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SOME REMARKS ON MINIMAX THEOREMS

BÙI CÔNG CƯỜNG
Institute of Mathematics, Hanoi

The minimax theorem, which plays a fundamental role in two-person game theory, has been generalized by many authors (see, for example [5,6]). Among these generalizations the theorems of Wu Wen Tsun [7], Ky Fan [2], Sion [4], and Hoàng Tụy [1] in topological spaces are of particular interest.

It is the object of the present paper to extend the results of Wu Wen Tsun and Ky Fan. Although the theorem recently discovered by Hoàng Tụy [1] is very general, our results can not be deduced from it as special cases. Moreover, in the proofs we shall use a different method.

1. MINIMAX THEOREMS FOR α- STRONGLY CONNECTED FUNCTIONS

1.1. Let X and Y be two Hausdorff topological spaces. Let $C \subset X$, $D \subset Y$ and let f be a real-valued function defined on $C \times D$. For every real number α and for every $x \in C$, $y \in D$ let us define the sets:

$$D^-(x) = \{ y \in D : f(x,y) < \alpha \},$$

$$D(x) = \{ y \in D : f(x,y) \geqslant \alpha \}, \quad C(y) = \{ x \in C : f(x,y) \geqslant \alpha \}.$$

First, we can state the following

LEMMA 1. *Let $C = [a,b] \subset R$, $D = Y$. Assume that: (i) for every $x \in C$, $y \in D$ the sets $D^-(x)$, $C(y)$ are connected or empty; (ii) f is upper semicontinuous separately in x and y; (iii) for every $y \in D$ the set $C(y)$ contains at least a or b. Then there exists $x^* \in C$ such that $D^-(x^*) = \varnothing$.*

Proof. Using conditions (i) — (iii) we first note that $C(y)$ is a nonempty closed interval of $[a, b]$ for every $y \in D$. Since C is compact and every $C(y)$ is closed, it suffices to show that for every finite set $y_1, y_2, \dots, y_k \in D$ we have $\bigcap_{i=1}^{k} C(y_i) \neq \varnothing$.

Indeed, assume that there exist $y_1, y_2, \ldots, y_k$ of D such that $\bigcap_{i=1}^{n} C(y_i) = \varnothing$. Let $C(y_p)$ be the smallest interval $C(y_i)$ containing a and $C(y_q)$ the smallest interval $C(y_i)$ containing b ($p, q \in \{1, 2, \ldots, n\}$). Then $C(y_p) \cap C(y_q) = \varnothing$ *) and $A = [a, b] \setminus (C(y_p) \cup C(y_q)) \neq \varnothing$. Now take an $x' \in A$ and define $J_p = D^-(x') \cap D(a)$, $J_q = D^-(x') \cap D(b)$. Obviously, $y_p \in J_p$, $y_q \in J_q$. Since f is upper semicontinuous separately in x and y, the sets $D(a)$, $D(b)$ are closed, $D^-(x')$ is open and J_p, J_q are nonempty closed subsets of $D^-(x')$. Since $D^-(x') \subset D(a) \cup D(b)$, we have $D^-(x') = J_p \cup J_q$, moreover the connectedness of $D^-(x')$ implies $J_p \cap J_q \neq \varnothing$. If $y' \in J_p \cap J_q$, then from the fact $a \in C(y')$, $b \in C(y')$ and the connectedness of $C(y')$ it follows that $[a, b] \subset C(y')$. Thus $x' \in C(y')$, which conflicts with $y' \in D^-(x')$ and so completes the proof.

As an immediate corollary of Lemma 1 we obtain :

THEOREM 1. *Under conditions stated in Lemma 1, if* $\alpha = \inf_{y \in D} \sup_{x \in C} f(x, y)$, *then*

$$v_1 = \inf_{y \in D} \sup_{x \in C} f(x, y) = \sup_{x \in C} \inf_{y \in D} f(x, y) = v_2 \tag{1.1}$$

1.2. A real-valued function f defined on $C \times D$ is said to be *α-strongly connected* if :

1) For any pair $c_1, c_2 \in C$ there exists a continuous mapping $u : [a, b] \to C$ such that $u(a) = c_1$, $u(b) = c_2$ and the set $u^{-1}(C(y))$ is connected or empty for every $y \in D$;

2) For any finite subset $\{c_1, c_2, \ldots, c_k\}$ of C one of the following conditions holds : 2a) the set $\bigcap_{i=1}^{k} D^-(c_i)$ is connected or empty ; 2b) the set $D^-(c_1) \cap (\bigcup_{i=1}^{l} D(c_i))$ is connected or empty for each $l = 2, 3, \ldots, k$.

