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STABILITY PROPERTY OF A SYSTEM OF INEQUALITIES

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In this paper we are concerned with the following stability problem for a system of inequalities: given a solution of such a system, when can one be sure that a « small » perturbation of the data will give rise to only a « small » variation of the solution? The importance of this problem for practical applications is obvious, since everyone is aware of the serious difficulties which may arise when the system we are solving fails to be stable.

In recent years stability properties for convex minimization problems have been investigated in connection with duality theory by a number of authors ([1], [2], [6], and others). As for the stability problem we are considering, here, it seems to have received up to now little attention from mathematicians although it is certainly not novel.

The purpose of the present paper is to develop some general criteria for the stability of a system of inequalities and/or equalities. In section 1, after a definition of the precise notion of stability we are dealing with, we shall state the main result for convex systems. Then, in section 2 a proof of this result will be provided which is based essentially on the same technique as that used in [3] and [4]. In section 3 the result will be specialized to the case where the system under consideration contains explicit equalities. In section 4 the present notion of stablity will be related to the standard notion of regularity in convex programming and also to the notion of stability of convex minimization problems that has been studied by some authors. The last section is devoted to nonconvex inequalities: here, using the method of convex approximation, we shall extend the main result to general non-convex systems.

Some of the results that follow are very near to those obtained recently by S.M. Robinson [5], who treated only the linear and finitedimensional case.

Although the present paper could be a natural development of our earlier paper [4], it is this work of S.N Robinson which has first stimulated our interest in the subject.

1. DEFINTIONS AND MAIN RESULT

Let us consider the system

$$x \in D, \quad G(x) \in N \tag{1}$$

where D is a given subset of a locally convex Hausdorff space X, N a given closed convex cone in a locally convex Hausdorff space Y, G a given mapping from D into Y.

Throughout the paper we shall assume $Y = Y_1 \times Y_2$ with $Y_1 = R^k$, $M = M_1 \times M_2$ with M_i a closed convex cone in Y_i, $i = 1,2$, and int $M_2 \neq \phi$. Accordingly, for any mapping $H : D \to Y$ we shall denote by $H_i : D \to Y_i$ $(i = 1,2)$ the components of H defined by $H(x) = (H_1(x), H_2(x))$. Also, for the sake of convenience we shall write $D_W = D \cap W$ for any neighbourhood W in X.

Definition 1. A solution $\overline{x}$ to system (1) is said to be *stable* if to every neighbourhood W of x one can associate a neighbourhood V of $O \in Y$ such that for any continuous mapping $A: D \to Y$ satisfying

$$A(D_W) \subset V + M \tag{2}$$

the system

$$x \in D_W, \qquad G(x) + A(x) \in M \tag{3}$$

has at least one solution.

The system (1) is said to be *stable* if every its solution is stable.

Definition 2. A solution $\overline{x}$ to system (1) is said to be *critical* if for some neighbourhood W of $\overline{x}$ we have

$$0 \in \partial\,(G(D_W) - M) \tag{4}$$

where ∂E denotes the set $E \setminus \text{int}\, E$.

The system (1) is said to be *critical* if

$$0 \in \partial(G(D) - M) \tag{5}$$

It turned out that for convex systems the two just defined notions are symmetric. Namely, the two following propositions hold:

THEOREM 1. *Assume that the system (1) is convex,* i.e. *D is a convex set and G an M-convex mapping. Then a solution $\overline{x}$ to this system is stable if and only if it is non-critical.*

We recall that a mapping $G: D \to Y$ is M-convex if for all $x^1, x^2 \in D$ and all t, $0 \leqslant t \leqslant 1$, we have

$$G(tx^1 + (1-t)x^2) \in tG(x^1) + (1-t)G(x^2) + M \tag{6}$$

THEOREM 2. *Under the same assumptions as in Theorem* 1, *the system* (1) *is stable if and only if it is non-critical.*

Since every solution to a critical system is necessarily critical, it follows from the previous theorems:

COROLLARY. *Under the above specified assumptions, if any one of the solutions to* (1) *is stable (critical), so are all others and the system is stable (critical,* resp.).

2. PROOF OF THE MAIN RESULT

We shall need the following

LEMMA 1. Let C be a subset of Y. Assume there are a k-simplex S in $Y_1 = R^k$, a neighbourhood V of the origin in each Y_i $(i = 1,2)$, and a set-valued mapping P from S into C such that:

1) $V_1 - V_1 \subset S$, $P(s) = P_1(s) \times P_2(s)$ with $P_i(s) \subset Y_i$, $P_1(s)$ being a non-empty convex set and the set-valued mapping P_1 from S into Y_1 being upper semi-continuous

2) For every $s \in S$ we have

$$P_1(s) \cap (s + V_1) \neq \phi \tag{6}$$

$$V_2 \subset P_2(s) . \tag{7}$$

Then C contains O in its interior.

