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ASYMPTOTIC EFFICIENCY IN THE BAHADUR SENSE FOR THE SIGNED RANK TESTS

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O. INTRODUCTION

The asymptotic efficiency in the Bahadur sense (the very exact slope, cf. [1]) of a sequence of statistics $\{ S_n \}$ in testing $H \{ f \}$ against $A \{ g \}$ is evaluated from

$$c(g) = 2\, K\, (\rho(g))\,, \tag{1}$$

when a particular density $g \in A$ obtains, provided the following two limits exist

$$\lim_{n \to \infty} \frac{1}{n}\, S_n = \rho(g) \tag{2}$$

with probability $1(g)$, and

$$\lim_{n \to \infty} - \frac{1}{n} \log \sup \{ P_f\, (S_n \geqslant n\, r) : f \in H \} = K\,(r), \tag{3}$$

where $0 < \rho < \infty$, and $K\,(r)$, $0 < K < \infty$, is continuous in some open interval including $\rho\,(g)$.

The constant r in the above limit is often replaced by r_n, where r_n tends to r as $n \to \infty$, for convenience of evaluating the limit. Note that in nonparametric cases the limit gets much simpler as

$$\lim_{n \to \infty} - \frac{1}{n} \log P \{ S_n \geqslant n\, r_n \} = K\,(r)\,, \tag{4}$$

where P denotes the probability measure under the hypothesis H. As proved by Bahadur [1] and Raghavachari [6], the exact slope is bounded above

$$c\,(g) \leqslant 2\, J\,(g) \tag{5}$$

for each $g \in A$, where $\quad J\,(g) = \inf \left\{ \int g \log\,(g/f)\, dx : f \in H \right\}.$

Several authors have investigated the limit (4) for nonparametric statistics, e.g. , M. Stone [7] (1967), [8] (1968) for the two-sample Wilcoxon statistic, G. G. Woodworth [9] (1970) for (not signed) linear rank statistics in the general case, etc.. J. Klotz [5] (1965) explored also the problem for the signed rank tests but only in a very simple case : the scores E_{ni} are to be expected values of the i-th smallest order statistics from a sample with $cdf\ R(x)$ on $(0, \infty)$, provided the 3-rd moment of it is finite.

The signed rank tests in the most general form will be the topic of the work. We shall be also concerned with the signed rank tests having the best exact slopes in testing the symmetry hypothesis $\mathcal{H}$ against the asymmetry alternative $\mathcal{A}$. This fact shows an asymptotically sufficient characteristic of vectors of signs and of ranks in the testing problem. The latter result is established similarly as in Hajek's work [2] (1971) regarding to the testing randomness against a general class of two-sample alternatives.

I. LARGE DEVIATION THEOREMS FOR THE SIGNED RANK TESTS

BASIC RESULTS

Let $X = (X_1, X_2, \ldots, X_n)$ be a sample of n independent observations from a continuous distribution. The symmetry hypothesis $\mathcal{H}$ asserts that the distribution is arbitrary but symmetric. Let $R^+ = (R_1^+, R_2^+, \ldots, R_n^+)$ be ranks of $|X| = (|X_1|, |X_2|, \ldots, |X_n|)$. Consider the following general signed rank test

$$S_n = \sum_{i=1}^{n} \alpha_n\left(\frac{i}{n+1}, \frac{R_i^+}{n+1}, W_i\right), \tag{6}$$

where $\alpha_n = \alpha_n(u, v, w)$ is a real function defined on the unit cube $I = \{0 \leqslant u < 1,\ 0 \leqslant v < 1,\ 0 \leqslant w \leqslant 1\}$, and $W_i = w(X_i)$ are random variables on [0,1], provided under the hypothesis $\mathcal{H}$

$$P(W_i \in \Delta w_k) = \frac{1}{2}, \quad 1 \leqslant i \leqslant n, \quad k = 1, 2, \tag{7}$$

where $\Delta w_1 = [0, \frac{1}{2})$, $w_2 = [\frac{1}{2}, 1]$. Note that (7) is held with $W_i = \frac{1}{2}(\text{sign } X_i + 1)$.

G. G. Woodworth [9] has explored the problem for the linear rank statistics

$$S_n = \sum_{i=1}^{n} a_n\left(\frac{i}{n+1}, \frac{R_i}{n+1}\right).$$

where R_i, $1 \leqslant i \leqslant n$, are ranks of identically distributed and independent observations. In spite of difficulties arised in our problem as the presence of component W_i in (6), all results in the Section are attained similarly as those in Woodworth's paper. Therefore we shall leave out all detail proofs where unnecessary.

