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# THE LOCALLY MOST POWERFUL SIGNED RANK TESTS

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## I. INTRODUCTION:

In [1] there are two theorems (Th. II. 4. 9 and Th. II. 4.10) on the locally most powerful signed rank tests in testing the symmetry hypothesis $\mathscr{H}$ against the location shift and the two samples differing in scale respectively. These theorems have been generalized in [2] (see Theorems 1. 1 and 1. 2) for the location and scale alternatives $K_1$ and $K_2$ with regression constants of the form

$$(1)\ K_1\{\Delta\} = \{q_\theta(x) = \prod_{i=1}^{N} f(x_i - \theta c_i),\ \theta \in \Delta\},$$

$$(2)\ K_2\{\Delta\} = \{q_\theta(x) = \prod_{i=1}^{N} e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i),\ \theta \in \Delta\},$$

where $c_1, \ldots, c_N$, $b_1, \ldots, b_N$ are known constants, $\theta$ is a parameter, $\theta \neq 0$, $f(.)$ is a given symmetric density, and $\Delta = (0, +\infty)$ or $\Delta = (-\infty, 0)$.

The aim of this paper is to establish the locally most powerful signed rank tests in testing $\mathscr{H}$, defined by

$$(3)\ \mathscr{H} = \{p(x) = \prod_{i=1}^{N} g(x_i),\ g \in \mathfrak{F}_0\},$$

where $\mathfrak{F}_0$ is a family of all symmetric densities $g(x)$, i.e., $g(-x) = g(x)$ a.e., against the alternatives.

$$(4)\ K_1\{\Delta\} = \{q_\theta(x) = \prod_{i=1}^{N} f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}),\ \theta \in \Delta\},$$

$$(5)\ K_2\{\Delta\} = \{q_\theta(x) = \prod_{i=1}^{N} e^{-\theta b_i} f\,(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta)\, c_{1i} - \dots - h_k(\theta)\, c_{ki}),\ \theta \in \Delta\},$$

and

$$(6)\ K_3\{\Delta\} = \{q_\theta(x) = \prod_{i=1}^{N} e^{-\theta b_i} f\,(e^{-\theta b_i}\,[x_i - \theta c_i - h_1(\theta)\, c_{1i} - \dots - h_k(\theta)\, c_{ki}]),\ \theta \in \Delta\},$$

which is equivalent of (5), where $f \in \mathcal{F}_0$ is known and $\Delta = (0, +\infty)$ or $\Delta = (-\infty, 0)$, $b_i$, $c_i$, $c_{1i}, \dots, c_{ki}$, $1 \leqslant i \leqslant N$, are known constants and functions $h_1(.), \dots, h_k(.)$ are more or less known.

It is shown in the COROLLARY in Section IV that the locally most powerful signed rank tests for $\mathcal{H}$ against $\mathcal{K}_2$ and $\mathcal{K}_3$ are the same.

## II. NOTATIONS AND DEFINITIONS:

Let $X = (X_1, \dots X_N)$ be N independent continuous random variables. X satisfies $\mathcal{H}$ iff the $N$-dimensional density of $X$ belongs to $\mathcal{H}$. Similarly for $X$ satisfying $\mathcal{K}_1$, $\mathcal{K}_2$, or $\mathcal{K}_3$. Let $X_{(.)} = (X_{(1)}, \dots, X_{(N)})$ be order statistics based on absolute values of $X = (X_1, \dots, X_N)$, i.e., $|X_{(1)}| \leqslant \dots \leqslant |X_{(N)}|$.

Denote $R^+ = (R_1^+, \dots, R_N^+)$ to be the ranks of $|X| = (|X_1|, \dots, |X_N|)$,

i.e., $R_i^+ = \sum_{j=1}^{N} u(|X_i| - |X_j|)$, where

$u(x) = 1$ or $0$ if $x \geqslant 0$ or $x < 0$,

and $V = (V_1, \dots, V_N)$ to be the signs of $X = (X_1, \dots, X_N)$,

i.e., $V_i = \operatorname{sgn} X_i$, where

$\operatorname{sgn} x = 1, 0,$ or $-1$ if $x > 0$, $x = 0$, or $x < 0$.