Proof of Lemma 1. Consider an arbitrary element $v_1 \in V_1$ and define a set-valued mapping T from S into itself by

$$T(s) = S \cap (s + v_1 - P_1(s)).$$

Then $T(s)$ is non-empty for every s since by virtue of (6) there is an element $v_1' \in V_1$ such that $s + v_1' \in P_1(s)$ and, consequently, $u \in s + v_1 - P_1(s)$ with $u = v_1 - v_1' \in V_1 - V_1 \subset S$. Further, $T(s)$ is a convex set, since S and $P_1(s)$ are convex. Finally, it is easy to see that T is upper semi-continuous. Indeed, if $u^n \in T(s^n)$, $u^n \to u^o$, $s^n \to s^o$, then $s^n + v_1 - u^n \in P_1(s^n)$ and so, by the upper semi-continuity of P_1, $s^o + v_1 - u^o \in P_1(s^o)$, which means that $u^o \in T(s^o)$. Therefore, by Kakutani's fixed point theorem there exists an element $s \in S$ such that $s + v_1 - s \in P_1(s)$, i.e.

$$v_1 \in P_1(s).$$

On the other hand, from (7) we have for this s and for any $v_2 \in V_2$:

$$v_2 \in P_2(s).$$

This shows that $(v_1, v_2) \in P(s) \subset C$ and since v_1, v_2 are arbitrary elements of V_1, V_2 we have $V_1 \times V_2 \subset C$. Q.E.D.

Proof of Theorems 1 *and* 2. Clearly these theorems follow from the next two propositions:

I. Every stable solution of (1) is non-critical.

II. If system (1) is non-critical, it is stable.

To prove I, consider a stable solution $\overline{x}$ of (1) and let W be any neighbourhood of $\overline{x}$. Then there exists a neighbourhood V of $0 \in Y$ such that for every $v \in V$ the system

$$x \in D_W, \quad G(x) - y \in M \tag{8}$$

will have at least one solution. This implies $V \subset G(D_W) - M$, hence $\overline{x}$ is not critical.

Turning to the proof of II, suppose the system (1) non-critical and consider any solution $\overline{x}$ of (1) (if such exists). Then, evidently,

$$0 \in \operatorname{int} (G(D) \rightarrow M). \tag{9}$$

Let W be any convex neighbourhood of $\overline{x}$. In view of (9) one can find a k-simplex S in Y_1 such that S is a neighbourhood of $0 \in Y_1$ and $S \times \{0\} \subset \operatorname{int} (G(D) - M)$. If $s^1, \ldots, s^{k+1}$ denote the vertices of S, then for each i we have $(s^i, 0) \in \operatorname{int} (G(D) - M)$ and the set $\{s^i\} \times \operatorname{int} M_2$ must intersect $G(D) - M$. In other words there exists an $y^i \in \operatorname{int} M_2$ and an $x^i \in D$ such that $(s^i, y^i) \in G(x^i) - M$, i.e.

$$s^i \in G_1(x^i) - M_1, \tag{10}$$

$$y^i \in G_2(x^i) - M_2. \tag{11}$$

Noting that $G(\overline{x}) \in N$, we have from (10), (11), for $0 \leqslant t \leqslant 1$:

$$G_1(tx^i + (1-t)\overline{x}) \in tG_1(x^i) + (1-t)\, G_1(\overline{x}) + M_1 \in ts^i + M_1;$$

$$G_2(tx^i + (1-t)\overline{x}) \in tG_2(x^i) + (1-t)\, G_2(\overline{x}) + M_2 \in ty^i + M_2.$$

Thus, relations (10), (11) still hold when we replace s^i, y^i, x^i by $ts^i, ty^i, tx^i + (1-t)\overline{x}$ resp. Since $tx^i + (1-t)\overline{x} \in W$ for $t > 0$ sufficiently small, we can assume, by performing if necessary the just described replacement, that all $x^i \in W$. Let V_1 be a neighbourhood of $0 \in Y_1$ and V_2 a neighbourhood of $0 \in Y$ small enough to ensure $V_1 - V_1 \subset S$, $V_2 - V_2 + y^i \subset M_2$ for all $i = 1, \ldots, k+1$. From (11) we have that

$$G_2(x^i) + V_2 - V_2 \subset y^i + V_2 - V_2 + M_2 \subset M_2. \tag{12}$$

Let now $\widetilde{G} = G + A$, where $A: D \rightarrow Y$ is any continuous mapping satisfying condition (2) with $V = V_1 \times V_2$. For each $s \in S$ of the form $s = \sum_i t_i s^i$ with t_i the barycentric coordinates of s in S, let us set

$$P(s) = \sum_i t_i G(x^i) + A(\sum_i t_i x^i) - M \subset \widetilde{G}(\sum_i t_i x^i) - M.$$