We are concerned first with a simple case. Assume that for all $n = 1, 2, \ldots$

$$\alpha_n = \alpha(u,v,w) = \alpha_{ijk} \tag{8}$$

constant, if $(u,v,w) \in I_{ijk} = I\{u \in \Delta u_i, v \in \Delta v_j, w \in \Delta w_k\}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$, $k = 1, 2$, where

$$\Delta u_i = [u_{i-1}, u_i) \text{ with } 0 = u_0 < u_1 < \ldots < u_s = 1,$$

$$\Delta v_j = [v_{j-1}, v_j) \text{ with } 0 = v_0 < v_1 < \ldots < v_t = 1,$$

$$\Delta w_1 = [0, \tfrac{1}{2}), \quad \Delta w_2 = [\tfrac{1}{2}, 1].$$

Let $Z^{(n)} = Z^{(n)}_{ijk}$ be a random « matrix » of three dimensions, where

$$Z^{(n)}_{ijk} = N\left\{ l \,\middle|\, \left(\frac{l}{n+1}, \frac{R_l^+}{n+1}, W_l\right) \in I_{ijk} \right\} \tag{9}$$

with $N\{.\}$ standing for « the number of integers in $\{.\}$ » Thus S_n defined by (6) can be presented as

$$S_n = \sum_{i,j,k} \alpha_{ijk} Z^{(n)}_{ijk}. \tag{10}$$

Since $R^+ = (R_1^+, \ldots, R_n^+)$ is a permutation of $(1, 2, \ldots, n)$, $Z^{(n)}$ must satisfy under $\mathscr{H}$

$$\left.\begin{aligned} \sum_{j,k} Z^{(n)}_{ijk} &= Z^{(n)}_{i..}, \text{ say, } = m_i, \ 1 \leqslant i \leqslant s, \\ \sum_{i,k} Z^{(n)}_{ijk} &= Z^{(n)}_{.j.}, \text{ say, } = n_j, \ 1 \leqslant j \leqslant t, \end{aligned}\right\} \tag{11}$$

where

$$m_i = N\{l \mid \frac{l}{n+1} \in \Delta u_i\} = N\{[(n+1)u_{i-1}, (n+1)u_i)\},$$

$$n_j = N\{l \mid \frac{R_l^+}{n+1} \in \Delta v_j\} = N\{[(n+1)v_{j-1}, (n+1)v_j)\}.$$

Clearly

$$\sum_i m_i = \sum_j n_j = n \quad ,$$

$$m_i/n \longrightarrow u_i - u_{i-1} = \delta u_i \text{ , say,}$$

$$n_j/n \longrightarrow v_j - v_{j-1} = \delta v_j \text{ , say, as } n \to \infty.$$

Putting

$$X^{(n)} = \{ X_{ij}^{(n)} \} = \{ Z_{ij1}^{(n)} + Z_{ij2}^{(n)} \} = \{ Z_{ij.}^{(n)} \}$$

it is satisfied that

$$X_{i.}^{(n)} = \sum_j X_{ij}^{(n)} = m_i \text{ , } 1 \leqslant i \leqslant s \text{ ,}$$

$$X_{.j}^{(n)} = \sum_i X_{ij}^{(n)} = n_j \text{ , } 1 \leqslant j \leqslant t \text{ ,}$$

Then under $\mathscr{H}$ the distribution of $X^{(n)}$ is multihypergeometric. In view of this fact and by the assumption (7), one can verify that under $\mathscr{H}$

$$P\{Z^{(n)} = z\} = \left(\prod_i m_i! \prod_j n_j! / n! \prod_{ij} z_{ij.}!\right)\left(\prod_{ij}\binom{z_{ij.}}{z_{ij1}} \left(\frac{1}{2}\right)^{z_{ij.}}\right) \tag{12}$$

$$= \prod_i m_i! \prod_j n_j! / 2^n n! \prod_{ijk} z_{ijk}!$$

or

$$= 0 \text{ if } z = \{ z_{ijk} \} \text{ satisfies (11) or not.}$$

THEOREM 1: *Let S_n be defined by (6) with $\alpha_n = \alpha$ satisfying (8) such that $r_0(\alpha) < R_0(\alpha)$, where*

$$r_0(\alpha) = \sum_{ijk} \alpha_{ijk} p_{ijk} \text{ ,} \quad p_{ijk} = \frac{1}{2} \delta u_i \delta v_j \text{ ,} \tag{13}$$

and

$$R_0(\alpha) = \sup\left\{\sum_{ijk} \alpha_{ijk} q_{ijk} \,\middle|\, q_{ijk} \geqslant 0,\ q_{i..} = \delta u_i \text{ , } q_{.j.} = \delta v_j \text{ , for all } i,j,k\right\} \tag{14}$$

Let $\{r_n\}$ be a sequence of constants approaching a constant r. Then under $\mathscr{H}$, for $r < R_0(\alpha)$

$$\lim_{n\to\infty} -\frac{1}{n} \log P(S_n \geqslant nr_n) = K(r\,;\alpha) \text{ ,} \tag{15}$$

say, where

$$K(r\,;\alpha) = \inf\Big\{\sum_{ijk} q_{ijk}\ \log\,(q_{ijk}/p_{ijk})\ \ \sum_{ijk}\alpha_{ijk}\,q_{ijk} \geqslant r,\ \ q_{ijk} \geqslant 0,\ q_{i..} = \delta u_i\,,$$

$$q_{.j.} = \delta v_j\,,\ \text{for all}\ \ i, j, k\Big\}.$$

The K just defined equals zero if $r \leqslant r_0\ (\alpha)$.

