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On pivotal methods for computing fixed points

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The problem of computing Brouwer fixed points, which arise in many fields of applied mathematics (such as mathematical economics, games theory, control systems theory, etc...), has received in recent years an intensive development. Among the methods now available for solving this problem, a particularly important class of algorithms is constituted by the so-called *pivotal algorithms.*

The first and fundamental variant of these algorithms was proposed in 1967 by H. Scarf ([5]; see also [1], or [3] for a complete bibliography). Since then, the basic ideas of H. Scarf have been extended and developed further by a number of other mathematicians and economists.

However, as have observed many authors, Scarf's algorithm suffered from the serious defect that it did not allow to take advantage of prior information and information progressively accumulated in the course of computation about the location of the fixed point to be sought. Because of this defect, Scarf's algorithm has appeared to be not enough effective when a high level of accuracy was required.

In order to overcome this difficulty, several approaches have been proposed, among which the most interesting and the best known are the sandwich method (Merrill, MacKinnon, Kuhn ; see [4]) and the *homotopy method* (Eaves [3], [4]).

The purpose of the present paper is to show that Scarf's algorithm can be modified and extended in a natural way, so as to overcome its mentioned defect and to solve a wider class of problems.

First, in § 1, we shall introduce a general problem of « fair sharing » which contains Brouwer fixed point problem as a special case. Then, in § 2, we shall present a method for finding such a « fair sharing ». This method extends Scarf's algorithm and is more flexible than the latter, in as much as it allows the computation procedure to start from *any* point one likes near to where the fixed point is expected to be. Finally, in § 3, as an illustration, we shall apply our method to variational inequalities and convex programming problems.

§ 1. FIXED POINTS AS FAIR SHARINGS

Brouwer fixed point theorem and its generalization by Kakutani are now of frequent use in nonlinear optimization theory (necessary conditions, stability questions), games-theory (saddle-point theorems) and mathematical economics (theory of general economic equilibrium).

We shall show in this section that these fixed point theorems can be viewed as special cases of a more general proposition which can be interpreted as a theorem on « fair sharing ». Hopefully this will shed some new light on the logical structure that underlies various equilibrium situations and problems.

1. Formulation of the main theorem

Let $S = [a^1, \ldots, a^n]$ be a closed $(n-1)$-simplex in R^{n-1}, with vertices $a^1, \ldots, a^n$. For every $x \in S$ we shall denote by x_i the barycentric i-coordinate of x in this simplex (so $x_i \geqslant 0, \sum_{i=1}^{n} x_i = 1$). Let $F_i = \{x \in S : x_i = 0\}$ be the i-face of S (the face opposite to vertex a^i) and let $\overset{\circ}{F}_i = \{x \in S : x_i = 0, x_j > 0$ for all $j \neq i\}$.

Consider now a symmetric *n-ary* predicate L over S, i.e. a mapping from the set of all unordered systems of n distinct elements of S to the set $\{$true, false$\}$. A unordered system $U = (u^1, \ldots, u^n)$ of n distinct elements of S is said to be a *L-system* if $L(U) = L(u^1, \ldots, u^n) =$ *true*. A subset E of S is said to be a *L-set* if it contains a *L-system*. A point $x^* \in S$ is called a *fixed point* of the predicate L if every neighbourhood of x^* in S is a *L-set*.

We shall be concerned with the conditions under which a given predicate L has a fixed point. Before stating these conditions, let us introduce a convenient terminology and notation. For any two sets of same cardinality, if $U' = (U \setminus \{u\}) \cup \{u'\}$, we shall say that U' obtains from U by the *pivoting* u/u' and shall write $U' = U\,(u/u')$, or $U \xrightarrow{u/u'} U'$, or simply $U \longrightarrow U'$ if there is no need to precise the pivoting. Here it is not excluded that $u' = u$, in which case $U' = U$.

The conditions we had in view can be formulated as follows:

(i) For every L - *system* U and for every $u' \in S \setminus U$ there exists a unique $u \in U$ such that $U' = U(u/u')$ is also a L - *system.*

(ii) The vertex set of $S = (a^1, \ldots, a^n)$ is a L - *system*, but no proper face of S is a L - *set.*

(ii') Every system $U = (u^1, \ldots, u^n)$ with $u^i \in \mathring{F}_i$ is a L - *system*, but no system $U = (u^1, \ldots, u^n)$ with $|U \cap \mathring{F}_i| > 1$ (for some i) is a L - *system.*

The predicate L is said to be *proper* if it satisfies: (i) and (ii), or (i) and (ii').

THEOREM 1. Every proper predicate L has a fixed point, which can be computed with any prescribed accuracy. More precisely, given any $\varepsilon > 0$ *one can find by a finite procedure a L - set U with* diam $U < \varepsilon$.

Proof. The first statement follows from the second one. Indeed, if the second statement is true, one can find for every positive integer k a L - *system* $U_k = (u^{k,1}, \ldots, u^{k,n})$ with diam $U_k < \varepsilon_k$, where $\varepsilon_k \searrow 0$. Using the compactness of S, one can assume, by taking subsequences if necessary, that $u^{k,1} \to x^* \in S$. Since diam $U_k \to 0$, we then have $u^{k,i} \to x^*$ for all $i = 1, \ldots, n$. This means that x^* is a fixed point of L.

Thus, it is enough to prove the second statement of the Theorem. We shall do it by pointing out in § 2 an algorithm for computing the desired L - *system.*

Note 1. In many applications, the predicate L of concern is not proper, but one can easily find a proper predicate L' implying L (i. e. such that every L' - *system* is a L - *system*). In these cases, *Theorem 1* obviously remains valid.