Then, obviously, $P(s) = P_1(s) \times P_2(s)$ with $P_j(s) = \sum_i t_i G_j(x^i) + A_j(\sum_i t_i x^i) - M_j$ $(j = 1, 2)$. From (2), (10), (11), (12) we can write for each $s \in S$:

$$\Sigma t_i \; G_1(x^i) + A_1(\Sigma t_i x^i) \subset M_1 + s + V_1 ;$$

$$\Sigma t_i G_2(x^i) + A_2(\Sigma t_i x^i) - V_2 \subset \Sigma t_i (G_2(x^i) + V_2 - V_2) + M_2 \subset M_2.$$

Therefore:

$$P_1(s) \cap (s + V_1) \neq \varnothing \tag{13}$$

$$V_2 \subset P_2(s). \tag{14}$$

But $P_1(s)$ is clearly a non-empty convex set and P_1 is an upper semi-continuous mapping from S into Y_1. Indeed, let $u^n \in P_1(s^h)$, $u^h \to u^o$, $s^h \to s^o$. If we denote by $t_1^h, \ldots, t_{k+1}^h$ and $t_1^o, \ldots, t_{k+1}^o$ the barycentric coordinates of s^n and s^o resp, then

$$u^n - \sum_i t_i^n G_1(x^i) - A_1(\Sigma t_i^n x^i) \in - M_1 ,$$

hence

$$\overset{o}{u} - \sum_i \overset{o}{t_i} G_1(x^i) - A_1(\Sigma \overset{o}{t_i} x^i) \in - M_1 ,$$

i. e. $u^o \in P_1(s^o)$, proving the upper semi-continuity of P_1. In view of (13) and (14) all conditions specified in the Lemma are fulfilled for the set $C = \widetilde{G}(D_W) - M$. Consequently, by the Lemma, $0 \in \operatorname{int}(\widetilde{G}(D_W) - M)$, which implies that the system (3) is consistent. Q. E. D.

Remark. From the proof of proposition I above it is clear that if to every neighbourhood W of $\bar{x}$ there corresponds a neighbourhood V of $0 \in Y$ such that the system (8) is consistent for every $y \in V$, then $\bar{x}$ is a non-critical solution of (1) and hence, a stable solution. In other words, if in Definition 1 we consider only constant mappings $A(x) = y$, then the corresponding notion of stability is equivalent to the above notion, as far as we are concerned with convex systems.

3. IMPORTANT SPECIAL CASES

The previous result shows that for consistent convex systems, stability holds if and only if condition (9) holds. we now proceed to specialize the latter condition, in order to obtain more convenient stability criteria for the most usual cases.

THEOREM 3. *Assume that the system* (1) *is convex and consistent. Then it is stable if and only if the partial system.*

$$x \in D, \qquad G_1(x) \in M_1 \tag{15}$$

is stable and there exists an element x^o verifying

$$x^o \in D, \qquad G(x^o) \in M_1, \qquad G_2(x^o) \in \text{int } M_2. \tag{16}$$

Proof. Consider a stable system (1) (which is supposed convex and consistent). Then (9) holds and so $0 \in \text{int } (G_1(D) - M_1)$ and the sets $G(D) - M$ and $\{0\} \times \text{int } M_2$ must have a common element, say $(0, y_2)$. Since $(0, y_2) \in G(D) - M$, an $x^o \in D$ can be found such that

$$G_1(x^o) \in M_1, \qquad G_2(x^o) \in y_2 + M_2 \subset \text{int } M_2,$$

proving the first part of the Theorem. Note that the proof makes no use of convexity assumptions, so that this part of the Theorem is true for arbitrary systems, not necessarily convex.

To prove the converse part, assume that (15) is stable and (16) is consistent. Since (15) is consistent, we have $0 \in \text{int } (G_1(D) - M_1)$. It is not hard to see that 0 in Y is an internal point of the set $Q = G(D) - M$, i.e. an interior point in the finest locally convex topology of Y.