Proof. The principal manner of the proof is to present the probability in (12) by means of the multinomial distribution of the random «matrix» $Y^{(n)} = \{Y^{(n)}_{ijk}\}$ with cell probabilities $p = \{p_{ijk}\}$, and one Theorem of Hoeffding (1965, Theorem 2.1 of [3]) employed for the distribution. So we have

$$P\,(Z^{(n)} = z) = \text{prob}\left(Y^{(n)} = z\,,\ Y^{(n)}_{i..} = m_i\,,\ Y^{(n)}_{.j.} = n_j\,,\ \text{for all}\ i,\,j\right) \cdot\ \ \exp\,(n\varepsilon_n),\quad (16)$$

where

$$\varepsilon_n = \frac{1}{n}\log\ \left[\prod_i (m_i\,!\ \delta u_i^{-m_i}) \prod_j (n_j\,!\ \delta v_j^{-n_j})\ /\ (n\,!)^2\right] = o\,(1)\ \text{as}\ n \to \infty\,,$$

and

$$-\ \frac{1}{n}\ \log\ P(S_n \geqslant nr_n)\ = -\ \frac{1}{n}\ \log\text{prob}\ \Big(\sum_{ijk}\ \alpha_{ijk}\,Y^{(n)}_{ijk} \geqslant\ nr_n\ ,$$

$$Y^{(n)}_{i..} = m_i\ ,\ Y^{(n)}_{.j.} = n_j\,\text{for all}\ \ i\ ,\ j\Big) + o\,(1) =$$

$$= \min\ \Big\{\sum_{ijk} (y_{ijk}/n)\ \log\ (y_{ijk}/np_{ijk})\ \Big|\ \ y_{ijk}\ \text{are natural numbers,}$$

$$\sum_{ijk} \alpha_{ijk}y_{ijk} \geqslant nr_n,\ y_{i..} = m_i, y_{.j.} = n_j\ \text{for all}\ \ i,j,k\Big\} + o\,(1). \qquad (17)$$

The assertion that $K(r\,;\alpha) = 0$ for $r \leqslant r_o\,(\alpha)$, and for all $r < R_0\,(\alpha)$ it is nonnegative, nondecreasing convex function of r, hence it is continuous where it is finite, is justified by the information inequality and the definition of K itself.

Let there be some $r < R_o\,(\alpha)$ and a subsequence $\{n'\}$ of $\{n\}$ such that $\lim\ -\frac{1}{n}\ \log\ P(S_n \geqslant nr_n) < K(r\,;\alpha)$ as $n \to \infty$ through $\{n'\}$. Clearly that there exists a subsequence $\{n''\}$ of $\{n'\}$ such that the value of y for which the minimum is attained on the right member of (17), say $y^{(n)} = \{y^{(n)}_{ijk}\}$, satisfies $y^{(n)}/n \to q^o = \{q^o_{ijk}\}$, say, as $n \to \infty$ through $\{n''\}$. Obviously q^o satisfies the constraint in the right member of (15). In view of this fact it follows from (15) and (17) that

$$\lim -\frac{1}{n}\log P(S_n \geqslant nr_n) = \sum_{ijk} q^{o}_{ijk} \log (q^{o}_{ijk}/p_{ijk}) \geqslant K(r;\alpha)$$

as $n \to \infty$ through $\{n''\}$. The contradiction affirms that

$$\lim_{n\to\infty} \inf -\frac{1}{n} \log P(S_n \geqslant nr_n) \geqslant K(r;\alpha) \text{ for all } r < R_o(\alpha). \tag{18}$$

Now assume there are some $r < R_o(\alpha)$, $\delta > 0$ and a subsequence $\{n^*\}$ of $\{n\}$ such that

$$\lim -\frac{1}{n} \log P(S_n \geqslant nr_n) > K(r;\alpha) + 4\delta$$

as $n \to \infty$ through $\{n^*\}$. Then for all sufficiently large n in $\{n^*\}$

$$\sum_{ijk} (y^{(n)}_{ijk} / n) \log (y^{(n)}_{ijk} / np_{ijk}) > K(r;\alpha) + 3\delta. \tag{19}$$

It follows from the behaviour of $K(r;\alpha)$ that there exists a constant ε_o, $0 < \varepsilon_o < R_o(\alpha) - r$, such that

$$K(r;\alpha) \leqslant K(r+\varepsilon;\alpha) \leqslant K(r;\alpha) + \delta \text{ for all } \varepsilon,\ 0 < \varepsilon < \varepsilon_o. \tag{20}$$