A generalization of Wu Wen Tsun's result [7] is the following

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*) Note that Helly's Theorem is not necessary for this proof.

THEOREM 2. *Assume that: a) D is compact; b) f is upper semicontinuous in x and continuous in y; c) there exists an increasing sequence of numbers α_j converging to v_1 such that f is α-strongly connected for each $\alpha = \alpha_j$. Then (1,1) holds.*

Proof. Note that v_1 may be finite or infinite. The inequality $v_1 \geqslant v_2$ being evident, let us suppose $v_1 > v_2$ and show that this leads to a contradiction. In fact, using assumption c) and the strict inequality $v_1 > v_2$ we can find a number α such that f is α-strongly connected and $v_1 > \alpha > v_2$.

Since $\inf_{y \in D} \sup_{x \in C} f(x,y) > \alpha$, it follows that for every $y \in D$ there is $x_y \in C$ such that $f(x_y, y) > \alpha$. Using the lower semicontinuity of f in y we can find a neighborhood U_y of y such that $f(x_y, y') > \alpha$ for every $y' \in U_y$. Since D is compact, the open cover $\{ U_y \}$ of D contains a finite subcover; in other words, there exists a finite system $y_1, y_2, \ldots, y_m \in D$ such that the open sets $U_j = U_{y_j}$ $(j = 1,2,\ldots,m)$ form a cover of D. Let $x_j = x_{y_j}$ $(j=1,\ldots,m)$. Since $U_j \subset D(x_j)$ $(j = 1,\ldots,m)$ we have $D = \bigcup_{j=1}^{m} D(x_j)$. Now using this last relation we can find by the following Lemma 2 a point $x_\alpha^* \in C$ such that $D^-(x_\alpha^*) = \varnothing$. Then for every $y \in D$, $f(x_\alpha^*, y) \geqslant \alpha$, so that $v_2 \geqslant \alpha$ contrary to the inequality $\alpha > v_2$.

LEMMA 2. *Assume that $\alpha < v_1$ and that: 1) f is upper semicontinous separately in x and y; 2) for any pair $c_1, c_2 \in C$ there is a continuous mapping $u : [a, b] \to C$ such that $u(a) = c_1$, $u(b) = c_2$ and for every $y \in D$ the set $\overset{-1}{u}(C(y))$ is connected or empty; 3) there exists a finite system $x_1, x_2, \ldots, x_m \in C$ such that $D = \bigcup_{i=1}^{m} D(x_i)$; 4) one of the following conditions holds: 4a) for every $x \in C$ the sets $D^-(x)$ and $D^-(x) \cap \left(\bigcap_{i=k}^{m} D^-(x_i) \right)$, $k = 2, \ldots, m$, are connected or* *empty* ; 4 b) *for every* $x \in C$ *the sets* $D^{-}(x) \cap \left(\bigcup_{i=1}^{k} D(x_i) \right)$, $k = 2, \ldots, m$, *are connected or empty. Then there exists an* x^*_α *such that* $D^{-}(x^*_\alpha) = \varnothing$.

Proof. First let us suppose the conditions 1, 2, 3 and 4a) to hold. To prove the existence of x^*_α we shall proceed by induction. For $m = 1$ the fact is evident since we may take $x^*_\alpha = x_1$; assuming that it holds for $m = n - 1$ let us prove it for $m = n$. Let $D_n = D^{-}(x_n)$. It is easy to verify that the conditions of the Lemma still hold for $C, D_n, x_1, x_2, \ldots, x_{n-1}$ and the restriction of f on $C \times D_n$. Therefore, there exists $x^*_{n-1} \in C$ such that $f(x^*_{n-1}, y) \geqslant \alpha$ for every $y \in D_n$. Let $u : [a, b] \to C$ be the continuous mapping that corresponds to the pair x^*_{n-1}, x_n according to assumption 2) and let g be the function defined on $[a, b] \times D$ by $g(z, y) = f(u(z), y)$ for every $z \in [a, b]$, $y \in D$. Since clearly, $[a, b]$, D and g satisfy the conditions of Lemma 1, there exists $z^*_n \in [a, b]$ such that $g(z^*_n, y) \geqslant \alpha$ for every $y \in D$, Putting $x^*_n = u(z^*_n)$, we have $f(x^*_n, y) \geqslant \alpha$ for every $y \in D$, i. e., $D^{-}(x^*_n) = \varnothing$, so that we can take $x^*_\alpha = x^*_n$.