Indeed, consider an arbitrary element $y = (y_1, y_2) \in Y_1 \times Y_2$. Since $0 \in \text{int } (G_1(D) - M_1)$, we have $-ty_1 \in G_1(D) - M_1$ for some $t > 0$. Let $x \in D$ be an element satisfying $-ty_1 \in G_1(x) - M_1$ and let $x' = sx + (1-s)x^o$, $0 \leqslant s \leqslant 1$. Then $x' \in D$ and

$$G_1(x') \in sG_1(x) + (1-s)G_1(x^o) + M_1 \subset -sty_1 + M_1 \tag{17}$$

On the other hand, as s approaches 0 we have

$$sty_2 + sG_2(x) + (1-s)G_2(x^o) \to G_2(x^o) \in \text{int } M_2,$$

so that, provided $s > 0$ be small enough,

$$sty_2 + sG_2(x) + (1-s)G_2(x^o) \in M_2.$$

Noting that $G_2(x') \in sG_2(x) + (1-s)G_2(x^o) + M_2$, we then deduce

$$G_2(x') \in -sty_2 + M_2. \tag{18}$$

This together with (17) shows that for every y there exists an $\varepsilon = st > 0$ such that

$$-\varepsilon y \in G(D) - M.$$

Therefore, 0 is an internal point of Q, as asserted.

Let now V_1 be a neighbourhood of $0 \in Y_1$ such that

$$V_1 \times \{0\} \subset Q \tag{19}$$

(Y_1 being finite-dimensional, the topology of Y_1 is the finest locally convex one). Further, since $G_2(x^o) \in \text{int } M_2$, a neigbbourhood V_2 of $O \in Y_2$ can be found satisfying

$$G_2(x^o) - V_2 \subset M_2.$$

Remembering that $G_1(x^o) \in M_1$, we have

$$\{0\} \times V_2 \subset Q,$$

which together with (19) yields, in view of the convexity of Q:

$$\frac{1}{2}(V_1 \times V_2) = \frac{1}{2}(V_1 \times \{0\}) + \frac{1}{2}(\{0\} \times V_2) \subset Q.$$

That is, O is an interior point of $G(D) - M$ in the given topology of Y, and hence, the system (1) is stable. Q.E.D.

Remark. From the above proof it is seen that Theorem 3 holds even if Y_1 is infinite-dimensional, provided its topology be the finest locally convex one.

Let us now apply the preceding Theorem to some important special cases.

I. *Case where* $M_1 = \{0\}$.

If the cone M_1 is the singleton $\{0\}$, system (1) becomes

$$x \in D, \qquad G_1(x) = 0, \qquad G_2(x) \in M_2 \tag{20}$$

we have for this system the following simple stability criterion.

THEOREM 4. *Assume that the system* (20) *is convex*, i. e. *D is a convex set,* G_1 *an affine mapping and* G_2 *an* M_2 *- convex mapping. Assume furthermore that this system is consistent. Then it is stable if and only if* G_1 *maps D onto a neighbourhood of* $0 \in Y_1$ *and there exists an* x^0 *satisfying*

$$x^0 \in D, \qquad G_1(x^0) = 0, \qquad G_2(x^0) \in int\, M_2 \tag{21}$$

Proof. This proposition follows from the previous results and the fact that the partial system (which is consistent)

$$x \in D, \qquad G_1(x) = 0$$

is stable if and only if $0 \in \text{int}\, G_1(D)$.

II. *Finite-dimensional case.*

If $D = X = R^n$, $M_1 = \{0\}$, $Y_2 = R^n$, $M_2 = R^m$ (the non-positive orthant), we have the ordinary system of equalities and inequalities:

$$x \in R^n, \qquad G_1(x) = 0, \qquad G_2(x) \leqslant 0 \tag{22}$$

where $G_1: R^n \to R^k$ is an affine mapping, $G_2: R^n \to R^m$ is a convex (in the usual sense) mapping.

As applied to this case, the notion of stability introduced above may be defined in the following way.

A system (22) is called *stable* if for every its solution x and for every positive number ε there exists a positive number δ such that, whenever the mappings A_1, A_2 are continuous and satisfy the conditions

$$\| x - \bar{x} \| \leqslant \varepsilon \Rightarrow \| A_1(x) \| \leqslant \delta,\ A_2(x) \leqslant \underline{\delta}$$

($\underline{\delta}$ denotes a vector whose every component is equal to δ; $\| . \|$ is the usual euclidean norm), then the system

$$\| x - \bar{x} \| \leqslant \varepsilon, \qquad G_1(x) + A_1(x) = 0, \qquad G_2(x) + A_2(x) \leqslant 0$$

has at least one solution.