By the definition of $K(r+\varepsilon_o;\alpha)$ there exists a $q^1 = \{q^1_{ijk}\}$ satisfying the constraint in the definition and

$$\sum_{ijk} q^1_{ijk} \log (q^1_{ijk} / p_{ijk}) \leqslant K(r+\varepsilon_o;\alpha) + \delta \leqslant K(r;\alpha) + 2\delta, \tag{21}$$

by (20). Since both $\sum_{ijk} \alpha_{ijk} q_{ijk}$ and $\sum_{ijk} q_{ijk} \log (q_{ijk}/p_{ijk})$ are continuous in $q \geqslant 0$, there exist natural numbers $\tilde{y}^{(n)}_{ijk}$ such that $\tilde{y}^{(n)} = \tilde{y}^{(n)}_{ijk}$ satisfies the constraint in the right member of (17) and for all sufficiently large n in $\{n^*\}$

$$\left| \sum_{ijk} (\tilde{y}^{(n)}_{ijk} / n) \log (\tilde{y}^{(n)}_{ijk} / np_{ijk}) - \sum_{ijk} q^1_{ijk} \log (q^1_{ijk} / p_{ijk}) \right| \leqslant \delta$$

Hence, by definition of $y^{(n)}$ and by (21)

$$\sum_{ijk} (y^{(n)}_{ijk} / n) \log (y^{(n)}_{ijk} / np_{ijk}) \leqslant K(r;\alpha) + 3\delta$$

for all sufficiently large n in $\{n^*\}$. That contradicts with the above assumption. Thus, for all $r < R_0(\alpha)$

$$\limsup_{n \to \infty} -\frac{1}{n} \log P(S_n \geqslant nr_n) \leqslant K(r;\alpha). \tag{22}$$

Note (18) and (22). Q.E.D.

The above result will be extended to the general case as follows. Assume that α_n satisfies

CONDITION 1: for each $n = 1, 2, \ldots, \alpha_n(u, v, w)$ is constant over the set of rectangular parallelepipeds

$$I_{ijk}^{(n)} = I\{u \in \Delta u_i, v \in \Delta v_j, w \in \Delta w_k\}, \quad 1 \leqslant i, j \leqslant n, k = 1, 2,$$

where $\Delta w_1 = [0, \frac{1}{2})$, $\Delta w_2 = [\frac{1}{2}, 1]$, $\Delta u_i = \Delta v_i = [\frac{i-1}{n}, \frac{i}{n})$.

CONDITION 2: there exists a function $\alpha(u, v, w)$ over the unit cube I such that $\alpha = \alpha^{(k)}(u, v)$ if $w \in \Delta w_k$, $k = 1, 2$, and

$$\begin{aligned}\sup\left\{\iiint |\alpha_n - \alpha| f\, du\, dv\, dw \;\middle|\; f \in \mathcal{F}\right\} &= \sup\left\{ d_f(\alpha_n, \alpha) \;\middle|\; f \in \mathcal{F}\right\} \\ &= d(\alpha_n, \alpha),\end{aligned} \tag{23}$$

say, converges to zero as $n \to \infty$, where $\mathcal{F}$ is a space of trivariate densities such that

$$\mathcal{F} = \left\{ f(u,v,w):\ f = f^{(k)}(u,v) \text{ if } w \in \Delta w_k,\ k = 1,2, \right.$$
$$\left. \iint f\, du\, dw = \iiint f\, dv\, dw = 1 \right\}. \tag{24}$$

Let us define

$$r_0(\alpha) = \iiint \alpha\, du\, dv\, dw, \tag{25}$$

$$R_0(\alpha) = \sup\left\{ \iiint \alpha f\, du\, dv\, dw \;\middle|\; f \in \mathcal{F} \right\}, \tag{26}$$

and for $r < R_0(\alpha)$

$$K(r;\alpha) = \inf\left\{ \iiint f \log f\, du\, dv\, dw \;\middle|\; \iiint \alpha f\, du\, dv\, dw \geqslant r, f \in \mathcal{F}\right\}. \tag{27}$$

If α_n is of a special form as in (8), relations (25) — (27) reduce to (13) — (15)

respectively (it is verified at once, putting $q/p = f$). One can check easily by means of the information inequality and (27) that the $K(r\,;\,\alpha)$ is nonnegative nondecreasing convex function of r, $r < R_0(\alpha)$, hence it is continuous where it is finite, and for $r \leqslant r_0(\alpha)$, $K(r\,;\,\alpha) = 0$. We have also for arbitrary r, $\varepsilon > 0$, $r + \varepsilon < R_0(\alpha)$,

$$K(r - \varepsilon\,;\,\alpha) \leqslant K(r\,;\,\alpha^{(\varepsilon)}) \leqslant K(r + \varepsilon\,;\,\alpha) \text{ if } d(\alpha^{(\varepsilon)};\alpha) < \varepsilon. \tag{28}$$

THEOREM 2 ; *Let S_n be defined by (6) where α_n satisfies Conditions 1 and 2 with α such that $r_0(\alpha) < R_0(\alpha)$, where $r_0(\alpha)$ and $R_0(\alpha)$ are defined by (25) and (26). Let $\{r_n\}$ be a sequence of constants approaching some constant r as $n \to \infty$. It is satisfied under $\mathscr{H}$ that*

$$\lim_{n\to\infty} -\frac{1}{n} \log P(S_n \geqslant nr_n) = K(r,\alpha) \tag{29}$$

for $r < R_0(\alpha)$, where $K(r\,;\,\alpha)$ is defined by (27). In particular, for $r \leqslant r_0(\alpha)$, the limit is zero.

