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ON A METHOD OF APPROXIMATE SOLUTION OF OPERATIONAL EQUATIONS AND ITS APPLICATION TO INTEGRAL EQUATIONS

PHAN VĂN HẠP
University of Hanoi

Consider the operational equation of the form

$$A\,x = 0 \tag{1}$$

where A — operator, which operates in the Banach space. Further, let X be the commutative ring with the ensemble of the units Y.

§ 1. THE ITERATED METHOD

Equation (1) will be solved by the iterated method as follows :

$$x_{\hat{1}} = x_0 + \mu\, A\, x_0 \,,$$

$$x_1 = x_0 - A^{-1}_{0\hat{1}}\, A\, x_0 \,,$$

$$x_{\hat{j}} = x_{j-1} + \mu\, \varepsilon_{j-1} \,,$$

$$x_j = x_0 - A^{-1}_{01}\Big[\varepsilon_0 + A_{012}\,(x_{\hat{j}} - x_0)\,(x_{\hat{j}} - x_1) + \ldots +$$

$$+ A_{\hat{j}01\ldots j-1}\,(x_{\hat{j}} - x_0)\,(x_{\hat{j}} - x_1)\ldots(x_{\hat{j}} - x_{j-1})\Big], \tag{1.1}$$

where $\varepsilon_j = A\,x_j$, μ is a positive constant, $A_{01\ldots j} = A\,(x_0\,;\, x_1\,;\ldots;\, x_j)$ — the divide differences of j-th order of the operator A.

Further we construct an infinite sequence of successive approximations defined by

$$x_k^{m+1} = x_o - A_{o1}^{-1}\Big[\varepsilon_o + A_{o12}(x_k^m - x_o)(x_k^m - x_1) + \dots +$$
$$+ A_{mo1\dots k-1}(x_k^m - x_o)(x_k^m - x_1)\dots(x_k^m - x_{k-1})\Big] \tag{1.2}$$

In the case $k = 1$ we have

$$x_1^{m+1} = x_o - A_{mo}^{-1} A x_o = x_1^m - A_{mo}^{-1} A x_1^m ; \tag{1.3}$$

this is the method of generalized regular falsi [1,2].

§ 2. THE THEOREMS OF CONVERGENCE

Suppose that the equation (1) has a unique solution x^*. Denote by δ_j the norm of $x^* - x_j$:

$$\| x^* - x_j \| = \delta_j , \quad (j = 0, 1, \dots)$$

and

$$\delta = \max_i \delta_i \qquad (i = 1, 2, \dots, k-1).$$

We obtain the following

THEOREM 2.1. *If the following conditions are satisfied*

(1) $\| A_{o1}^{-1} \| \leqslant C_o ,$

(2) $\| A_{o1} \| \leqslant C_1 , \dots , \| A_{o1\dots j} \| \leqslant \frac{1}{j!} C_j \quad (j = 1, 2, \dots, k)$

for all x_i *, such that*

$$\| x^* - x_i \| \leqslant \delta$$

and

(3) $q = C_o \alpha \sum_{j=1}^{k-1} \frac{C_{j+1}}{j!} \alpha^{j-1} < 1$ *where* $\alpha = (2 + \mu C_1)\delta$

Then $\{ x_k^m \}$ *converges in norm to the solution of the equation (1) and we have*

$$\| x^* - x_k^m \| \leqslant \frac{\gamma.\delta.q^m}{(1+\gamma)^m} \exp\left[\gamma . \frac{q(1-\beta^m)}{1-\beta}\right] \leqslant$$
$$\leqslant \gamma.\delta.q^m \exp\left\{-\frac{\gamma}{1+\gamma}\left(m - q\,\frac{1-\beta^m}{1-\beta}\right)\right\} \tag{2.1}$$

where $\gamma = 1 + \mu C_1$, $\beta = \dfrac{1}{1 + (1-q)\gamma}$.