Noting that a linear mapping from R^n to R^k is surjective if and only if the corresponding matrix has rank k, we get from Theorem 4:

COROLLARY. *The system* (22) *is stable if and only if the matrix associated with the linear mapping $G_1(x) - G_1(0)$ has rank k and there exists an x^0 satisfying*

$$x^0 \in R^n, \qquad G_1(x^0) = 0, \qquad G_2(x^0) < 0. \tag{23}$$

For linear systems (G_2 affine) results very near to these have been obtained earlier by S. M. Robinson [5], as we have said in the introduction. The reader who is familiar with the theory of mathematical programming could notice the similarity between the stability condition described here and the well known regularity condition in convex programming. As we shall show in the next section, it turned out that the relationship between the two notions is far deeper than it seems.

4. APPLICATIONS TO CONVEX PROGRAMMING

In this section we shall relate the present notion of stability with the notion of regularity in convex programming and also with the notion of stability developed for convex minimization problems in some previously published works.

Definition 3. We shall say that system (1) is *regular* if there exists a subspace Y' of Y containing $G(D)$ such that:

1) The system

$$x \in D, \qquad G(x) \in M'$$

is stable, where $M' = M \cap Y'$ and G is to be regarded as a mapping from D into Y';

2) Every continuous linear functional on Y' which is non-negative on M' can be extended to a continuous linear functional on Y, non-negative on M.

Observe that condition 2) amounts to

$$M'^* = M^* + Y'^* \tag{24}$$

(for every cone E in Y we denote by B^* the cone formed by all continuous linear functionals on Y which take on only non-negative values on E), it is necessarily fulfilled in each of the following situations:

a) $M \subset Y'$, so that $M = M'$ (Hahn-Banach theorem);

b) (int M) $\cap Y' \neq \phi$ (Krein's theorem);

c) M is a polyhedral cone, i.e. is defined by an inequality of the form $Cy \leqslant 0$, where C is a linear mapping from Y into a finite-dimensional space (Farkas-Minkowski theorem).

On the other hand, from Theorem 3 we can easily deduce that system (1) will be regular, if the partial system (15) is regular and there is an x^0 satisfying (16). Thus, the just defined regularity condition contains as special cases most of the regularity conditions introduced in convex programming literature.

Let now $F: D \to R^1$ be a given function and consider the mathematical programming problem:

$$F(x) \to \inf, \text{ subject to constraints (1)} \tag{P}$$

Assume that the problem is convex, i.e. the system of constraints (1) and the objective function F are convex. Then we have the next proposition which extends a classical result.

THEOREM 5. *If the system of constraints* (1) *is regular, there exist for every optimal solution* $\bar{x}$ *of problem (P) a continuous linear functional* $L \in M^*$ *such that*

$$L\,(G(\bar{x})) = 0 \tag{25}$$

$$F(\bar{x}) - L(G(\bar{x})) = \min \{F(x) - L(G(x)) : x \in D\} \tag{26}$$

Proof. Consider first the case where the space Y' mentioned in Definition 3 coincides with Y. Let ε be an arbitrary positive number and

$$\bar{F}(x) = F(x) - F(\bar{x}),\ \bar{F}_\varepsilon(x) = F(x) - F(\bar{x}) - \varepsilon$$

Since system (1) is stable, it follows from Theorem 3 that the partial system (15) is stable and there exists an x^0 satisfying (16). Then for $t > 0$ small enough the element $\hat{x} = tx^0 + (1-t)\,\bar{x}$ satisfies

$$\hat{x} \in D, \quad G_1(\hat{x}) \in M_1, \quad G_2(\hat{x}) \in \operatorname{int} M_2, \quad \bar{F}_\varepsilon(\hat{x}) < 0$$

so that by the same theorem the system

$$x \in D, \quad G(x) \in M, \quad \bar{F}_\varepsilon(x) \leqslant 0 \tag{27}$$

is stable. By Theorem 2 the origin in $Y \times R^1$ is an interior point of the set $(G \times \bar{F}_\varepsilon)(D) - M \times R^1$, where $G \times \bar{F}_\varepsilon$ denotes the mapping $x \mapsto (G(x), \bar{F}_\varepsilon(x))$. Hence, $(0, \varepsilon) \in Y \times R^1$ is an interior point of the set $E = (G \times \bar{F})(D) - M \times R^1_-$. But $\bar{x}$ being optimal for (P), the origin in $Y \times R^1$ cannot be an interior point of E. Therefore, by the separation theorem, we can find a continuous functional L on Y and a number u, not both zero, such that for all $y \in Y$, $t \in R^1$ we have