Let $\mathcal{R} = \{r\}$ be the space of all $N!$ permutations of $(1, \dots, N)$ and $\mathcal{V} = \{v\}$ of all $2^N$ sequences of size $N$ from 1's and $-1$'s. It is well known that under $\mathcal{H}$, $|X_{(.)}| = (|X_{(1)}|, \dots, |X_{(N)}|)$, $R^+$ and $V$ are mutually independent and

$$(7)\quad P(R^+ = r, V = v) = \frac{1}{2^N N!},\quad r \in \mathcal{R},\ v \in \mathcal{V}.$$

Here and throughout the paper P is used to denote the probability measure belonging to $\mathcal{H}$.

Let $f(.)$ in the definitions of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ is absolutely continuos. Define the scores

$a(i) = E_f\,\{-f'(|X_{(i)}|)\,/\,f(|X_{(i)}|)\}$, $1 \leqslant i \leqslant N$,

(8) $a^+(i) = E_f\{-1 - |X_{(i)}| f'(|X_{(i)}|) / f(|X_{(i)}|)\}$, $1 \leqslant i \leqslant N$,

where $E_f$ denotes the expectation provided $X$ has density $\prod_{i=1}^{N} f(x_i)$

**Definition 1.** A test is defined to be a signed rank test iff it is determined by the statistic of the form $T = T(R^+, V)$.

**Definition 2.** Consider an indexed set of densities

$\mathcal{K}^+ = \{q_\theta(x) = q_\theta(x_1, \ldots, x_N), \theta > 0\}$ and assume that $q_0 \in \mathcal{H}$. A test will be called locally most powerful for $\mathcal{H}$ against $\mathcal{K}^+$ at a level $\alpha$, $0 < \alpha < 1$, iff it is uniformly most powerful at the level $\alpha$ for $\mathcal{H}$ against $\mathcal{K}_\varepsilon^+ = \{q_\theta, 0 < \theta < \varepsilon\}$, for some $\varepsilon > 0$. If it holds among signed rank tests, we speak of a locally most powerful signed rank test. Similar definitions would formulated for $\mathcal{H}$ against $\mathcal{K}^- = \{q_\theta(x), \theta < 0\}$.

## III. LEMMAS:

The following two lemmas will be used to prove Theorems in Section IV. The first is an immediate consequence of Neyman-Pearson Lemma and of (7), while the second is a useful convergence theorem for statistics formulated by Scheffé (1947) [3].

**Lemma 1.** In testing $\mathcal{H}$ against a simple alternative $q(x)$ at level $\alpha$, $0 < \alpha < 1$ the most powerful signed rank test is given by critical function

$$\Phi(r, v) = \begin{cases} 1 & \text{if } Q(R^+ = r, V = v) > \lambda, \\ \delta & \text{if } Q(R^+ = r, V = v) = \lambda, \\ 0 & \text{if } Q(R^+ = r, V = v) < \lambda, \end{cases}$$

where $dQ = q\,dx$, and the constants $\lambda$, $\delta$, $0 < \delta < 1$, can be determined so that $E\{\Phi(R^+, V)\} = \alpha$ under $\mathcal{H}$.

**Lemma 2.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $\sigma$-finite measure $\mu$ and, $\Omega = \{\omega\}$. Let a sequence of $\mathcal{A}$-measurable functions $h(\omega)$, $h_1(\omega)$, $h_2(\omega)$,... be such that $\lim\limits_{n\to\infty} h_n(\omega) = h(\omega) \pmod{\mu}$,

and $\limsup\limits_{n\to\infty} \int |h_n| \, d\mu \leqslant \int |h| \, d\mu < \infty$.

Then for each $A \in \mathcal{A}$

$$\lim_{n\to\infty} \int_A h_n \, d\mu = \int_A h \, d\mu.$$

In this Section there are two Theorems and a Corollary performed, regarding alternatives $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ respectively.