Note 2. If condition (i) holds, then condition (ii') is equivalent to the following one:

(ii^*) Every system $U = (u^1, \ldots, u^n)$ with $u^i \in \mathring{F}_i$ is a L - *system*; if U is a L — *system*, if $u \in \mathring{F}_i \cap U$ and $u' \in \mathring{F}_i \setminus U$, then $U' = U(u/u')$ is also a L - *system.*

Indeed, assuming (i) and (ii') to hold, let U be a L - *system and* $u \in \mathring{F}_i \cap U$, $u' \in \mathring{F}_i \setminus U$. Then there is an element $v \in U$ such that $U(v/u')$ is a L - *system.* One must have $v = u$, other wise $U(v/u')$ would contain two elements $u, u' \in \mathring{F}_i$, contrary to (ii'). Therefore (ii^*) holds. Conversely, assuming (i) and (ii^*) to hold, let $U = (u^1, \ldots, u^n)$ be a L — *system* such that $U \cap \mathring{F}_i$ contains two distinet elements u,v. Then for any $u' \in \mathring{F}_i \setminus U$ both $U(u/u')$ and $U(v/u')$ are L-*systems*, contrary to (i).

2. Corollaries : The Brouwer's fixed point theorem

We shall see later (§ 3) that *Theorem 1* can be used to derive easily Kakutani's fixed point theorem. Here let us derive from *Theorem 1* two propositions which are known to be equivalent forms of Brouwer's fixed point theorem.

COROLLARY 1. Let $S = [a^1, \ldots, a^n]$ be a closed $(n-1)$ - simplex, whose i-face is denoted by F_i. If the sets $L_1, \ldots, L_n$ are such that:

$$1)\ S \subset \bigcup_{i=1}^{n} L_i\ ; \qquad 2)\ a^i \in L_i,\ F_i \cap L_i = \varnothing\ (\forall_i),$$

then there exists in S a point $x^* \in \bigcap_{i=1}^{n} \bar{L}_i$.

Here $\bar{L}_i$ denotes the closure of L_i.

Proof. Using the hypotheses: $S \subset \bigcup_{i=1}^{n} L_i$, $a^i \in L_i$, we can define a mapping $l : S \rightarrow \{1, \ldots, n\}$ such that $l(a^i) = i$, and $l(x) = i$ only if $x \in L_i$. Let L denote the symmetric n-*ary* predicate over S, such that a set U of n distinct elements of S is a L—*system* if and only if $l(U) = \{1, \ldots, n\}$. Then it can be easily verified that L fulfils conditions (i) and (ii). Hence, by *Theorem 1*, there exists in S a point x^*, every neighbourhood of which contains n points $u^i \in L_i$ $(i = 1, \ldots, n)$. This implies $x^* \in \bar{L}_i$ $(i = 1, \ldots, n)$, as required.

COROLLARY 2. Let $S = [a^1, \ldots, a^n]$ be a closed $(n-1)$ - simplex, whose i-face is denoted by F_i. If the sets $L_1, \ldots, L_n$ are such that :

$$1)\ S \subset \bigcup_{i=1}^{n} L_i, \qquad 2)\ F_i \subset L_i\ (\forall_i),$$

then there exists in S a point $x^* \in \bigcap_{i=1}^{n} \bar{L}_i$.

Proof. Using the hypothesis $S \subset \bigcup_{i=1}^{n} L_i$, we can define a mapping $l : S \rightarrow \{1, \ldots, n\}$ such that $l(x) = i$ implies $x \in L_i$. Let L denote the symmetric n-*ary* predicate over S, such that a set U of n distinct elements of S is a L-*system* if and only if $l(U) = \{1, \ldots, n\}$. Then it is readily seen that L fulfils conditions (i) and (ii'). The proof can be completed just in the same way as in the previous case.

Brouwer's fixed point theorem follows easily from any of the preceding corollaries. Indeed, assume for example that *Corollary 2* holds and consider a continous mapping $f: S \to S$. Setting $L_i = \{x \in S : x_i \leqslant f_i(x)\}$ ($f_i(x)$ denotes the barycentric *i-coordinate* of $f(x)$ in S), one can verify that $L_1, \ldots, L_n$ are closed sets satisfying conditions 1) and 2) of *Corollary* 2. Hence, there is in S a point $x^* \in \bigcap_{i=1}^{n} L_i$, i.e. such that $x_i^* \leqslant f_i(x^*)$ for every $i = 1, \ldots, n$. Since $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} f_i(x^*) = 1$, this implies $x^* = f(x^*)$.

3. « Economic » interpretation

Assume that some utility (which may be positive, like a profit, or negative, like a lass) is to be shared among a group of n persons $1, 2, \ldots, n$, and we are looking for a «*fair sharing*», i.e. a sharing acceptable for everybody (in some sense to be made precise).

Let us represent every sharing by a point x of the simplex $S = [a^1, \ldots, a^n]$, such that the barycentric i-coordinate x_i of x is equal to the share of person i in this sharing.

Imagine that the same utility is to be shared not just once, but n times successively. Then each sequence $U = (u^1, \ldots, u^n)$ of n such sharings may be fair or not, and we can describe the family of all fair systems U by giving a symmetric n-ary predicate L over S, such that a (unordered) set $U = (u^1, \ldots, u^n)$ of n sharings $u^1, \ldots, u^n$, is fair if and only if U is a *L-system*. Having thus defined the notion of fairness for systems of n sharings, we can accept as fair any sharing x^*, in every neighbourhood of which, as small as we like, there are n sharings $u^1, \ldots, u^n$ forming a fair system. In other words, a fair sharing is a fixed point of the predicate L, in the sense defined above.

THEOREM 1: can thus be interpreted as pointing out the conditions under which a fair sharing exists.

Condition (i) expresses a rather common feature of many real situations. Assume, for example, that each person i has chosen a set $L_i \subset S$, which represents the set of all sharings good for him, and that a set U of n sharings is regarded as fair if and only if for each $i = 1, \ldots, n$ the set U contains just one $u^i \in L_i$. In that case condition (i) reduces to

$$(i^*) \quad \bigcup_{i=1}^{n} L_i = S, \; L_i \cap L_j = \emptyset \text{ for } i \neq j,$$

which means that: every sharing is good for exactly one person in the group.