$$(y, t) \in E \Rightarrow L(y) - ut \leqslant 0. \tag{28}$$

This implies, by a standard argument, $u \geqslant 0$, $L \in M^*$ and

$$(\forall x \in D)\ L(G(x)) - u(F(x) - F(\bar{x})) \leqslant 0. \tag{29}$$

If $u = 0$, then from (28) we have $L(y) \leqslant 0$ for all $y \in G(D) - M$. But the system (1) being stable and consistent, the set $G(D) - M$ contains $0 \in Y$ in its interior, and so $L = 0$ identically, contradicting the fact that L and u are not both zero. Consequently, $u > 0$ and we may assume $u = 1$, so that (29) becomes

$$(\forall\ x \in D)\ L(G(x)) - F(x) \leqslant -F(\overline{x})$$

which yields relation (26) if we note that $L(G(\overline{x})) \geqslant 0$ because $G(\overline{x}) \in M$. Since we have from (29) $L(G(\overline{x})) \leqslant 0$, relation (25) follows.

So Theorem 5 holds if $Y' = Y$. In the general case, the above argument shows that a continuous linear functional L on Y' can be found such that $L \in M'^*$ and properties (25), (26) hold. Then, by using condition 2) in Definition 3, one can extend L over the whole Y. Q.E.D.

Thus, the regularity of the system of constraints (1) ensures the existence of Kuhn-Tucker multipliers for any optimal solution of *any* convex problem (P) with constraints (1). The above results shed a new light on the nature of the regularity condition in convex programming: indeed, regularity means, essentially, *stability of the system of constraints*, which is a very natural condition to impose upon the data of a practical problem, at least at the first stage of investigation.

Let us now say a few words about the relationship between the stability of a problem (P), as has been considered by some authors ([6], [1]) and the stability defined here for the system of constraints (1).

For every $u \in Y$ let (P_u) denote the problem that differs from (P) only in that $G(x)$ is replaced by $G(x)-u$, and let $f(u)$ be the optimal value of $F(x)$ for problem (P_u). Then $f(u)$ can take infinite values, but if we assume, as previously, the convexity of (P), it can be easily shown that $f(u)$ is a convex function.

Definition 4. We shall say that problem (P) is *stable* if $f(0)$ is finite and $f(u)$ is continuous at $u = 0$.

THEOREM 6. *A problem (P) with finite $f(0)$ is stable if and only if the system of constraints* (1) *is stable.*

Proof. Suppose the system (1) stable. Since $f(0)$ is finite, for every $\varepsilon > 0$ there is a solution $\overline{x}$ of (P) such that $F(\overline{x}) < f(0) + \frac{1}{2}\varepsilon$. Then, as has been showne in the proof of Theorem 5, the system

$$x \in D, \quad G(x) \in M, \quad \overline{F}_{\varepsilon/2}(x) \leqslant 0 \tag{30}$$

is stable. Hence there is a neighbourhood V of $0 \in Y$ such that for every $u \in V$ the system

$$x \in D, \quad G(x) - u \in M, \quad \overline{F}_{\varepsilon/2}(x) \leqslant 0$$

is consistent. This means $f(u) \leqslant f(0) + \varepsilon$. But by virtue of a well known property of convex functions, the latter inequality implies the continuity of f on the set V.

Conversely, suppose the problem (P) stable. Then there is a neighbourhood V of $0 \in Y$ such that $f(0) - 1 \leqslant f(u) \leqslant f(0) + 1$ for all $u \in V$. Since $f(0)$ is finite, so is $f(u)$ for $u \in V$. But this means in particular that for every $u \in V$ the system

$$x \in D, \quad G(x) - u \in M$$

is consistent, and hence, that the system (1) is stable (see remark at the end of section 2). Q. E. D.

COROLLARY. If a problem (P) is stable, then for every $\varepsilon > 0$ there exists a neighbourhood V of $0 \in Y$ such that for any continuous mapping $A(x)$ satisfying $A(D) \subset V + M$ the optimal value of the perturbed problem

$$F(x) \to \inf: \quad x \in D, \quad G(x) + A(x) \in M$$

differs from the initial optimal value by no more than ε.

Indeed, using the continuity of $f(u)$ and the stability of (30), we can take V to be such that $|f(u) - f(0)| \leqslant \varepsilon$ for all $u \in V$ and such that the system

$$x \in D, \quad G(x) + A(x) \in M, \quad \overline{F}_{\varepsilon/2}(x) \leqslant 0$$

is consistent for all continuous mappings A satisfying $A(D) \subset V + M$.

5. EXTENSION TO NON-CONVEX SYSTEMS

So far we have assumed the system (1) to be convex. In the general case the procedure of convex approximation can be used to reduce the stability problem to the convex case, as we now show.