It remains to consider the case where conditions 1, 2, 3 and 4 b) hold. Let $D_k = \bigcup_{i=1}^{k} D(x_i)$, we shall prove by induction, that for every $k = 1, \ldots, m$ there exists a point x'_k such that $f(x'_k, y) \geqslant \alpha$ for every $y \in D_k$. For $k = 1$ the fact is evident since $D(x_1) = \left\{ y \in D : f(x_1, y) \geqslant \alpha \right\}$. Suppose that an $x'_{k-1} \in C$ has been found such that $f(x'_{k-1}, y) \geqslant \alpha$ for every $y \in D_{k-1}$. Let $v : [a, b] \to C$ be the continuous mapping that corresponds to the pair x'_{k-1}, x_k according to assumption 2) and let h be the function defined on $[a, b] \times D_k$ by $h(z, y) = f(v(z), y)$ for every $z \in [a, b]$, $y \in D_k$. Since $[a, b]$, D_k and h obviously satisfy the conditions of Lemma 1, there exists an $z'_k \in [a, b]$ such that $h(z'_k, y) \geqslant \alpha$ for every $y \in D_k$, i. e., $f(v(z'_k), y) \geqslant \alpha$ for every $y \in D_k$. We may take $x'_k = v(z'_k)$ and $x^*_\alpha = x'_m$. The proof is complete.

From Lemma 2 it follows that:

Let $\alpha < v_1$ and let C, D, f satisfy assumptions 1 and 3 of Lemma 2. If f is α-strongly connected, then there exists an $x^*_\alpha \in C$ such that $D^-(x^*_\alpha) = \varnothing$.

A function f defined on $C \times D$ is said to be *strongly connected* if it is α-strongly connected for every $\alpha \in R$.

COROLLARY 2.1. *Assume that: D is compact, f is strongly connected and continuous separately in x and y. Then* (1.1) *holds.*

The above defined notion of strong connectedness is a generalization of the corresponding notion introduced by Wu Wen Tsun in [7]. A function f defined on $C \times D$ is said to be strongly connected in the sense of Wu Wen Tsun if condition 1) and 2a) of α-strongly connectedness hold for every $\alpha \in R$. Therefore the main result of Wu Wen Tsun [7] is included in Corollary 2.1. Since a function $f(x,y)$ which is quasiconvex in x on C and quasiconcave in y on D is obviously strongly connected, Corollary, 2.1. contains also as a special case the well-known minimax theorem of Nikaido in [3].

1.3. In many questions, we are interested in the existence of a saddle point for $f(x, y)$. From the previous results we can deduce the following

THEOREM 3. *Let D and f satisfy all conditions specified in Theorem 2 and let the set C be sequentially compact. Then f has a saddle point on $C \times D$, i. e.*

$$\min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) \tag{1.2}$$

Proof. From Theorem 2 we know that (1.1) holds. Since C is sequentially compact, for every $y \in D$ the function $f(x, y)$ which is upper semicontinuous in x attains a maximum on C. Since $f(x, y)$ is lower semicontinuous in y, so is the function $\sup f(x, y)$. The set D being compact, there exists a point y^* such that

$$\sup_{x \in C} f(x, y^*) = \inf_{y \in D} \sup_{x \in C} f(x, y) = v_1$$

and therefore $f(x, y^*) \leqslant v_1$ for every $x \in C$ and $v_1 < +\infty$.

On the other hand, for each $\alpha_j < v$ the function f satisfies the conditions of Lemma 2. Hence there is x^*_j such that $f(x^*_j, y) \geqslant \alpha_j$ for evrey $y \in D$. Since C is sequentially compact the sequence $\{x^*_j\}$ contains a cluster point $x^* \in C$. From $\alpha_j \to v_1$ and the semicontinuity of f in x we then have $f(x^*, y) \geqslant v_1$ for every $y \in D$. Clearly (x^*, y^*) is a saddle point of f.