Indeed, if (i^*) holds then for every fair system $U = (u^1, \ldots, u^n)$ and every $v \in S \setminus U$, we have $u^i \in L_i$ $(i = 1, \ldots, n)$ and $v \in L_{i_0}$ for just one i_0, ience u^{i_0} is the only element of U for which $U\,(u^{i_0}/v)$ is again a fair system. Therefore (i^*) implies (i). Conversely, if (i) holds, then it is easily seen that each $v \in S$ belongs to exactly one L_i, i.e. (i^*) holds.

So, in this particular case, condition (i) amounts to requiring simply that each possible sharing is good for just one person. Perhaps the requirement $L_i \cap L_j = \varnothing\ (i \neq j)$ is a too stringent one; however, if $\bigcup_{i=1}^{n} L_i = S$, one can always find subsets $L'_i \subset L_i$ such that $\bigcup_{i=1}^{n} L'_i = S$ and $L'_i \cap L'_j = \varnothing (i \neq j)$, and by this way it is possible to define a proper predicate L' implying L. As was pointed out in *Note 1*, this will suffice to ensure the existence of a fair sharing.

Conditions (ii) and (ii') concern the cases when the utility to be shared is a positive or a negative one, respectively.

If the utility is positive, then $a^i = (0,.., \underset{i}{1},..., 0)$ represents for person i the best sharing, while every $x \in F_i$ represents for him the worst one, so it is natural that the system $(a^1, \ldots, a^n)$ is fair, and that any system U contained in a face F_i cannot be fair: this is just what is stated in condition (ii). On the other hand, if the utility is negative, then a^i represents for person i the worst sharing, while every $x \in F_i$ represents for him the best one, so it is reasonable to assume, as was expressed in condition (ii'), that any system $(u^1, \ldots, u^n)$ with $u^i \in \mathring{F}_i$ is fair, but any system $(u^1, \ldots, u^n)$ having more than one element in some face F_i (i.e. such that some person i gets the best sharing at least twice) is not fair.

Thus, the conditions under which there is a fair sharing according to *Theorem 1*, appear to be quite natural.

It is worth while noticing also that, in the context of the interpretation given above. *Corollaries 1* and *2* could be restated as follows. Assume that each person i has chosen a closed set $L_i \subset S$ representing the collection of all sharings acceptable for him. Then, for the case of a positive utility, *Corollary 1* says that a fair sharing always exists, provided every possible sharing is acceptable for at least someone in the group (condition 1) and every sharing in which a person i has whole part is acceptable for him, while every sharing in which he has no part is not acceptable (condition 2). For the case of a negative utility. *Corollary 2* says that a fair sharing always exists, provided every possible sharing is acceptable for at least someone in the group (condition 1) and every sharing in which a person i has no part is acceptable for him (condition 2).

As we see, a common sense underlies Brouwer fixed point theorem. Perhaps it is this common sense that makes fixed point methods so useful in the study of various equilibrium models.

§ 2. AN ALGORITHM FOR FINDING THE "FAIR SHARING"

To complete the proof of *Theorem 1*, we proceed now to describe a finite algorithm for finding, for a given proper predicate L and a given number $\varepsilon > 0$, a L-set U with diam $U < \varepsilon$.

We may restrict ourselves to the case where the predicate L satisfies conditions (i) and (ii'), because the case where L satisfies (i) and (ii) can be reduced to the previous one.

The main idea of the method is to take a finite grid Q, fine enough, of the set int S and to define two families of subsets of cardinality $n+1$ of the set $\bar{Q} = Q \cup \{\bar{0}, \bar{1}, \ldots, \bar{n}\}$, where $\bar{0}, \bar{1}, \ldots, \bar{n}$ are arbitrary elements not belonging to Q. These subsets will be called « primitive sets » and « complete sets », respectively. To each subset V of $\bar{Q}$ is associated a simplex $\Delta(V) \subset S$, which will be a L-set if V is complete, and will have a diameter less than ε if, moreover, V is primitive. Then the problem reduces to finding a set V which is both primitive and complete (so that $\Delta(V)$ will be the desired L-set). This can be done by a so-called «pivotal» procedure, consisting of a finite number of operations similar to the pivot steps in the simplex method for solving linear programs.

1. Primitive sets

Consider $n+1$ arbitrary orderings $\underset{i}{\leqslant}$ $(i = 0, 1, \ldots, n)$ on the set Q, and define, additionnally, for every $i = 0, 1, \ldots, n$ and for every $x \in Q$:

$$\bar{i} \underset{i}{<} x \underset{i}{<} (\overline{i+1}) \underset{i}{<} \cdots \underset{i}{<} \bar{n} \underset{i}{<} \bar{0} \underset{i}{<} \bar{1} \underset{i}{<} \cdots \underset{i}{<} (\overline{i-1}) \tag{1}$$

In this way we obtain $n+1$ orderings (denoted also by $\underset{i}{\leqslant}$) on the set

$$\bar{Q} = Q \cup \{\bar{0}, \bar{1}, \ldots, \bar{n}\}$$

A set V of $n+1$ distinct elements of $\bar{Q}$ is said to be *primitive* if $(\forall x \in Q)\ (\exists i)\ (\forall v \in V)\ x \underset{i}{\leqslant} v$, or, equivalently, if there is no $x \in Q$ such that

$$(\forall i) \quad x \underset{i}{>} c^i(V)$$

where $c^i(V) = i\text{-}min\ V$, the minimal element of V in the i-ordering.

For example, the set $V_0 = (q, \bar{1}, \ldots, \bar{n})$, with $q = 0\text{-}max\ Q$, the maximal element of Q in the 0-ordering, is obviously primitive.

For our purpose the most important property of primitive sets is the following.

LEMMA 1. Let V be a primitive set and let v be an element of V. If $(V \setminus \{v\}) \cap Q \neq \varnothing$, there is exactly one $u \in \bar{Q}$ such that $V(v/u)$ is again primitive.; otherwise, there is no such u.