Proof. We can write

$$S_n = \sum_{i=1}^{n} \alpha_n\left(\frac{i}{n+1}, \frac{R_i^+}{n+1}, W_i\right) = n \int\int\int \alpha_n f_n \, du\, dv\, dw, \tag{30}$$

where $f_n \in \mathscr{F}$ as defined by $f_n = 2n$ or 0 if $(u, v, w) \in I_{(i)}^{(n)}$, $1 \leqslant i \leqslant n$, or not, where $I_{(i)}^{(n)} = I_{i\,R_i^+\,k_i}^{(n)}$ with k_i determined by $W_i \in \Delta w_{k_i}$. Let an arbitrary ε be given, $0 < 2\varepsilon < R_0(\alpha) - r$. In view of (23) there exists an index n_ε such that

$$d(\alpha^{(\varepsilon)},\alpha) < \varepsilon \quad \text{and} \quad d(\alpha_n, \alpha^{(\varepsilon)}) < \varepsilon \text{ for all } n \geqslant n_\varepsilon \tag{31}$$

where $\alpha^{(\varepsilon)}$ stands for α_{n_ε}. Clearly $\alpha^{(\varepsilon)}$ is bounded, say $|\,\alpha^{(\varepsilon)}\,| \leqslant M$, and satisfies (8) with $s = t = n_\varepsilon$, and $u_i = v_i = \dfrac{i}{n_\varepsilon}$, $0 \leqslant i \leqslant n_\varepsilon$.

Let

$$S_n^{(\varepsilon)} = \sum_{i=1}^{n} \alpha^{(\varepsilon)}\left(\frac{i}{n+1}, \frac{R_i^+}{n+1}, W_i\right), \; n \geqslant n_\varepsilon .$$

Since $$\alpha^{(\varepsilon)}\left(\frac{i}{n+1}\,;\,\frac{R_i^{+}}{n+1}\,,\,W_i\right) \neq 2n^2 \iiint\limits_{I_{(i)}^{(n)}} \alpha^{(\varepsilon)}\,du\,dv\,dw =$$

$$= n \iiint\limits_{I_{(i)}^{(n)}} \alpha^{(\varepsilon)} f_n\,du\,dv\,dw$$

only if there is some plane $u = \beta/n_\varepsilon$ or $v = \gamma/n_\varepsilon$, $0 \leqslant \beta$, $\gamma \leqslant n_\varepsilon$, which cuts through $I_{(i)}^{(n)}$. Since this can happen at most once for each plane $u = \beta/n_\varepsilon$ or $v = \gamma/n_\varepsilon$, $1 \leqslant \beta, \gamma \leqslant n_\varepsilon - 1$, it is plain that

$$\left| S_n^{(\varepsilon)}/n - \iiint \alpha^{(\varepsilon)} f_n\,du\,dv\,dw \right| \leqslant 2M\,.\,2\,(n_\varepsilon - 1)/n = \delta_n, \tag{32}$$

say, $\to 0$ as $n \to \infty$. It follows from (30) — (32) that

$$\left| \frac{S_n}{n} - \frac{S_n^{(\varepsilon)}}{n} \right| \leqslant \iiint |\alpha_n - \alpha^{(\varepsilon)}|\, f_n\,du\,dv\,dw + \delta_n \leqslant \varepsilon + \delta_n$$

for $n \geqslant n_\varepsilon$. Consequently for $n \geqslant n_\varepsilon$

$$P\left(S_n^{(\varepsilon)} \geqslant n\,(r_n + \varepsilon + \delta_n)\right) \leqslant P\,(S_n \geqslant nr_n)$$
$$\leqslant P\left(S_n^{(\varepsilon)} \geqslant n(r_n - \varepsilon - \delta_n)\right). \tag{33}$$

Since $r_n \pm \varepsilon \pm \delta_n \to r \pm \varepsilon$ as $n \to \infty$, it follows from (28), (33) and Theorem 1 applied to $S_n^{(\varepsilon)}$ that

$$K(r - 2\varepsilon\,;\,\alpha) \leqslant K(r - \varepsilon\,;\,\alpha^{(\varepsilon)}) \leqslant \lim_{n\to\infty} -\frac{1}{n} \log P(S_n \geqslant nr_n)$$
$$\leqslant K(r + \varepsilon\,;\,\alpha^{(\varepsilon)}) \leqslant K(r + 2\varepsilon\,,\,\alpha)\,.$$