**Theorem 1.** *Let the symmetric density $f(x)$ be absolutely continuous and satisfy*

$$(9) \qquad \int_{-\infty}^{+\infty} | f'(x) | \, dx < \infty .$$

*Let the alternative $\mathcal{K}_1\{\Delta\}$ be defined by* (4) *with $h_1, \dots h_k$ satisfying*

$$(10) \qquad h_i(0) = 0, \quad \lim_{\theta \in \Delta,\, \theta \to 0} (h_i(\theta) / \theta) = d_i \text{ , finite, } 1 \leqslant i \leqslant k.$$

*Let scores $a(i)$, $1 \leqslant i \leqslant N$, be defined by* (8). *Then the test determined by critical region*

$$(11) \qquad \sum_{i=1}^{N} \gamma_i \, a(\overset{+}{R_i}) \operatorname{sgn} X_i = \sum_{i=1}^{N} \gamma_i a(\overset{+}{R_i}) V_i \geqslant \lambda \ (\leqslant \lambda), \text{ for any constant } \lambda, \text{ where}$$

*$\gamma_i = c_i + d_1 c_{1i} + \dots + d_k c_{ki}$, $1 \leqslant i \leqslant N$, is the locally most powerful signed rank test at the respective level in testing $\mathcal{H}$ against $\mathcal{K}_1\{(0, +\infty)\}$ ($\mathcal{K}_1\{(-\infty, 0\}$).*

*Proof*: Let us prove for the case $\Delta = (0, +\infty)$. (For the case $\Delta = (-\infty, 0)$, put $\theta' = -\theta$, $c'_i = -c_i$, $c'_{si} = -c_{si}$, $1 \leqslant s \leqslant k$, $1 \leqslant i \leqslant N$). Since for arbitrary $a_1, \dots, a_N$ and $b_1, \dots b_N$

$$\prod_{s=1}^{N} a_s - \prod_{s=1}^{N} b_s = \left( \prod_{s=1}^{N} a_s - b_1 \prod_{s=2}^{N} a_s \right) + \left( b_1 \prod_{s=2}^{N} a_s - \prod_{j=1}^{2} b_j \prod_{s=3}^{N} a_s \right) +$$

$$+ \dots + \left( \prod_{j=1}^{i-1} b_j \prod_{s=i}^{N} a_s - \prod_{j=1}^{i} b_j \prod_{s=i+1}^{N} a_s \right) + \dots$$

$$+ \left( \prod_{j=1}^{N-1} b_j \cdot a_N - \prod_{s=1}^{N} b_s \right) =$$

$$= \sum_{i=1}^{N} \left\{ (a_i - b_i) \prod_{j=1}^{i-1} b_j \cdot \prod_{s=i+1}^{N} a_s \right\}, \text{ where } b_0 = a_{N+1} = 0,$$

one has

$$q_\theta(x) = \prod_{i=1}^{N} f(x_i) + \sum_{i=1}^{N} \left\{ \left[ f(x_i - \theta c_i - h_1(\theta) c_{1i} - \dots - h_k(\theta) c_{ki}) - f(x_i) \right] . \right.$$

$$\left. \cdot \prod_{j=1}^{i-1} f(x_j) \cdot \prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \dots - h_k(\theta) c_{ks}) \right\} .$$

Denote $dQ_\theta = q_\theta dx$, $q_0 = p$, $B(r,v) = \{x : R^+ = r, V = v\}$, $r \in \mathfrak{R}$, $v \in \mathcal{V}$.

It follows from (7) that

(12) $Q_\theta\ (B(r,v)) = Q_\theta(r,v)$, denote,

$$= \int \cdots \int_{B(r,v)} q_\theta(x)\, dx$$

$$= 1/(2^N N!) + \theta \sum_{i=1}^{N} \int \cdots \int_{B(r,v)} (1/\theta\, [f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki})$$

$$- f(x_i)].\ \prod_{j=1}^{i-1} f(x_j).\ \prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \ldots - h_k(\theta) c_{ks})\ dx$$

In view of (10) and of the absolute continuity of $f(x)$

(13) $$\lim_{\theta \to +0} (1/\theta)\ [f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}) - f(x_i)].\ \prod_{j=1}^{i-1} f(x_j).$$

$$\prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \ldots - h_k(\theta) c_{ks}) =$$

$$= -(c_i + d_1 c_{1i} + \ldots + d_k c_{ki}) f'(x_i) \prod_{j \neq i} f(x_j)$$

$$= -\gamma_i f'(x_i) \prod_{j \neq i} f(x_j), \text{ a.e., } 1 \leqslant i \leqslant N.$$