Proof. We first observe that if V is a primitive set, then for each $v \in V$ there is exactly one i such that $v = c^i(V)$. Indeed, if $c^k(V) = c^j(V)$ for $k \neq j$, there would exist an element $v \in V$ such that $v >_i c^i(V)$ for every i; this would imply, in view of (1), $v \in Q$, and would contradict the definition of primitive sets.

Consider now any primitive set V and any $v \in V$. Let $v = c^s(V)$. If there is $u \in \bar{Q}$ such that $V' = V(v/u)$ is again primitive, let $u = c^r(V')$. We must have $r \neq s$, because $r = s$ would imply for all $i \neq s$:

$$c^i(V) = i\text{-}min\,(V \setminus c^s(V)) = i\text{-}min\,(V' \setminus c^r(V')) = c^i(V'), \qquad (2)$$

and hence, $c^r(V') >_i c^i(V)$ for all i (if $c^r(V) <_r c^r(V')$), or $c^r(V) >_i c^i(V')$ for all i (if $c^r(V') <_r c^r(V)$), contradicting the primitiveness of V and V'. Furthermore, by (2), $c^i(V) = c^i(V')$ for all $i \notin \{r, s\}$, so

$$c^r(V) = c^s(V') \qquad (3)$$

If $c^r(V) \notin Q$, then by (1), $c^r(V) = \bar{r}$, and hence, $c^r(V) = r\text{-}min\ V' = c^r(V')$, i.e. $r = s$, which is not true. Therefore, $c^r(V) \in Q$. Thus, if there is u such that $V' = V(v/u)$ is primitive, then, necessarily

$$(V \setminus \{v\}) \cap Q \neq \varnothing$$

To complete the proof of the *Lemma*, it remains to show that if the above condition is fulfilled, then there is exactly one u such that $V' = V(v/u)$ is primitive.

Let $v^* = s\text{-}min\,(V \setminus \{c^s(V)\})$. Since $V' \setminus \{c^r(V')\} = V \setminus \{c^s(V)\}$ (with $r \neq s$), we deduce $v^* = s\text{-}min\,(V' \setminus \{c^r(V')\}) = c^s(V')$, and hence, by (3), $v^* = c^r(V)$. This completely determines r, since $c^s(V)$ is given. Define now the set

$$R = \{x \in \bar{Q} : x >_s v^*,\ x >_i c^i(V) \text{ for all } i \neq r, s\}.$$

This set is not empty (at least it contains $\bar{r}$) and it follows from the above that

$$c^r(V') = r\text{-}max\ R,$$

which completely determines $u = c^r(V')$, because R and r are known. Finally, it is easy to see that the set $V' = V(v/u)$, with $u = c^r(V')$, is actually primitive. Indeed, if there was $x \in Q$ with $x >_i c^i(V')$ for all i, then, since $c^i(V') = c^i(V)$ for all $i \notin \{r, s\}$, $c^s(V') = c^r(V)$, $c^r(V') = r\text{-}max\ R$, we would have $x \in R$ and $x > r\text{-}max\ R$, which is absurd.

Q. E. D.

The previous argument parallels the proof of the replacement theorem in Scarf's theory ([1] or [6]). The only difference is that we consider here arbitrary orderings on Q. This some what simplifies the reasoning and substantially widens the range of applicability of the model. As will be seen later, *lemma* 1 is crucial for the extension and the improvement of Scarf's algorithm.

We shall assume in the sequel that the n orderings $i = 1, \ldots, n$ satisfy, for all $x, x' \in Q$:

$$x \underset{i}{\leqslant} x' \Rightarrow x_i \leqslant x'_i \tag{4}$$

For example, one might define, as did Scarf in [6]

$$x \underset{i}{<} x' \Leftrightarrow (x_i, \ldots, x_n, x_1, \ldots, x_{i-1}) < (x'_i, \ldots, x'_n, x'_1, \ldots, x'_{i-1})$$

in the lexicographic sense.

Let us associate to each subset V of $\bar{Q}$ a simplex $\Delta(V)$ defined as being the smallest simplex, positively homothetic to S, containing all elements of $V \cap Q$, and meeting every $\overset{\circ}{F}_i$ such that $i > 0$, $\bar{i} \in V$. If V is primitive, then $\Delta(V)$ is just the simplex, positively homothetic to S, with the i-face passing through $c^i(V)$ if $c^i(V) \in Q$ or contained in the i-face of S if $c^i(V) = \bar{i}$ (note that if $c^i(V) \notin Q$, i. e. if $c^i(V) \in \{\bar{0}, \bar{1}, \ldots, \bar{n}\}$, then, in view of (1), necessarily $c^i(V) = \bar{i}$).

LEMMA 2. If V is a primitive set containing $\bar{0}$ and if Q is a η-net of S, then diam $\Delta(V) < N \,.\, \eta$, *where N is some positive constant* (depending only upon S).

(By a η-net of S we mean a set Q such that every ball of radius η around any point of S contains at least one point of Q; since S is compact, such a η-net exists for any given $\eta > 0$).

Proof. If V is a primitive set containing $\bar{0}$, then it is easily seen that the simplex $\Delta(V)$ contains no point of Q in its interior. Therefore, the greatest ball contained in $\Delta(V)$ contains no point of Q and hence, must have a radius less than η. Let N denote the ratio of the diameter of S to the radius of the greatest ball contained in S. Then, obviously, diam $\Delta(V) < N. \eta$.

2. Complete sets

A set V of $n+1$ distinct elements of $\bar{Q}$ is said to be *complete* if V contains the element $\bar{0}$ and if every set obtained from $V \setminus \{\bar{0}\}$ by replacing each $v \in V$ of form $v = \bar{i}$ $(i = 1, \ldots, n)$ by an arbitrary element of $\overset{\circ}{F}_i$ is a *L-system*. Clearly, if V is complete, then $\Delta(V)$ is a L-set.