Proof. In view of the conditions (1), (2), (3) of Theorem 2.1. and by formula (1.2) , we obtain

$$\delta_{k+1} = \| x^* - x_k^1 \| \leqslant C_o \, \delta_k \sum_{j=1}^{k-1} \frac{C_{j+1}}{j\,!} \, (\delta + \delta_k)^j$$

In fact, we have

$$\| A_{o1...j} (x^* - x_o) ... (x^* - x_{j-1}) - A_{o1...j} (x_k - x_o) ... (x_k - x_{j-1}) \| \leqslant$$

$$\leqslant \frac{C_j}{j\,!} \cdot j \cdot (\max_i \| \zeta - x_i \|)^{j-1}, \; (i = o , 1 , ... , j-1)$$

$$\zeta = x_k + \theta (x^* - x_k) , \; 0 \leqslant \theta \leqslant 1$$

and by the triangle inequality

$$\| \zeta - x_i \| \leqslant \| \zeta - x^* \| + \| x^* - x_i \| \leqslant \| x_k - x^* \| + \| x^* - x_i \| \leqslant \delta_k + \delta$$

Further, from

$$\delta_k = \| x_k - x \| \leqslant \| x - x_{k-1} \| + \mu \| \varepsilon_{k-1} \| \leqslant \delta + \mu \| \varepsilon_{k-1} \| ,$$

$$\| \varepsilon_{k-1} \| = \| A x_{k-1} \| = \| A x^* - A x_{k-1} \| \leqslant C_1 \, \delta_{k-1} \leqslant C_1 \, \delta,$$

we obtain

$$\delta_k \leqslant (1 + \mu C_1) \, \delta = \gamma \delta \tag{2.2}$$

and

$$\delta_{k+1} = \| x^* - x_k^1 \| \leqslant q \delta_k \leqslant q \gamma \delta \tag{2.3}$$

Analogously, we obtain

$$\delta_{k+m+1} = \| x^* - x_k^{m+1} \| \leqslant \mathscr{E}_{k+m} \cdot \delta_{k+m} , \tag{2.4}$$

where

$$\mathscr{E}_{k+m} = C_o \delta_{k+m} \sum_{j=1}^{k-1} \frac{C_{j+1}}{j\,!} \, (\delta + \delta_{k+m})^j ,$$

Using the inequalities (2.2) — (2.4) we obtain

$$\mathscr{E}_k = q , \quad \delta_{k+1} \leqslant \mathscr{E}_k \cdot \delta_k , \quad \delta + \delta_{k+1} \leqslant (1 + \gamma \mathscr{E}_k) \, \delta ,$$

$$\frac{\mathscr{E}_{k+1}}{\mathscr{E}_k} < \frac{1 + \gamma \mathscr{E}_k}{1 + \gamma} = 1 - (1 - \mathscr{E}_k) \frac{\gamma}{1 + \gamma} ,$$

which implies

$$\delta_{k+m+1} = \| x^* - x_k^{m+1} \| \leqslant \left(\prod_{j=0}^{m} \mathcal{E}_{k+j} \right) \delta_k \leqslant \left(\prod_{j=0}^{m} \mathcal{E}_{k+j} \right) \Upsilon \delta, \tag{2.5}$$

where

$$\prod_{j=0}^{m} \mathcal{E}_{k+j} = \mathcal{E}_k^{m+1} \prod_{j=0}^{m} \left(1 - \lambda_{k+j} \right) = q^{m+1} \prod_{j=0}^{m} \left(1 - \lambda_{k+j} \right)$$

and

$$\lambda_{k+j} = \frac{(1 - \mathcal{E}_{k+j}) \Upsilon}{1 + \Upsilon} .$$

Let.

$$U_m = \prod_{j=0}^{m} \left(1 - \lambda_{k+j} \right)$$

we have

$$\ln U_m < - \sum_{j=0}^{m} \lambda_{k+j} ,$$

hence

$$U_m < \exp \left(- \sum_{j=0}^{m} \lambda_{k+j} \right)$$

Further, we obtain

$$\sum_{j=0}^{m} \lambda_{k+j} = \frac{\Upsilon}{1+\Upsilon} \left(m + 1 - \sum_{j=0}^{m} \mathcal{E}_{k+j} \right)$$

and

$$\mathcal{E}_{k+1} < \mathcal{E}_k (1 - \lambda_{k+1}) = q \left(\mathcal{E}_{k+1} + \frac{1 - \mathcal{E}_{k+1}}{1 + \Upsilon} \right).$$