Moreover for arbitrary $\varepsilon > 0$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Big| (1/\theta)\ [f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}) - f(x_i)].\ \prod_{j=1}^{i-1} f(x_j).$$

$$\prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \ldots - h_k(\theta) c_{ks}) \Big| dx$$

$$= (1/|\theta|) \int_{-\infty}^{+\infty} \Big| f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}) - f(x_i) \Big| dx_i$$

$$= (1/|\theta|) \int_{-\infty}^{+\infty} \Big| \int_0^{[\theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}]} f'(x_i - y)\, dy \Big|\ dx_i$$

$$\leqslant \Big| (1/\theta) \int_0^{[\theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}]} \Big\{ \int_{-\infty}^{+\infty} \Big| f'(x_i - y) \Big| dx_i \Big\}\ dy \Big|$$

$$= \Big| (1/\theta) \int_0^{[\theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}]} \Big\{ \int_{-\infty}^{+\infty} \Big| f'(x_i) \Big| dx_i \Big\} dy \Big|$$

$$\leqslant (|\gamma_i| + \varepsilon) \int_{-\infty}^{+\infty} | f'(x_i) | dx_i$$

$$= \int_{-\infty}^{+\infty} \left| (|\gamma_i| + \varepsilon) f'(x_i) \right| dx_i, \; 1 \leqslant i \leqslant N,$$

if $\theta$ is sufficiently small, by (10).

Consequently, since $\varepsilon > 0$ is arbitrary,

$$(14) \; \limsup_{\theta \to +0} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Big| (1/\theta) \, [f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}) - f(x_i)] .$$

$$\prod_{j=1}^{i-1} f(x_j) \prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \ldots - h_k(\theta) c_{ks}) \Big| dx$$

$$\leqslant \int_{-\infty}^{+\infty} \left| \gamma_i f'(x_i) \right| dx_i .$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left| \gamma_i f'(x_i) \prod_{j \neq i} f(x_j) \right| dx.$$

In view of (13) and (14), it follows from Lemma 2 that

$$(15) \; \lim_{\theta \to +0} \sum_{i=1}^{N} \int \ldots \int_{B(r,v)} (1/\theta) [f(x_i - \theta c_i - h_1(\theta) c_{1i} - \ldots - h_k(\theta) c_{ki}) - f(x_i)] .$$

$$\prod_{j=1}^{i-1} f(x_j) . \prod_{s=i+1}^{N} f(x_s - \theta c_s - h_1(\theta) c_{1s} - \ldots - h_k(\theta) c_{ks}) dx =$$

$$= \sum_{i=1}^{N} \gamma_i \int \ldots \int_{B(r,v)} - f'(x_i) . \prod_{j \neq i} f(x_j) dx$$

$$= \sum_{i=1}^{N} \gamma_i \int \ldots \int_{B(r,v)} (-v_i f'(|x_i|) / f(|x_i|)) \, p(x) dx, \text{ by the symmetry of } f(x_i)$$

$$= (1/2^N N!) \sum_{i=1}^{N} \gamma_i \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (-v_i f'(|x_i|) / f(|x_i|)) \, p(x \mid R^+ = r, V = v) dx, \text{ by (7)},$$

$$= (1/2^N N!) \sum_{i=1}^{N} \gamma_i E_f \{ -V_i f'(|X_i|) / f(|X_i|) \mid R^+ = r, V = v \}$$

$$= (1/2^N N!) \sum_{i=1}^{N} \gamma_i v_i E_f \{ -f'(|X_{(r_i)}|) / f(|X_{(r_i)}|) \}$$

$$= (1/2^N N!) \sum_{i=1}^{N} \gamma_i a(r_i) v_i .$$

It follows from (12) and (15) that for any $\varepsilon > 0$ there exists $\delta(\varepsilon, r, v) > 0$ such that

$$(16) \quad \left| (1/\theta)\left[ Q_\theta(r, v) - (1/2^N N!)\right] - (1/2^N N!) \sum_{i=1}^{N} \gamma_i a(r_i) v_i \right| < \varepsilon/2 \text{ for } \theta,\ 0 < \theta < \delta.$$

Let $\mathcal{C} = \{ t_1, \dots, t_m \}$ be a set of all distinct values of

$(1/2^N N!) \sum_{i=1}^{N} \gamma_i a(r_i) v_i$, $r \in R$, $v \in \mathcal{V}$. Obviously, $1 \leqslant m \leqslant 2^N N!$.