For example, the set $V_0^* = (\bar{0}, \bar{1}, \ldots, \bar{n})$ is obviously complete (see condition (ii^*)). This complete set and the primitive set $V_0 = (q, \bar{1}, \ldots, \bar{n})$, with $q = 0\text{-}max\ Q$, will play a special role in the algorithm to be described below.

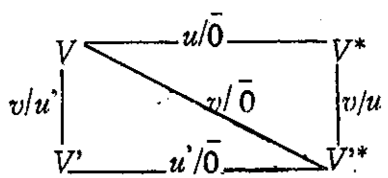
LEMMA 3. For every complete set V and for every element $v' \in \bar{Q} \setminus V$ there is exactly one $v \in V$ such that $V(v/v')$ is again complete.

Proof. Denote by $\Upsilon : \bar{Q} \to \mathring{S}$ an arbitrary injective mapping such that $\Upsilon(x) = x$ for every $x \in Q$ and $\Upsilon(\bar{i}) \in \mathring{F}_i$ for every $i = 1, \ldots, n$. It is easy to see that a set V containing $\bar{0}$ $(V \subset \bar{Q},\ |V| = n+1)$ is complete if and only if $U = \Upsilon(V \setminus \{\bar{0}\})$ is a *L-system.* Indeed, suppose U to be a *L-system* and consider an arbitrary set $\tilde{U}$ obtained from $V \setminus \{\bar{0}\}$ by replacing each $v \in V$ of form $v = \bar{i}$ by an element $\tilde{v} \in \mathring{F}_i$. For each $v \in V$ of form $v = \bar{i}$ we have $\Upsilon(v) \in \mathring{F}_i$ and so, by (ii^*), the pivoting $\Upsilon(v)/\tilde{v}$ performed on U yields again a *L-system.* Since $\tilde{U}$ obtains from U by a finite sequence of pivotings of the form just described, we see that $\tilde{U}$ is a *L-system.* Hence, if U is a *L-system,* then V is complete. The converse being obvious, our assertion is established.

Let now V be any complete set, and let $v' \in \bar{Q} \setminus V$. Then $v' \neq \bar{0}$ (because $\bar{0} \in V$), $u' = \Upsilon(v') \in S \setminus U$, where $U = \Upsilon(V \setminus \{\bar{0}\})$. Since U is a *L-system,* there is, by virtue of property (i), just one $v \in U$ such that $U(u/u')$ is a *L-system.* Hence, $v = \Upsilon^{-1}(u)$ is the only element of V such that $V(v/v')$ is complete. This proves the *Lemma.*

3. The algorithm

We shall say that (V, V^*) is a *p.c. couple* if V is primitive, V^* is complete and $V^* = V(u/\bar{0})$ for some $u \in V$. If for two p.c. couples (V, V^*), (V', V'^*) such that $V' \neq V$, $V'^* \neq V^*$ we have



then we shall say that (V', V'^*) *follows* (V, V^*).

LEMMA 4. For every p.c. couple (V, V^*) such that V is not complete and $V \neq V_0$ or $V^* = V_0^*$ there is exactly one p.c. couple following it.

Proof. Assume that (V, V^*) is a p.c. couple and V is not complete. We have $V^* = V(u/\bar{0})$ with $u \neq \bar{0}$, so $u \notin V^*$ and hence, by *Lemma 3,* there is a uniquely determined $v \in V^*$ such that $V'^* = V^*(v/u)$ is complete. We can not have $v = \bar{0}$, since then $V = V'^*(\bar{0}/v) = V'^*$, i.e. V would be complete. Thus $v \neq \bar{0}$ and hence $v \in V$. Since $V'^* = V(v/\bar{0})$ and $v \neq \bar{0}$, if $V \setminus \{v\} \subset \{\bar{0}, \bar{1}, \ldots, \bar{n}\}$, then necessarily $V \setminus \{v\} = \{\bar{1}, \ldots, \bar{n}\}$, and so $V'^* = V_0^*$ (hence $V^* \neq V_0^*$) and, by *Lemma 1,* $V = V_0$. Therefore, if $V \neq V_0$ or $V^* = V_0^*$, then $(V \setminus \{v\}) \cap Q \neq \emptyset$

and by *Lemma 1* there is a uniquely determined u' such that $V' = V(v/u')$ is primitive. Obviously, $V'^{*} = V'(u'/\bar{0})$ and $V' \neq V$, $V'^{*} \neq V^{*}$, so the *Lemma* is proved.

THEOREM 2. There exists a uniquely determined sequence of distinct p.c. couples

$$(V_0, V_0^{*}), (V_1, V_1^{*}), \ldots, (V_m, V_m^{*}) \qquad (^{*})$$

such that (V_{i+1}, V_{i+1}^{*}) *follows* (V_i, V_i^{*}) $(i = 0, 1, \ldots, m-1)$ *and* V_m *is complete, while no* V_i $(i = 0,1, \ldots, m-1)$ *is complete.*

Proof. Since (V_0, V_0^{*}) is a p.c. couple and V_0 is not complete (because $\bar{0} \notin V_0$), it follows from *Lemma 4* that there exists a uniquely determined p.c. couple (V_1, V_1^{*}) following (V_0, V_0^{*}). We now observe that in any sequence of successive p.c. couples of form $(^{*})$ all V_i are distinct. Indeed, assuming the contrary, let k denote the smallest integer such that $V_k = V_h$ for some $h < k$. Since $V_k^{*} = V_h(u^k/\bar{0})$, $V_h^{*} = V_h(u^h/\bar{0})$ we have $V_k^{*} = V_h^{*}(u^k/u^h)$. But $V_{h+1}^{*} = V_h^{*}(v^{h+1}/u^h)$ for some v^{h+1}, hence, by *Lemma 3*, $u^k = v^{h+1}$, $V_k^{*} = V_{h+1}^{*}$. The latter equality implies $h + 1 < k-1$, since $V_i^{*} \neq V_{i+1}^{*}$ for every i. On the other hand, $V_{h+1} = V_h(v^{h+1}/u^{h+1})$, whereas $V_{k-1} = V_k(u^k/v^k)$, and since $V_k = V_h$, $u^k = v^{h+1}$, it follows from *Lemma 1* that $V_{k-1} = V_{h+1}$. This contradicts the definition of k, because $h + 1 < k-1$. Therefore, all V_i are distinct. In particular, $V_i \neq V_0$ for every $i > 0$, and so, by *Lemma 4*, if the sequence $(^{*})$ has been constructed up to (V_i, V_i^{*}) and if V_i is not complete, then the sequence can be continued in a unique way till (V_{i+1}, V_{i+1}^{*}). Since there are only finitely many primitive sets (Q being finite), the sequence must terminate at some (V_m, V_m^{*}) with V_m complete.