Note that $K(r\,;\alpha)$ is continuous in $r < R_0(\alpha)$ and $\varepsilon > 0$ is arbitrary. *Q. E. D.*

EVALUATION OF $K(r\,;\,\alpha)$

Given a constant r, $r_0(\alpha) < r < R_0(\alpha)$. Let f belong to $\mathcal{F}$ and $\iiint \alpha f\,du\,dv\,dw \geqslant r$. Let a constant $\lambda > 0$ and a $\beta(v)$ on (0,1) be such that

$$\Upsilon(u) = \frac{1}{\lambda} \log \int\int \exp\left\{\lambda\left(\alpha(u, v, w) - \beta(v)\right)\right\} dv\, dw < \infty \quad \text{a.e.}$$

Put

$$g(u, v, w) = \exp\left\{\lambda\left(\alpha(u, v, w) - \beta(v) - \Upsilon(u)\right)\right\}.$$

Then $\int\int g\, dv\, dw = 1$. If $\lambda > 0$ and $\beta(v)$ can be chosen such that $\int\int g\, du\, dw = 1$, $\int\int\int \alpha\, g\, du\, dv\, dw = r$, then $g \in \mathcal{F}$ and $\int\int\int f \log f\, du\, dv\, dw \geqslant \int\int\int f \log g\, du\, dv\, dw \geqslant \int\int\int g \log g\, du\, dv\, dw$. Thus in view of Theorem 2 we get

THEOREM 3: *If there exists a solution* $(\lambda, \beta(v))$, $\lambda > 0$, *of*

$$\int \frac{exp\left(-\lambda\beta(v)\right)\left\{exp\left(\lambda\alpha^{(1)}(u, v)\right) + exp\left(\lambda\alpha^{(2)}(u, v)\right)\right\}}{\int exp\left(-\lambda\beta(v)\right)\left\{exp\left(\lambda\alpha^{(1)}(u, v)\right) + exp\left(\lambda\alpha^{(2)}(u, v)\right)\right\} dv} du = 1, \qquad (34)$$

and

$$\int \frac{\int exp(-\lambda\beta)\left\{\alpha^{(1)} exp(\lambda\alpha^{(1)}) + \alpha^{(2)} exp(\lambda\alpha^{(2)})\right\} dv}{\int exp(-\lambda\beta)\left\{exp(\lambda\alpha^{(1)}) + exp(\lambda\alpha^{(2)})\right\} dv} du = r, \qquad (35)$$

then for r, $r_0(\alpha) < r < R_0(\alpha)$

$$K(r; \alpha) = \lambda(r - \int \beta\, dv) -$$

$$- \int \left\{\log \frac{1}{2} \int \exp(-\lambda\beta) \left[\exp(\lambda\alpha^{(1)}) + \exp(\lambda\alpha^{(2)})\right] dv\right\} du. \qquad (36)$$

REMARK 1: The roles of u and v can be exchanged in Theorem 3.

A SPECIFIC STATEMENT FOR THE VALIDITY OF CONDITIONS 1 AND 2

THEOREM 4: *Let* α_n *be of the form*

$$\alpha_n = \sum_{i=1}^{l} \psi_{in}(u)\ \varphi_{in}(v)\ \zeta_{in}(w), \qquad (37)$$

where ψ_{in} *and* φ_{in} *are constant over intervals like* $\left(\frac{j-1}{n}, \frac{j}{n}\right)$, $1 \leqslant j \leqslant n$, *and*

$$\zeta_{in}(w) = b_{in}^{(1)} \text{ or } b_{in}^{(2)} \text{ if } w \in (0, \tfrac{1}{2}) \text{ or } w \in (\tfrac{1}{2}, 1).$$

Let , as $n \to \infty$, for $1 \leqslant i \leqslant l$:

$$\left.\begin{array}{l} b_{in}^{(k)} \to b_i^{(k)} \quad , \quad k = 1,2, \\ \\ \psi_{in}(u) \to \psi_i(u) \ \text{ in } \ L_2\ (0,1), \\ \\ \zeta_{in}(v) \to \zeta_i(v) \ \text{ in } \ L_2\ (0,1). \end{array}\right\} \tag{38}$$

Then α_n satisfies Conditions 1 and 2 with

$$\alpha = \sum_{i=1}^{l} \psi_i(u)\ \varphi_i(v)\ \zeta_i(w)\ , \tag{39}$$

where $$\zeta_i(w) = b_i^{(k)} \ \text{ if } w \in \Delta w_k\ ,\ k = 1,2.$$

Especially, if ψ_{in} (or φ_{in}), $1 \leqslant i \leqslant l$, approach step functions, it is sufficient to assume only that $\varphi_{in} \to \varphi_i$ (or $\psi_{in} \to \psi_i$) in L_1 (0,1) , $1 \leqslant i \leqslant l$.