Denote $\varepsilon_0 = \min \{ | t_i - t_j | , i \neq j = 1, \dots, m \}$, $(\varepsilon_0 > 0)$,

and $\delta_0 = \min \{ \delta(\varepsilon_0, r, v), r \in \mathfrak{R}, v \in \mathcal{V} \}$, $(\delta_0 > 0)$.

In view of (16) one finds easily that the inequality

$$\sum_{i=1}^{N} \gamma_i a(r_i) v_i > \sum_{i=1}^{N} \gamma_i a(r'_i) v'_i, \quad r, r' \in \mathfrak{R}, \quad v, v' \in \mathcal{V},$$

implies

$$Q_\theta(r, v) > Q_\theta(r', v') \text{ for all } \theta,\ 0 < \theta < \delta_0.$$

Hence the critical region defined by

$$\sum_{i=1}^{N} \gamma_i a(R_i^+) V_i \geqslant \lambda, \text{ for a given constant } \lambda,$$

is equivalent to the one by

$$Q_\theta(R^+, V) \geqslant \lambda^* \text{ for all } \theta,\ 0 < \theta < \delta_0,$$

where $\lambda^*$ is a constant compatible with $\lambda$ and independent of $\theta$, $0 < \theta < \delta_0$. Now Theorem 1 follows from Lemma 1. Q.E.D.

**Remark 1.** If $h_i(\theta) = 0(\theta)$ as $\theta \to 0$, $1 \leqslant i \leqslant k$, the locally most powerful signed rank test for $\mathcal{H}$ against $\mathcal{K}_1\{\Delta\}$ defined by (4) is the same as the one for $\mathcal{H}$ against $K_1$ defined by (1), and it is given in Theorem 1 1 of [2]. Especially, it is the case if

$$\mathcal{K}_1\{\Delta\} = \left\{ q_\theta(x) = \prod_{i=1}^{N} f(x_i - \theta c_i - \theta^2 c_{1i} - \dots - \theta^{k+1} c_{ki}),\ \theta \in \Delta \right\}.$$

**Theorem 2.** *Let $f$ be a symmetric and absolutely continuous density satisfying*

$$(17) \quad \int_{-\infty}^{+\infty} | x f'(x) | \, dx < \infty.$$

*Let $K_2\{\Delta\}$ be defined by (5) with $h_i$ such that*

$$(18) \quad \begin{cases} h_i(0) = 0,\ h'_i(\theta) \text{ exists for } \theta \in (-a, a) \cap \Delta \text{ for some } a > 0, \\ \lim\limits_{\substack{\theta \in \Delta \\ \theta \to 0}} h'_i(\theta) = \lim\limits_{\substack{\theta \in \Delta \\ \theta \to 0}} h_i(\theta)/\theta = d_i, \text{ finite}, \quad 1 \leqslant i \leqslant k. \end{cases}$$

*Let the scores $a(i)$ and $a^+(i)$ be defined by (8). Then the test determined by critical region*

$$\sum_{i=1}^{N} \{ b_i a^+(R_i^+) + \gamma_i a(R_i^+) V_i \} \geqslant \lambda \ (\leqslant \lambda) \tag{19}$$

*for any constant $\lambda$, where $\gamma_i = c_i + d_1 c_{1i} + \dots + d_k c_{ki}$, is the locally most powerful signed rank test at the respective level in testing $\mathscr{H}$ against*

$$\mathscr{K}_2\{(0, +\infty)\} \ (\mathscr{K}_2\{(-\infty, 0)\}).$$

**Proof** : It this quite the same as the proof of Theorem 1. Therefore we shall leave out all detail explainations. Note first that (17) implies (9).