Theorem 2 provides the following algorithm for solving our problem:

ALGORITHM. Take a grid $Q \subset \text{int } S$, which is a η - net of S, with $\eta < \frac{\varepsilon}{N}$ (N being the constant mentioned in *Lemma 2*). Start from the p.c. couple (V_0, V_0^{*}), where $V_0 = (q, \bar{1}, \ldots, \bar{n})$ with $q = 0\text{-}max\ Q$, $V_0^{*} = (\bar{0}, \bar{1}, \ldots, \bar{n})$. At step k, one has a p.c. couple (V_k, V_k^{*}), such that $V_k^{*} = V_k(u^k/\bar{0})$. If V_k is not complete, determine the following p.c. couple (V_{k+1}, V_{k+1}^{*}) by the rule: $V_{k+1}^{*} = V_k^{*}(v^k/u^k)$, where v^k is determined by u^k according to *Lemma 3*; then $V_{k+1} = V_k(v^k/u^{k+1})$, where u^{k+1} is determined by v^k according to *Lemma 1*. Otherwise, V_k is complete and the procedure terminates: since V_k is complete, the set $U = \Delta(V_k)$ is a L-set; since V_k is primitive, diam $U \leq$ diam $V_k < N.\ \eta < \varepsilon$ (*Lemma 2*).

Note 3. The effectiveness of the algorithm highly depends upon the *choice of the* 0*-ordering*, in particular, upon the choice of the maximal element of Q in this ordering. In practice, prior information may suggest to seek the L-set in the proximity of some point $x^0 \in \text{int } S$. In this case x^0 should be included in the grid Q and the 0-ordering should be defined such that $x^0 = q = 0\text{-}max\ Q$, and $x' \underset{0}{\leqslant} x$ only if $|x' - x^0| \geqslant |x - x^0|$. So the starting-primitive set should be

$$V_0 = (x^0, \bar{1}, \ldots, \bar{n}).$$

Note 4. If the 0-ordering on Q is taken to be the same as one of the i-orderings $(i = 1, \ldots, n)$, i.e. such that $x' \underset{0}{\leqslant} x$ implies $x'_i \leqslant x_i$, then the above algorithm, as applied to the Brouwer fixed point problem, will coincide with the *algorithm of Scarf* ([1] or [6]). In this respect the above algorithm can be considered as an extension of the algorithm of Scarf. There is, however, an essential difference between the two approaches whereas in Scarf's one the procedure always begins near to a vertex of the simplex S (which constitutes, as is well known, a serious difficulty of this method), our approach allows the procedure to start from *any* point one likes in the region where the fixed point is expected to be. This improvement is possible owing to the generalization of the replacement theorem in Scarf's theory, provided by *Lemma 1.* On the basis of this crucial *Lemma*, a 0-ordering has been introduced on the set Q, allowing a large freedom in the choice of the starting primitive set V_0.

§3. APPLICATIONS TO VARIATIONAL INEQUALITIES AND CONVEX PROGRAMMING

As an illustration, we shall consider in this section some applications of the previous results.

1. A particular class of proper predicates

Let us first prove a lemma pointing out an important class of proper predicates.

As before, $S = [a^1, \ldots, a^n]$ denotes a closed $(n-1)$ - simplex, with vertices $a^1, \ldots, a^n$; $\mathring{F}_i$ denotes the relative interior of the face of S opposite to a^i.

LEMMA 5. Let b be an interior point of S, $B: S \to R^{n-1}$ a mapping such that $B(x) = a^i$ for every $x \in \mathring{F}_i$. Let L be a symmetric n-ary predicate over S, such that a set U of n distinct elements of S is a L-system if and only if the vectors $\{B(u), u \in U\}$ span a $(n-1)$-simplex containing b. Then L is implied by a proper predicate over S (and hence, by Theorem 1, has a fixed point).

Proof. To each $x \in R^{n-1}$ associate $x' = \begin{pmatrix} x \\ 1 \end{pmatrix} \in R^n$. Then we have a vector $b' \in R^n$ and a mapping $B': S \to R^n$, such that $\{B(u), u \in U\}$ span a $(n-1)$-simplex containing b if and only if the system

$$\sum_{u \in U} t(u) B'(u) = b', \qquad t(u) \geqslant 0 (u \in U) \tag{5}$$

has a unique solution. Define a predicate L' over S such that a set U of n distinct elements of S is a *L'-system* if and only if for all $\varepsilon > 0$ small enough the system

$$\sum_{u \in U} t(u)\, B'(u) = b' + \sum_{i=1}^{n} \varepsilon^i (a^i)', \qquad t(u) > 0 \, (u \in U) \tag{6}$$

has a unique solution. Then L' obviously implies L and it is easy to see that L' is proper.

