERENCES

[1] F. H. CLARKE. *Generalized Gradients and Applications.* Trans. Amer. Math. Soc. 205 (1975), 247–262

[2] F.H. CLARKE. *A New Approach to Lagrange Multipliers.* Math. of Operations Research 1 (1976), 165–174.

[3] A. D. IOFFE AND V.M. TIKHOMIROV *The Theory of Extremum Problems.* Moscow 1974 (in russian)

[4] L.V. KANTOROVITCH and. G.P. AKILOV. *Functional Analysis in Normed Spaces.* Moscow 1959 (in russian)

[5] L.A. LJUSTERNIK and V.I. SOBOLEV. Elements of Functional *Analysis.* Moscow 1965 (in russian)

[6] S.B. NADLER, Jr. *Multivalued Contraction Mappings.* Pacific J. of Math. 30 (1969), 475–488.

[7] S.M. ROBINSON. *Normed Convex Processes.* Trans. Amer. Math. Soc. 174 (1972), 127–140.

[8] S.M. ROBINSON. *An Inverse – Function Theorem for a Class of Multivalued Mappings.* Proc. Amer Math. Soc. 41 (1973), 211—217.

[9] S.M. ROBINSON. *Regularity and Stability for Convex Multivalued Functions.* Math. of Operations Research 1(1976), 130—143.

[10] S.M. ROBINSON *Stability Theory for Sytems of Inequalities,* Siam J. Numer Anal 13 (1976), 497—513.

[11] R.T. ROCKAFELLAR. *Convex Analysis.* Princeton University Press. Princeton 1970.

[12] L SCHWARTZ *Analyse Mathématique.* Paris Hermann. 1967

[13] H. TỤY. *On the Convex Approximation of non Linear Inequalities.* Math. Operations Forsch.u. Statist 5 (1974), 451—466

[14] H. TỤY. *On necessary conditions for optimality.* Progress in Operations Research. ed. A. Prekopa. North. Holland American Elsever. Amsterdam – London 1976; 921 – 941.

[15] H. TỤY. *Stability Property of a System of Inequalities.* Math. Operations Forsch. u. Statist 8 (1977), 27 – 39.