Let us consider an arbitrary system (1) and let $\overline{x}$ be a solution of it.

Definition 5. A convexs ubset D' of X is said to be *k-contingent* to D at $\overline{x}$ if for every triple (Σ, U, δ), where Σ is the convex hull of any $k + 1$ points of D', U is a neighbourhood of $0 \in X$, δ a positive number, there exist a continuous mapping $v: \Sigma \to D$ and a number $\varepsilon \in (0, \delta)$ such that

$$(\forall\ x \in \Sigma) \qquad v(x) \in \overline{x} + \varepsilon(x + U).$$

Definition 6. An M-convex mapping $G': D' \to Y$ (where D' is a convex subset of X) is said to be an *M-differential* of $G: D \to Y$ at $\overline{x}$ if for every pair (x, V), where $x \in D'$, V is a neighbourhood of $0 \in Y$, there exist a neighbourhood U of $0 \in X$ and a number $\delta > 0$ such that

$$\frac{G(x + \varepsilon z) - G(\overline{x})}{\varepsilon} - G'(x) \in V + M \tag{31}$$

whenever

$$z \in x + U, \ \overline{x} + \varepsilon z \in D, \ 0 < \varepsilon < \delta. \tag{32}$$

LEMMA 2. Let $G': D' \to Y$ be an M-differential of $G: D \to Y$ at $\overline{x}$, Σ a compact subset of D'. If G' is continuous then for every neighbourhood V of $0 \in Y$ there exist a neighbourhood U of $0 \in X$ and a number $\delta > 0$ such that for every $x \in \Sigma$ we have (31) whenever (32) holds.

Proof. Denote by V' a neighbourhood of $0 \in Y$ such that $V' + V' \subset V$. For every $x \in \Sigma$ one can choose a neighbourhood U_x and a number $\delta_x > 0$

such that $G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon(G'(x) + V' + M)$ whenever $z \in x + U_x$, $\bar{x} + \varepsilon z \in D$, $0 < \varepsilon < \delta_x$. Since G' is continuous, one can find a neighbourhood U'_x of $0 \in X$ such that $U'_x + U'_x \subset U_x$ and $G'(q) - G'(x) \in V'$ whenever $x - q \in U'_x$. Let $\{q + U'_q : q \in Q\}$ be a finite family covering the compact set Σ, and let $U = \bigcap_{q \in Q} U'_q$, $\delta = \min_{q \in Q} \delta_q$. Then for every $x \in \Sigma$ there is $q \in Q$ such that $x \in q + U'_q$ and hence, $G'(q) - G'(x) \in V'$. So if $z \in x + U$, $\bar{x} + \varepsilon z \in D$, $0 < \varepsilon < \delta$, then $z \in q + U'_q + U \subset q + U_q$, $0 < \varepsilon < \delta_q$ and we have $G(\bar{x}) + \varepsilon z) - G(\bar{x}) \in \varepsilon(G'(q) + V' + M) \in \varepsilon(G'(x) + V' + V' + M) \in \varepsilon(G'(x) + V + M)$.

THEOREM 7. *Assume that G_1 is continuous and there exists a consistent convex system*

$$x \in D', \qquad G'(x) \in M, \tag{33}$$

such that D' is k-contingent to D at $\bar{x}$ and G' is continuous and is an M-differential of G at $\bar{x}$. If the system (33) *is stable, then $\bar{x}$ is a stable solution of system* (1).

Proof. Let W be any neighbourhood of $\bar{x}$. We shall show that

$$0 \in \operatorname{int}(\tilde{G}(D_W) - M) \tag{34}$$

for every mapping $\tilde{G} = G + A$, where $A : D \to Y$ is a continuous, sufficiently « small » perturbation.

From the consistency and stability of system (33) we have by Theorem 2: $0 \in \operatorname{int}(G'(D') - M)$, and so there exists a k-simplex S' in Y_1 such that S' is a neighbourhood of $0 \in Y_1$ and $S' \times \{0\} \subset \operatorname{int}(G'(D') - M)$. By a simple argument (similar to that used in the proof of Theorems 1 and 2), we can find for each vertex s^i of S' two elements $x^i \in D'$, $y^i \in \operatorname{int} M_2$ such that

$$s^i \in G'_1(x^i) - M_1, \qquad y^i \in G'_2(x^i) - M_2, \tag{35}$$

$(i = 1,\dots, k+1)$. Then $G'_2(x^i) \in y^i + M_2 \subset \operatorname{int} M_2$ and we can select a convex neighbourhood V'_2 of $0 \in Y_2$ such that