Indeed, if $u^i \in \mathring{F}_i$ $(i = 1, \ldots, n)$, then $U = (u^1, \ldots, u^n)$ is a L'-set, since $B(u^i) = a^i$ and b is by hypothesis an interior point of S. Moreover, if $|U \cap \mathring{F}_i| > 1$, then $B'(U)$ has at least two equal elements and so $B'(U)$ cannot be a set of n independent vectors in R^n, i.e. U cannot be a *L'-system*. Thus L'satisfies condition (*ii'*). On the other hand, if U is a *L'-system* and $v \notin U$, then the system

$$\sum_{u \in U} t(u)\, B'(u) + t(v)\, B'(v) = b' + \sum_{i=1}^{n} \varepsilon^i (a^i)'$$
$$t(u) \geqslant 0 \; (u \in U), \quad t(v) \geqslant 0 \tag{7}$$

admits $\{B'(u),\, u \in U\}$ as a non-degenerate feasible basis (i.e. (7) has a unique solution, suth that $t(v) = 0$, $t(u) > 0$ for all $u \in U$). Since (7) implies

$$\sum_{u \in U} t(u) + t(v) = 1 + \sum_{i=1}^{n} \varepsilon^i,$$

the set of solutions to (7) is bounded; hence, by a known result of linear programming theory, there is a unique $w \in U$ such that $\{B'(u),\, u \in U(w/v)\}$ will be a non-degenerate feasible basis for (7), i.e. such that $U(w/v)$ is a *L'-system* (note the fact, established in linear programming theory, that for all $\varepsilon > 0$ small enough the polytope (7) is non-degenerate). Thus L' satisfies also condition (*i*), and the *Lemma* is proved.

2. Fixed points and variational inequalities

Consider the following problem, the interest of which in many fields of applied mathematics, is well known.

Given a set C in $X = R^n$ and a set-valued mapping $f: C \to 2^x$, find a point $x^* \in C$ for which there exists y^* satisfying

$$y^* \in f(x^*), \qquad (\forall x \in C) \quad \langle x - x^*, y^* \rangle \geqslant 0 \tag{8}$$

where, as usually, $\langle \ldots \rangle$ denotes the inner product.

Such a point x^* is called a solution of the *variational inequality* (8).

For each $x \in X$ let $T(x)$ denote the normal cone of C, i.e. the set of all $y \in X$ such that $(\forall x' \in C) \langle x' - x, y \rangle \leqslant 0$. Then solving the variational inequality (8) amounts to finding a point $x^* \in C$ satisfying the inclusion

$$0 \in f(x^*) + T(x^*). \tag{9}$$

We shall assume that: 1) C is a compact convex set; 2) for every $x \in C$ the set $f(x)$ is nonempty, compact and convex; 3) f is an upper semi-continuous set-valued mapping. Under these assumptions it is known that a solution x^* to (8) always exists. We shall provide here a new proof of this fact, based upon the method presented in the previous section. The proof is thus constructive and yields a new algorithm for computing the solution x^*.

Clearly, without loss of generality, we may assume, additionnally, that C has a nonempty interior and is contained in the interior of the simplex $S = [a^0, a^1, \ldots, a^k]$ with $a^0 = 0 \in R^k$, a^i = the i-unit vector in R^k. Let $\mathring{F}_i = \{x \in S: x_i = 0,\ x_j > 0 \text{ for } j \neq i\}$, where x_i $(i = 1, 2, \ldots, k)$ is the i-coordinate of x and $x_0 = 1 - \sum_{j=1}^{k} x_j$. Let $e = (1/\sqrt{k}, 1/\sqrt{k}, \ldots, 1/\sqrt{k}) \in R^k$.

Consider now a mapping $B: S \to X$ satisfying:

1) $B(x) = -a^i$ for $x \in \mathring{F}_i$ $(i = 1, \ldots, k)$, $B(x) = e$ for $x \in \mathring{F}_0$;

2) $B(x) \in f(x)$ for $x \in C$; $B(x) \in T(x)$ and $|B(x)| = 1$ for $x \in S \setminus C$.

Since $-a^i \in T(x)$ for $x \in \mathring{F}_i$ $(i = 1, \ldots, k)$ and $e \in T(x)$ for $x \in \mathring{F}_0$, conditions 1) and 2) are consistent. Furthermore, the simplex generated by $\{-a^1, -a^2, \ldots, -a^k, e\}$ contains 0 in its interior, and so by *Lemma 5* one can associate with B a proper predicate L such that a set U of $n+1$ distinct elements of S is a *L-system* if and only if the vectors $\{B(u), u \in U\}$ span a n-simplex containing 0.

THEOREM 3. Every fixed point of the predicate L just defined is a solution of the variational inequality (8).

Thus, in order to solve the variational inequality (6) it suffices to compute a fixed point of the predicate L (for example, by the method described previously).

Proof. Let x^* be any fixed point of L (such a point exists, by *Lemma 5*). Then there is a sequence of *L-systems* $U_\gamma = (u^{0\gamma}, u^{1\gamma}, \ldots, u^{k\gamma}) \subset S$ such that $u^{i\gamma} \to x$ as $\gamma \to \infty$ $(i = 0, 1, \ldots, k)$. Denote by N_γ the set of all i for which $u^{i\gamma} \in C$.

By definition of *L-systems*, there exist, for each γ, numbers $t_{i\gamma}$ satisfying

$$t_{i\gamma} \geqslant 0, \quad \sum_{i=0}^{k} t_{i\gamma} = 1, \quad \sum_{i=0}^{k} t_{i\gamma} B(u^{i\gamma}) = 0.$$

Since C is compact and f is upper semi-continuous, $f(C)$ is compact and hence, the set $B(S)$ is bounded. We may then assume, by taking subsequences if necessary,

$$N_\gamma = N \quad \text{for all } \gamma$$

$$t_{i\gamma} \to t_{i*}, \quad B(u^{i\gamma}) \to v^{i*} \text{ as } \gamma \to \infty.$$

Obviously,

$$t_{i*} \geqslant 0, \quad \sum_{i=0}^{k} t_{i*} = 1, \quad \sum_{i=0}^{k} t_{i*} v^{i*} = 0 \tag{10}$$

We contend that $\theta = \sum_{i \in N} t_{i*} > 0$. Indeed, if $\theta = 0$ then we have from (10):

$$\sum_{i \notin N} t_{i*} v^{i*} = 0$$

and hence,
$$(\forall x \in C) \quad \sum_{i \notin N} t_{i*} \langle x - x^*, v^{i*} \rangle = 0$$