It follows that

$$\mathcal{E}_{k+1} < \frac{q}{1 + \Upsilon (1 - q)} = q \cdot \beta = q_1 ,$$

where

$$\beta = \frac{1}{1 + \Upsilon (1 - q)} .$$

Analogously, we have

$$\mathcal{E}_{k+1+j} < \frac{q_j}{1 + \Upsilon (1 - q_j)} = q_j \cdot \beta_j = q_{j+1} ,$$

where

$$\beta_j = \frac{1}{1 + \Upsilon (1 - q_j)} .$$

It is not difficult to show that

$$\beta_j < \beta_{j-1} < \dots < \beta ,$$

Hence

$$\sum_{j=o}^{m} \delta_{k+j} < q + q_1 + \dots + q_m < q \; \frac{1-\beta^m}{1-\beta}$$

Consequently

$$C_m \leqslant \exp \left[- \frac{\gamma}{1+\gamma} \left(m + 1 - q \; \frac{1-\beta^{m+1}}{1-\beta} \right) \right]$$

Putting this expression in (2.5) we obtain inequality (2.1). Finally, note that the member x_k^m of the sequence (1.2), under the assumptions of the theorem, belongs to the sphere $\| x^* - x \| \leqslant \delta$. This proves the theorem.

Remark. If the operator A is differentiable of k-th order then we can show that

$$\| A_{o1} \| \leqslant \| A'_{\zeta} \| , \quad \zeta = x_o + \theta \, (x_1 - x_o) , \quad 0 \leqslant \theta \leqslant 1 ;$$

$$\| A_{o12} \| \leqslant \left(\frac{1}{2} \right) \| A''_{\eta} \| , \qquad \eta = x_o + \theta_o \, (\eta_1 - x_o) ,$$

$$\eta_1 = x_1 + \theta_1 \, (x_2 - x_1) , \quad 0 \leqslant \theta_o , \theta_1 \leqslant 1 ;$$

. . . .

The following theorem shows the convergence of sequence (1.2) without any assumption on the existence of the solution of equation (1).

THEOREM 2.2. *Suppose that the following conditions are satisfied*

(1) $\| A_{o1}^{-1} \| \leqslant C_o$

(2) $\| A_{o1} \| \leqslant C_1 , \quad \| A_{o12} \| \leqslant \frac{1}{2} C_2$ *for all* x

such that

$$\| x - x_o \| < \frac{C \delta_o}{1 - q_1} + \eta$$

where

$$\delta_o = \mu \, C_o \, C_1 \, \| \varepsilon_o \| , \quad \eta = \gamma \, C_o \, \| \varepsilon_o \| ,$$

$$q_1 = C_o \, C_2 \, (\eta + \delta_o) < 1 .$$

Then

$$x_2^{m+1} = x_o - A_{o1}^{-1} \left[\varepsilon_o + A_{m12} \left(x_2^m - x_o \right) \left(x_2^m - x_1 \right) \right] \tag{2.6}$$

converges in norm to the solution x^* of the equation (1) which is in the domain

$$\| x^* - x_o \| \leqslant C \delta_o \frac{q_1}{1-q_1} + \eta , \tag{2.7}$$

where

$$C = \prod_{i=1}^{\infty} \left(1 + \frac{q_1^i}{1+\nu} \right) < \exp \left[\frac{q_1}{(1+\nu)\ (1-q_1)} \right]$$

and

$$\nu = \frac{\Upsilon}{\Upsilon - 1}$$

Proof. We have

$$\| x_1 - x_o \| \leqslant C_o \ \| \varepsilon_o \| \ , \ \| x_{\hat{1}} - x_1 \| \leqslant C_o C_1 \mu \| \varepsilon_o \| ,$$

$$\| x_{\hat{1}} - x_o \| \leqslant \mu \| \varepsilon_o \| , \ \| x_{\hat{2}} - x_o \| \leqslant C_o \ \| \varepsilon_o \| + \mu C_o C_1 \ \| \varepsilon_o \| \ ;$$

$$\delta_o = \| x_{\hat{2}} - x_1 \| \leqslant \mu \| \varepsilon_1 \| \leqslant \mu C_o C_1 \ \| \varepsilon_o \| . \tag{2.8}$$