Note 1. Using the following

LEMMA 3. *Let the sets C and D be sequentially compact and let f be lower semicontinuous in x and upper semicontinuous in y, then* $v'_1 = \inf_{x \in C} \sup_{y \in D} f(x, y)$ *is finite and there exists an* $x^* \in C$ *such that* $f(x^*, y) \leqslant v'_1$ *for every* $y \in D$.

we can deduce the following Hoàng Tuy's modification :

Assume that: (i) *D is compact, C is sequentially compact*; (ii) *f is strongly* $(v'_1)^-$*-connected in the sense of* [1]; (iii) *f is lower semicontinuous in x and upper semicontinuous in y. Then*

$$\min_{x \in C} \max_{y \in D} f(x, y) = \max_{y \in D} \min_{x \in C} f(x, y) \tag{1.3}$$

Note 2. From Note 1 one can easily deduce the following modification of Sion's theorem [4] :

Assume that C, D are convex sets of two linear topological spaces X, Y resp. If (i) *the set C is sequentially compact, D is compact*; (ii) *the function f is lower semicontinuous in x and upper semicontinuous in y*; (iii) *f is quasiconvex in x and quasiconcave in y, then f has a saddle point.*

2. A GENERALIZATION OF KY FAN'S RESULT

2. 1. From the above established results we can deduce some minimax theorems, which immediately generalize the result of Ky Fan.

Let f be a function defined on $C \times D$ and k be a positive integer and let R^k be a Euclidian space, R^k_+ be its non-negative orthant. Define now $S_k = \{ z \in R^k_+ : \sum_{i=1}^{k} z_i = 1 \}$, $C_k = \{ c_1, c_2, \ldots, c_k \} \subset C$, $\mathcal{C}_k = \{ C_k \subset C : \bigcap_{c_i \in C_k} D(c_i) = \varnothing \}$ and $f(C_k, D) = \{ z \in R^k : (z)_i = f(c_i, y) \text{ for some } y \in D \}$. For each $\delta \in R$ denote $\tilde{\delta} = (\delta, \ldots, \delta) \in R^k$.

A function f is said to be *α-strongly connected in y* if for each positive integer k and for every $C_k \in \mathcal{C}_k$ there exists a set $E \subset R^k$ and a number $\delta > 0$ such that :

a) $f(C_k, D) + \tilde{\delta} - \tilde{\alpha} \subset E$ (2.1)

b) $v_2(E) = \sup_{z \in E} \inf_{t \in S_k} \langle t, z \rangle < 0$ (2.2)

c) the mapping $\varphi(t, z) = \langle t, z \rangle$ defined on $E \times S_k$ is ν-strongly connected for every $\nu : v_1(E) - \delta < \nu < v_1(E)$ where $v_1(E) = \inf_{t \in S_k} \sup_{z \in E} \varphi(t, z)$ (2.3)

Further we denote by Cov(A) the convex hull of A in R^k. A function f defined on $C \times D$ is said to be *α-locally pseudoconcave in y* if for every positive integer k and for every $C_k \in \mathcal{C}_k$ there is a number $\varepsilon > 0$ such that $(\mathrm{Cov}(f(C_k, D)) + \tilde{\varepsilon}) \cap (R^k_+ + \tilde{\alpha}) = \varnothing$. The relationship between this notion and that of α-strongly connectedness is summarized in the following

LEMMA 4 *Any function α - locally pseudoconcave in y is α - strongly connected in y.*

Proof. Let f be an α - locally pseudoconcave in y and $C_k \in \mathcal{C}_k$. Now take the number ε as in the definition of α - locally pseudo-concaveness and $E =$ $\mathrm{Cov}(f(C_k, D)) + \tilde{\varepsilon} - \tilde{\delta}$. Then (2.1) holds with $\varepsilon = \delta$. Since the sets E and R^k_+ are convex in R^k and $E \cap R^k_+ = \varnothing$, there exists an $t^0 \in S^k_k$ such that $\langle t^0, z \rangle < 0$ for every $z \in E$, and therefore (2.2) also holds. To complete the proof it suffices to observe that any linear mapping $\varphi(t, z) = \langle t, z \rangle$ defined on convex sets is strongly connected, which implies that condition c) is satisfied.