We have successively

$$Q_\theta(r, v) = \int_{B(r,v)} \dots \int q_\theta(x) dx$$

$$= (1/2^N N!) + \theta \sum_{i=1}^{N} \int_{B(r,v)} \dots \int (1/\theta) \left[ e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta) c_{1i} - \dots \right.$$

$$\left. - \dots - h_k(\theta) c_{ki}) - f(x_i) \right] . \prod_{j=1}^{i-1} f(x_j).$$

$$\prod_{s=i+1}^{N} . e^{-\theta b_s} f(e^{-\theta b_s} x_s - \theta c_s - h_1(\theta) c_{1s} - \dots - h_k(\theta) c_{ks}) dx,$$

$$\lim_{\theta \to +0} (1/\theta)[e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta) c_{1i} - \dots - h_k(\theta) c_{ki}) - f(x_i)].$$

$$. \prod_{j=1}^{i-1} f(x_j).$$

$$. \prod_{s=i+1}^{N} e^{-\theta b_s} f(e^{-\theta b_s} x_s - \theta c_s - h_1(\theta) c_{1s} - \dots - h_k(\theta) c_{ks})$$

$$= [-b_i f(x_i) - (b_i x_i + c_i + d_1 c_{1i} + \dots + d_k c_{ki}) f'(x_i)] \prod_{j \neq i} f(x_j)$$

$$= [- b_i (f(x_i) + x_i f'(x_i)) - \gamma_i f'(x_i)] \prod_{j \neq i} f(x_j), \ 1 \leqslant i \leqslant N,$$

and for any $\varepsilon > 0$,

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left| \frac{1}{\theta} \left[ e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta) c_{1i} \dots - h_k(\theta) c_{ki}) - f(x_i) \right] . \right.$$

$$\left. . \prod_{j=1}^{i-1} f(x_j) . \prod_{s=i+1}^{N} e^{-\theta b_s} f(e^{-\theta b_s} x_s - \theta c_s - h_1(\theta) c_{1s} - \dots - h_k(\theta) c_{ks}) \right| dx$$

$$=\int_{-\infty}^{+\infty}\left|\frac{1}{\theta}[e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta)c_{1i} - \dots - h_k(\theta)c_{ki}) - f(x_i)]\right| dx_i$$

$$=\int_{-\infty}^{+\infty}\left|\frac{1}{\theta}\int_{\theta}^{\theta}[-b_i e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta) c_{1i} - \dots - h_k(\theta)c_{ki}) -\right.$$

$$-(b_i e^{-b\theta i} x_i + c_i + h'_1(\theta) c_{1i} + \dots + h'_k(\theta)c_{ki}).e^{-\theta b_i}.$$

$$\left. .f'(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta)c_{1i} - \dots - h_k(\theta)c_{ki})]\, d\theta \right| dx_i$$

$$\leqslant \left|\frac{1}{\theta}\int_0^{\theta} d\theta \left\{ \int_{-\infty}^{+\infty} \right| - b_i e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta)c_{1i} - \dots - h_k(\theta) c_{ki}) -$$

$$- (b_i e^{-\theta bi} x_i + c_i + h'_1(\theta)c_{1i} + \dots + h_k(\theta)c_{ki}).$$

$$.e^{\theta - b_i} f'(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta)c_{1i} - \dots - h_k(\theta)c_{ki}) \mid dx_i \Big\}$$

$$=\left|\frac{1}{\theta}\int_{\theta}^{\theta} d\theta \left\{\int_{-\infty}^{+\infty}\right| - b_i f(x_i) - [b_i (x_i + \theta c_i + h_1(\theta) c_{1i} + \dots + h_k(\theta) c_{ki}) + c_i + h'_1(\theta)c_{1i} + \dots$$

$$+ \dots + h'_k(\theta) c_{ki}] f'(x_i) \mid dx_i \Big\} \Big|.$$

$\leqslant \int_{-\infty}^{+\infty} | -b_i [f(x_i) + x_i f'(x_i)] - \gamma_i f'(x_i) \; | \, dx_i + \varepsilon$ if $\theta$ is sufficiently small, by (17) and (18).