Proof. Given a $f \in \mathcal{F}$, let us consider

$$d_f(\alpha_n, \alpha) = \iiint \left| \sum_{i=1}^{l} \psi_{in}\varphi_{in}\ \zeta_{in} - \sum_{i=1}^{l} \psi_i\varphi_i\ \zeta_i \right| f\, du\, dv\, dw$$

$$\leqslant \frac{1}{2} \sum_{i=1}^{l} \sum_{k=1}^{2} \Big\{ | b_{in}^{(k)} | \iint | (\psi_{in} - \psi_i)\ \varphi_{in} f^{(k)} |\, du\, dv +$$

$$+ | b_{in}^{(k)} | \iint | \psi_i(\varphi_{in} - \varphi_i) f^{(k)} |\, du\, dv +$$

$$+ | b_{in}^{(k)} - b_i^{(k)} | \iint | \psi_i\, \varphi_i f^{(k)} |\, du\, dv \Big\} .$$

Since

$$\sum_{k=1}^{2} | b_{in}^{(k)} | \iint | (\psi_{in} - \psi_i)\ \varphi_{in} f^{(k)} |\, du\, dv$$

$$\leqslant (\max_k | b_{in}^{(k)} |) \sum_{k=1}^{2} \iint | (\psi_{in} - \psi_i)\ \varphi_{in} f^{(k)} |\, du\, dv,$$

and

$$\sum_{k=1}^{2} \left(\iint | (\psi_{in} - \psi_i)\ \varphi_{in} f^{(k)} |\, du\, dv \right)^2 \leqslant$$

$$\leqslant \sum_{k=1}^{2} \left\{ \int\int (\psi_{in} - \psi_i)^2 f^{(k)} du\, dv. \int\int \varphi_{in}^2 f^{(k)} du\, dv \right\}$$

$$\leqslant \left\{ \sum_{k=1}^{2} \int\int (\psi_{in} - \psi_i)^2 f^{(k)} du\, dv \right\} \left\{ \sum_{k=1}^{2} \int\int \varphi_{in}^2 f^{(k)} du\, dv \right\}$$

$$= \left\{ 2 \int\int\int (\psi_{in} - \psi_i)^2 f\, du\, dv\, dw \right\} \left\{ 2 \int\int\int \varphi_{in}^2 f\, du\, dv\, dw \right\}$$

$$= 4 \int (\psi_{in} - \psi_i)^2 du. \int \varphi_{in}^2 dv \to 0 \quad \text{as} \quad n \to \infty$$

uniformly in $\mathcal{F}$, by (38), $1 \leqslant i \leqslant l$, we have

$$\sum_{k=1}^{2} | b_{in}^{(k)} | \int\int | (\psi_{in} - \psi_i) \varphi_{in} f^{(k)} | du\, dv \to 0 \quad \text{as } n \to \infty$$

uniformly in $\mathcal{F}$, $1 \leqslant i \leqslant l$. Similarly, both

$$\sum_{k=1}^{2} | b_{in}^{(k)} - b_i^{(k)} | \int\int | \psi_i \ \varphi_i \ f^{(k)} | du\ dv \to 0 \text{ as } n \to \infty$$

and

$$\sum_{k=1}^{2} | b_{in}^{(k)} | \int\int | (\varphi_{in} - \varphi_i) \ \psi_i f^{(k)} | du\ dv \to 0 \text{ as } n \to \infty$$

uniformly in $\mathcal{F}$, $1 \leqslant i \leqslant l$. The last assertion of the Theorem is evident.

APPLICATION TO THE SIGNED RANK TESTS WITH REGRESSION CONSTANTS

Theorem 4 is a bridge connecting results for the general signed rank test with the ones in the sequence. Putting

$$W_i = \frac{1}{2} (\text{sgn } X_i + 1),$$

$$\alpha_n = c_n (1 + [nu])\ a_n (1 + [nv])\ \text{sgn}\ (w - \frac{1}{2}),$$

where $[\ .\ ]$ indicates the integer function and $c_{ni} = c_n(i)$, $a_{ni} = a_n(i)$, the statistic defined by (6) reduces to the test with regression constants

$$S_n = \sum_{i=1}^{n} c_{ni} a_n (\overset{+}{R_i}). \text{ sgn } X_i . \tag{40}$$

If

$$c_n(1+[nu]) \to \psi(u)\ ,\ a_n(1+[nv]) \to \varphi(v) \text{ in } L_2\,(0,1), \tag{41}$$

α_n satisfies Conditions 1 and 2 with

$$\alpha = \begin{cases} -\ \psi(u)\ \ \varphi(v) & \text{if } w \in \Delta w_1\ , \\ \ \ \ \psi(u)\ \ \varphi(v) & \text{if } w \in \Delta w_2\ , \end{cases}$$

Hence we have from (25) — (27)

$$r_0(\alpha) = 0, \tag{42}$$

$$R_0(\alpha) = \sup \left\{ \frac{1}{2} \int\int \psi\,\varphi\,(f^{(2)} - f^{(1)})\,du\,dv \,\middle|\, f \in \mathcal{F} \right\}, \tag{43}$$

and for $r < R_0(\alpha)$,

$$K(r\,;\,\alpha) = \inf \left\{ \int\int\int f \log f\,du\,dv\,dw \,\middle|\, \frac{1}{2}\int\int \psi\,\varphi\,(f^{(2)} - f^{(1)})\,du\,dv \geqslant r, f \in \mathcal{F} \right\}. \tag{44}$$