A function defined on $C \times D$ is said to be *locally pseudoconcave in y* if for any pair $y_1, y_2 \in D$ and for any real $\mu \in [0, 1]$ and for any positive integer k, for any $C_k \subset C$, there exists a $y \in D$ such that $\mu . f(c_i, y_1) + (1 - \mu) f(c_i, y_2) \leqslant f(c_i, y)$ for each $c_i \in C_k$. A function f defined on $C \times D$ is said to be *pseudoconcave in y* (Ky Fan [2]) if for any $y_1, y_2 \in D$ and for any real $\mu \in [0, 1]$ there is a $y \in D$ such that $\mu f(x, y_1) + (1 - \mu) f(x, y_2) \leqslant f(x, y)$ for every $x \in C$. A function f is said to be *pseudoconvex in y* if $-f$ is pseudoconcave in y. It is easy to verify the following

LEMMA 5. *Any function pseudoconcave in y is locally pseudoconcave in y. Let the function f be upper semicontinuous in y and locally pseudoconcave in y. Then f is α -locally pseudoconcave in y for every α ∈ R.*

2.2. THEOREM 4. *Assume that:* a) *the set D is compact,* b) *the function f is upper semicontinuous in y,* c) *the function f is pseudoconvex in x and v'_1 -strongly connected in y. Then*

$$v'_1 = \inf_{x \in C} \sup_{y \in D} f(x, y) = \sup_{y \in D} \inf_{x \in C} f(x, y) = v'_2 \tag{2.4}$$

Proof. Since the set D is compact and the function f is upper semicontinuous in y and $\alpha = v'_1$, for every $x \in C$ the set $D(x)$ is nonempty and compact. Therefore to prove the Theorem we need only to show that $\bigcap_{i=1}^{k} D(x_i) \neq \varnothing$ for any finite system of C.

Suppose that for some finite system $C_k = \{c_1, c_2, \ldots, c_k\}$ of C we have $\bigcap_{i=1}^{k} D(c_i) = \varnothing$, i. e. $C_k \in \mathcal{C}_k$. Since the function f is v'_1-strongly connected in y, there exist a number $\delta > 0$ and a set $E \subset R^k$ such that (2.1), (2.2) hold and the mapping $\varphi(x, y) = \langle t, z \rangle$ defined on $E \times S_k$ satisfies the conditions of Theorem 2. This implies $\inf_{t \in S_k} \sup_{z \in E} \varphi(t, z) = \sup_{z \in E} \inf_{t \in S_k} \varphi(t, z) < 0$ and so there exists a $t^* \in S_k$ such that $\langle t^*, z \rangle \leqslant 0$ for every $z \in E$. If now we take $z \in E$ with $z_i = f(c_i, y) - v'_1 + \varepsilon$ we have $\sum_{i=1}^{k} t_i^* (f(c_i, y) - v'_1 + \varepsilon) \leqslant 0$ for every $y \in D$. Since f is pseudoconvex in x, there is an $x^* \in C$ such that for every $y \in D$, $f(x^*, y) - v'_1 + \varepsilon \leqslant \sum_{i=1}^{k} t_i^* f(c_i, y) - v'_1 + \varepsilon \leqslant 0$ and hence $\sup_{y \in D} f(x^*, y) \leqslant v'_1 - \varepsilon$. This is contrary to the definition of v'_1 and completes the proof.

COROLLARY. (Ky Fan [2]). *Assume that*: a) *The set D is compact*; b) *the function f is upper semicontinuous in y*; c) *the function f is pseudoconvex in x and pseudoconcave in y. Then* (2.4) *holds.*

Note 3. From Lemma 3 and Theorem 4 we can deduce the following:

Assume that: a) *the set C is sequentially compact and the set D is compact*; b) *the function f is lower semicontinuous in x and upper semicontinuous in y*; c) *f is pseudoconvex in x and v'_1-strongly connected in y. Then f has a saddle point on $C \times D$.*

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REFERENCES

[1] HOÀNG TỤY, *On the general minimax theorem*, Doklady Akad. Nauk, 219, 4, *1974*, 818-821 (in russian).

[2] KY FAN, *Minimax theorems*, Proc. Math. Acad. Sci. USA 39, *1953*, 42-47.

[3] NIKAIDO, H., *On Von Neumann's minimax theorems*, Pacific J. Math. 4, *1954*, 65-72.

[4] SION, M. *On general minimax theorems*, Pacific J. Math. 8, *1958*, 171-176.

[5] VOROBIOV, N., *Infinite antagonistic games*. Math. Physic. Press, Moscow, *1963*, 7-23. (in russian).

[6] VOROBIOV, N., *The present state of the theory of games*. Theory of games. Armen Acad. Press. Erevan., *1973*, 5-57, (in russian).

[7] WU WEN TSUN, *A remark on the fundamental theorem in the theory of games*. Science Record. III, *1959*, 229-233.