$$G'_2(x^i) + 3V'_2 \subset M_2 \ (i = 1,\dots, k+1) \tag{36}$$

Denote by Σ the convex hull of $x^1,\dots,x^{k+1}$. From (36) it follows that

$$(\forall\, x \in \Sigma) \qquad G'_2(x) + 3V'_2 \subset M_2 \tag{37}$$

Let V'_1 be a convex neighbourhood of $0 \in Y_1$ such that $2(V'_1 - V'_1) \subset S'$ and let $V' = V'_1 \times V'_2$. Since Σ is compact and contained in D', there corresponds to V', according to the previous Lemma, a neighbourhood U of $0 \in X$ and a number $\delta > 0$ such that for every $x \in \Sigma$ we have

$$G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon(G'(x) + V' + M) \tag{38}$$

whenever

$$z \in x + U, \quad \bar{x} + \varepsilon z \in D, \quad 0 < \varepsilon < \delta. \tag{39}$$

On the other hand, the set D' being k-contingent to D at $\bar{x}$, we can find a continuous mapping $v: \Sigma \rightarrow D$ and a number $\varepsilon \in (0, \delta)$ such that

$$(\forall\, x \in \Sigma) \quad v(x) \in \bar{x} + \varepsilon(x + U). \tag{40}$$

It can of course be arranged that $v(x) \in D_W$, because we can always take v to be the mapping that corresponds in Definition 5 to (Σ, U', δ') where $U' = U \cap \frac{1}{2}(W - \bar{x})$ and δ' is such that $\delta' \Sigma \subset \frac{1}{2}(W - \bar{x})$, W being assumed to be convex. Thus $v: \Sigma \rightarrow D_W$ and we have from (40) and (38)

$$(\forall\, x \in \Sigma) \quad G(v(x)) \in \varepsilon(G'(x) + V') + M. \tag{41}$$

Now let $\widetilde{G} = G + A$, where $A: D \rightarrow Y$ is any continuous mapping satisfying

$$A(D_W) \subset \varepsilon V' + M \tag{42}$$

Then we can write, according to (41), (42), (37):

$$\widetilde{G}_1(v(x)) \in \varepsilon(G_1'(x) + 2V_1') + M_1$$

$$\widetilde{G}_2(v(x)) + \varepsilon V_2' \subset \varepsilon(G_2'(x) + 3V_2' + M_2) \subset M_2$$

So if we put $S = \varepsilon S'$, $V_1 = 2\varepsilon V_1'$, $V_2 = \varepsilon V_2'$ and define the mapping

$$s = \varepsilon \sum_i t_i\, s^i \mapsto (s) = P(s) = \widetilde{G}\,(v(\sum_i t_i\, x^i)) - M,$$

then we have (see (35)):

$$\widetilde{G}_1(v(\sum t_i\, x^i)) \in \varepsilon\, G_1'\,(\sum t_i x^i)\; V_1 + M_1 \subset \varepsilon(\sum t_i\, G_1'\,(x^i)) + V_1 + M_1$$

$$= \varepsilon \sum_i t_i\, s^i + V_1 + M_1 = (s + V_1) + M_1,$$

$$\widetilde{G}_2(v\,(\sum t_i\, x^i)) - V_2 \subset M_2\,.$$

Therefore, relations (6) and (7) hold for every $s \in S$. Since $V_1 - V_1 \subset S$ and P_1 is obviously upper semi-continuous (because $\widetilde{G}_1$ and v are continuous), it follows from Lemma 1 that (34) must hold. Q.E.D.

As an application, let us again consider the system (22), where the mappings G_1, G_2 are Fréchet differentiable at $\bar{x}$, with derivatives G_1', G_2'. Then R^n is obviously k-contingent to itself at $\bar{x}$ and $G' = G_1' \times G_2'$ is an M-differential of $G = G_1 \times G_2$ for every convex cone M in Y, so that:

COROLLARY 1. *If G_1' has rank k and if there exists an z^0 verifying*

$$z^0 \in R^n, \quad G_1'(z^0) = 0, \quad G_2'(z^0) < 0$$

then $\bar{x}$ is a stable solution of (22).

Finally, let us notice that a necessary condition for a system (1) to be stable is that it is non-critical (the proof of this fact given in section 2 for convex systems is valid for the general case). Hence we get from Theorem 7:

COROLLARY 2. *Under the same assumptions as in Theorem 7, if the system (1) is critical, then so is the approximate system (33) and hence there exists a non-zero continuous linear functional $L \in M^*$ such that*

$$(\forall\, x \in D') \qquad L(G'(x)) \leqslant 0.$$

As was shown in [3], this proposition could be given a basic role in the theory of extremal problems.

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