On the other hand, the set-valued mapping $x \to T(x)$ being closed (as can be easily proved) it follows that $v^{i*} \in T(x^*)$ for all $i \notin N$, i.e.

$$(\forall x \in C) \quad \langle x - x^*, v^{i*} \rangle \leqslant 0 \quad (i \notin N).$$

Hence, for every $i \notin N$,

$$(\forall x \in C) \quad \langle x - x^*, t_{i*} v^{i*} \rangle = 0.$$

But, the set C having a nonempty interior can be contained in none of the manifolds $\langle x - x^*, t_{i*} v^{i*} \rangle = 0$, unless $t_{i*} v^{i*} = 0$ for all $i \notin N$. Since the numbers t_{i*} $(i \notin N)$ sum up to one, there must be at least one $i_0 \notin N$ such that $t_{i_0} > 0$. Then $v^{i_0 *} = 0$, which conflicts with the fact $v^{i_0 *} = \lim_{\gamma \to \infty} B(u^{i_0 \gamma}) \,|B(u^{i_0 \gamma})| = 1$.

Thus we have proved that $\theta > 0$. Let now

$$y^* = \sum_{i \in N} (t_{i*}/\theta) v^{i*}, \qquad z^* = \sum_{i \notin N} (t_{i*}/\theta) v^{i*},$$

so that, from (10), $0 = y^* + z^*$. Since, for $i \in N$, $u^{i\gamma} \in C$, $u^{i\gamma} \to x^*$, $B(u^{i\gamma}) \in f(u^{i\gamma})$, $B(u^{i\gamma}) \to v^{i*}$, it follows from the closedness of the set C and the mapping f that $x^* \in C$, $v^{i*} \in f(x^*)$ for $i \in N$ and hence, $y^* \in f(x^*)$, by the convexity of $f(x^*)$. Furthermore, for $i \notin N$, the relations $u^{i\gamma} \to x^*$, $B(u^{i\gamma}) \to v^{i*}$, $B(u^{i\gamma}) \in T(u^{i\gamma})$ imply, in view of the closedness of T, that $v^{i*} \in T(x^*)$. Hence, $z^* \in T(x^*)$, because $T(x^*)$ is a convex cone. Therefore,

$$0 \in f(x^*) + T(x^*),$$

as was to be proved.

Note 5. The implementation of the procedure described above requires an effective method for constructing at each point $u \in S \setminus C$ a vector $B(u) \in T(u)$, i.e. a vector satisfying

$$(\forall x \in C) \ \langle x - u, B(u) \rangle \leqslant 0.$$

But this is an easy task, at least in usual cases. Indeed, assume C to be given by a system of inequalities of the form

$$g_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m, \tag{11}$$

where each g_i is a continuous convex function. Then $u \notin C$ implies $g_i(u) > 0$ for some i. Let $\partial g_i(u)$ be the subdifferential of g_i at point u (as it is known, $\partial g_i(u)$ is nonempty) and let $B(u)$ be an arbitrary element of the set $\partial g_i(u)$. Since $x \in C$ only if $g_i(x) \leqslant 0$, we have, for all $x \in C$:

$$\langle x - u, B(u) \rangle \leqslant g_i(x) - g_i(u) < 0 .$$

Note 6. The previous theorem can be used to derive easily *Kakutani's fixed point theorem.* Indeed, if $F: C \rightarrow 2^C$ is a set-valued mapping satisfying all conditions in Kakutani's theorem, then the mapping $f(x) = x - F(x)$ will satisfy all conditions in *Theorem 3*. Applying the latter theorem yields a point $x^* \in C$ such that, for some $z^* \in F(x^*)$, $(\forall x \in C)\ \langle x - x^*,\ x^* - z^* \rangle \geqslant 0$ hence, in particular, $\langle z^* - x^*, x^* - z^* \rangle \geqslant 0$, which implies $x^* = z^* \in F(x^*)$.

3. Fixed points and convex programming

To conclude the paper, let us consider the *convex programming* problem

$$\min F(x) \ : \quad x \in C \tag{12}$$

where C is a compact convex set in R^k, $F(x)$ is a continuous convex function defined in some open set $C' \supset C$.

Under these hypotheses the subdifferential $\partial F(x)$ of F at every $x \in C$ is a nonempty convex set and, as can be easily shown, the set-valued mapping $x \longmapsto \partial F(x)$ is upper semi-continuous. Therefore, the previous algorithm can be applied to find a point $x^* \in C$ solving the variational inequality

$$y^* \in \partial F(x^*),\ (\forall x \in C)\ \langle\, x - x^*, y^* \,\rangle \geqslant 0.$$

Since $F(x) - F(x^*) \geqslant \langle\, x - x^*, y^* \,\rangle \geqslant 0$ for every $x \in C$, it follows that x^* is an optimal solution of (12).

It should be pointed out that, in the case where C is given by a system of inequalities of the form (11), this algorithm does not require the functions F and g_i to be differentiable.

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REFERENCES

[1] K. J. Arrow, H. H. Hahn. *General Competitive Analysis.* Oliver and Boyd. Edinburgh. 1971.

[2] B. C. Eaves. *Homotopies for Computation of Fixed Points.* Mathematical Programming. Vol. 3, 1–22, 1972.

[3] B. C. Eaves. *A Short Course in Solving Equations with PL Homotopies.* SIAM - AMS Proceedings. Vol. 9, 73 - 143, 1976.

[4] H. W. Kuhn, J. G. MacKinnon. *Sandwich Methods for Finding Fixed Points.* Journal of Optimization Theory and Applications. Vol. 17, 189 - 204, 1975.

[5] H. E. Scarf. *The Approximation of Fixed Points of a Continuous Mapping.* SIAM Journal on Appl. Math. Vol. 15, 1323 - 1343, 1967.

[6] H. E. Scarf, with the collaboration of T. Hansen. *The Computation of Economic Equilibrium.* Yale University Press. New Haven. 1973.