Hence

$$\limsup_{\theta \to +0} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left| \frac{1}{\theta} [e^{-\theta b_i} f(e^{-\theta b_i} x_i - \theta c_i - h_1(\theta) c_{1i} - \dots - h_k(\theta) c_{ki}) - f(x_i)]. \right.$$

$$\left. . \pi_{j=1}^{i-1} f(x_j) . \pi_{s=i+1}^{N} e^{-\theta b_s} f(e^{-\theta b_s} x_s - \theta c_s - h_1(\theta) c_{1s} - \dots - h_k(\theta) c_{ks}) \right| dx$$

$$\leqslant \int_{-\infty}^{+\infty} | - b_i [f(x_i) + x_i f'(x_i)] - \gamma_i f'(x_i) \mid dx_i$$

$$= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} | \{- b_i [f(x_i) + x_i f'(x_i)] - \gamma_i f'(x_i)\} \pi_{j \neq i} f(x_j) \mid dx$$

Consequently, by Lemma 2,

$$\lim_{\theta\to+0}\sum_{i=1}^{N}\int\limits_{B(r,v)}\!\!\dots\!\int \frac{1}{\theta}[e^{-\theta b_i} f(e^{-\theta b_i} x_i-\theta c_i-h_1(\theta)c_{1i}-\dots-h_k(\theta)c_{ki})-f(x_i)]$$

$$\cdot\prod_{j=1}^{i-1} f(x_j)\prod_{s=i+1}^{N} e^{-\theta b_s} f(e^{\theta-b_s} x_s-\theta c_s-h_1(\theta)c_{1s}-\dots-h_k(\theta)c_{ks})\,dx$$

$$=\sum_{i=1}^{N}\int\limits_{B(r,v)}\!\!\dots\!\int \{-b_i[f(x_i)+x_i f'(x_i)]-\gamma_i f'(x_i)\}\prod_{j\neq i} f(x_j)\,dx$$

$$=\sum_{i=1}^{N}\int\limits_{B(r,v)}\!\!\dots\!\int \{b_i[-1-x_i f'(x_i)/f(x_i)]+\gamma_i[-f'(x_i)/f(x_i)]\}p(x)dx$$

$$=\sum_{i=1}^{N}\int\limits_{B(r,v)}\!\!\dots\!\int \{b_i[-1-|x_i|f'(|x_i|)/f(|x_i|)]+\gamma_i[-v_i f'(|x_i|)/f(|x_i|)]\}p(x)dx$$

$$=(1/2^N N!)\sum_{i=1}^{N}\{b_i E_f[-1-|X_{(r_i)}|f'(|X_{(r_i)}|)/f(|X_{(r_i)}|)]+$$

$$+\gamma_i v_i E_f[-f'(|X_{(r_i)}|)/f(|X_{(r_i)}|)]\}$$

$$=(1/2^N N!)\sum_{i=1}^{N}\{b_i a^+(r_i)+\gamma_i a(r_i)v_i\}.$$

The proof wil be fulfilled as in the proof of Theorem 1.

**Corollary:** Theorem 2 remains true when $\mathcal{K}_2\,\{\Delta\}$ replaced by $\mathcal{K}_3\,\{\Delta\}$ defined in (6).

**Proof:**

Put $h_i^*(\theta)=e^{-\theta b_i}h_i(\theta)$, $1\leqslant i\leqslant k$. If $h_i(\theta)$, $1\leqslant i\leqslant k$, satisfy (18) then $h_i^*(\theta)$, $1\leqslant i\leqslant k$, do also.

**Remark 2.** Theorem 1 is not a consequence of Theorem 2, as Conditions (9) and (10) are weaker than (17) and (18) respectively,

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## REFERENCES

[1] HÁJEK,J. and SIDÁK,Z. *Theory of Rank Tests.* (Academia, Praha, 1967)

[2] NGUYỄN VĂN HỘ. *Probabilities of large deviations and asymptotic Efficiency in the Bahadur sense for the signed rank tests.* (PhD. Dissertation, Charles Univ., Prague, 1971)

[3] SCHEFFÉ,H. *A useful convergence theorem for probability distributions.* (Ann. Math. Stat., V. 18 (1947) pp. 434–438)