Let

$$\eta = \max \ \| x_i - x_j \| \ , \quad i, j = o, 1, \hat{1}, \hat{2},$$

we have

$$\eta = \Upsilon C_o \ \| \varepsilon_o \| \ , \qquad \Upsilon = 1 + \mu C_1 . \tag{2.9}$$

In view of the (2.6) we obtain

$$\delta_1 = \| x_2^1 - x_{\hat{2}} \| \leqslant C_o C_2 (\eta + \delta_o) \delta_o = q_1 \cdot \delta_o .$$

Analogously,

$$\delta_2 = \| x_2^2 - x_2^1 \| \leqslant C_o C_2 (\eta + \delta_o + \delta_1) \delta_1 .$$

Let

$$\eta_o = \eta + \delta_o \ , \quad \eta_1 = \eta_o + \delta_1 , \ \dots , \ \eta_j = \eta_{j-1} + \delta_j$$

Using (2.8) and (2.9) we have

$$\delta_o = \frac{\eta_o}{1+\nu} , \ \nu = \frac{\Upsilon}{\Upsilon - 1} .$$

Consequently

$$\eta_1 \leqslant \eta_o \left(1 + \frac{q_1}{1+\nu} \right)$$

and

$$\delta_2 \leqslant q_1^2 \left(1 + \frac{q_1}{1+\nu}\right) \delta_0$$

Further, we get the following estimates

$$\eta_j \leqslant \prod_{i=0}^{j} \left(1 + \frac{q_1^i}{1+\nu}\right) \eta_0 ,$$

$$\delta_j \leqslant q_1^j \prod_{i=1}^{j-1} \left(1 + \frac{q_1^i}{1+\nu}\right) \delta_0 .$$

Furthermore we have

$$\prod_{i=1}^{j} \left(1 + \frac{q_1^i}{1+\nu}\right) < \prod_{i=1}^{\infty} \left(1 + \frac{q_1^i}{1+\nu}\right) = C < \infty$$

Hence

$$\delta_{j+1} < C\, q_1^{j+1}\, \delta_0 .$$

For any positive integers n and p the inequality

$$\| x_2^{n+p} - x_2^n \| \leqslant \sum_{j=n}^{n+p-1} \| x_2^{j+1} - x_2^j \| = \sum_{j=n}^{n+p-1} \delta_{j+1} <$$

$$< C\, \delta_0 \sum_{j=n}^{n+p-1} q_1^{j+1} < C\, \delta_0 \frac{q_1^{n+1}}{1-q_1}$$

holds.

Hence for an arbitrary $\varepsilon > 0$ we can find N_ε such that $\| x_2^{n+p} - x_2^n \| < \varepsilon$ for $n > N_\varepsilon$, and so the Cauchy condition is satisfied. According to the definition of a complete space X, there exists a point x^* of the space to which the sequence $\{ x_2^m \}$ converges.

We now prove the estimate (2.7). In fact, we have

$$\| x^* - x^0 \| \leqslant C\, \delta_0 \sum_{j=0}^{\infty} q_1^{j+1} + \| x_{\hat{2}} - x_0 \| \leqslant C\, \delta_0 \frac{q_1}{1-q_1} + \eta .$$

Because $\varepsilon_j = A x_j$ as $j \to \infty$, x^* is the solution of the equation (1). This proves the theorem.

§ 3. — APPLICATION TO INTEGRAL EQUATIONS

We observe that the above described method may be applied to the following functional integral equation of resolvent (see [3])

$$\Gamma(x, y, z) = K(x, y) + \int_0^z \Gamma(x, s, s)\ \Gamma(s, y, s)\ ds .$$

Suppose that

$$\Gamma_0 = K(x, y) \qquad \text{and} \qquad \| K(x, y) \|_2 < 1$$

We can prove that the conditions of Theorem 2.2 are then satisfied and hence the sequence (2.6) converges in norm to the solution of this equation.

The problem of solving the integral equations with (weakly and strongly) singular kernels can be reduced to finding solutions by the described iterated method [4].

This method may also be applied to the problem of solving a class of singular integral equations with displacement kernels [5].

REMARK. This method is very convenient in the case when the differentiation of the operator A is complicated or the operator A is not differentiable.

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