Theorems 2 and 3 applied to (40) are of the form

THEOREM 2* : *Let S_n be defined by (40). Let (41) be satisfied with $\psi(u)\ \varphi(v)$ not being a.e. zero. Then $r_0(\alpha) < R_0(\alpha)$ and for $\{r_n\}: r_n \to r < R_0(\alpha)$*

$$\lim_{n \to \infty} -\frac{1}{n} \log\ P\,(S_n \geqslant nr_n) = K(\,r;\,\alpha)\,. \tag{45}$$

In particular, for $r \leqslant r_0(\alpha)$, the limit is zero. $r_0(\alpha)$, $R_0(\alpha)$ and $K\,(r\,;\,\alpha)$ are defined by (42) — (44).

THEOREM 3* : *If there exists a solution $(\lambda\,,\,\beta(v))\,,\,\lambda > 0$, of*

$$\int \frac{exp\Big(-\lambda\,\beta(v)\Big)\,cosh\Big(\lambda\,\psi(u)\ \ \varphi(v)\Big)}{\int exp\Big(-\lambda\,\beta(v)\Big)\,cosh\Big(\lambda\,\psi(u)\ \ \varphi(v)\Big)dv}\,du = 1, \tag{46}$$

$$\int \frac{\int exp\Big(-\lambda\,\beta\Big)\,\psi\,\varphi\ \ sinh\,\Big(\lambda\ \psi\ \varphi\Big)\ dv}{\int exp\Big(-\lambda\,\beta\Big)\ \ cosh\ \ \Big(\lambda\ \psi\ \varphi\Big)\ dv}\,du = r, \tag{47}$$

$0 < r < R_0(\alpha)$, where $R_0(\alpha)$ defined by (43), then $K(r\,;\,\alpha)$ defined by (44) is evaluated from

$$K(r;\alpha) = \lambda\left(r - \int \beta\, dv\right) - \int\left[\log \int \exp(-\lambda\beta)\cosh(\lambda\psi\varphi)\, dv\right] du, \tag{48}$$

for $0 < r < R_0(\alpha)$. For $r \leqslant 0$, $K(r;\alpha) = 0$.

REMARK 2. In view of Theorem 4, the convergence in (41) is required only in $L_1(0, 1)$ if one limit function is a step function. In particular it is the case for tests without regression constants.

REMARK 3. Putting $W_i = u(X_i)$, where $u(x) = 0$ or 1 if

$x < 0$ or $\geqslant 0$, and $\alpha_n = c_n(1 + [nu])\; a_n(1 + [nv])\, u(w - \frac{1}{2})$,

the statistic (6) derives to

$$S_n = \sum_{i=1}^{n} c_{ni}\; a_n\; (R_i^+)\, u(X_i), \tag{40*}$$

which is not equivalent to (40) with exception the case $c_{n1} = \ldots = c_{nn}$ or $a_{n1} = \ldots = a_{nn}$. One can form Theorems for (40*) similarly as for (40).

Example 1. Let us consider an example in order to illustrate the role of Theorem 3*. We are concerned with (40), where (41) is satisfied with $\psi(u) = \Phi^{-1}(u)$, $\varphi(v) = \Phi^{-1}\left(\frac{1}{2} + \frac{v}{2}\right)$ where Φ is the standardized normal distribution function and Φ^{-1} is its reverse function. We try to find $\beta(v)$ in the form $\beta(v) = b\varphi^2$. The constant b attained by solving (46) is

$$b = (-1 + \sqrt{1 + 4\lambda^2})/4\lambda.$$

Solving (47) for $0 < r < 1$, λ is found to be $\lambda = r/(1 - r^2)$, then $b = r/2$. Consequently (48) leads to

$$K(r;\alpha) = -\frac{1}{2}\log(1 - r^2).$$

If $\psi(u) = \Phi^{-1}(u)$, $\varphi(v) = \Phi^{-1}(v)$, by similar calculation, we get $\lambda = r/(1 - r^2)$, $\beta(v) = \frac{r}{2}(\Phi^{-1}(v))^2$, and also $K(r;\alpha) = -\frac{1}{2}\log(1 - r^2)$. Both the K's are equal to that attained by Woodworth for the Fisher — Yates (normal-scores) correlation coefficient (cf. Example 1 § 3 of [9]).

Let $\alpha_n = \sum_{i=1}^{l} \psi_{in}(u)\ \varphi_{in}(v)\ \text{sgn}\,(w - \frac{1}{2})$, where $\psi_{in}(u)$, $\varphi_{in}(v)$ are constant over intervals like $\left(\frac{j-1}{n}, \frac{j}{n}\right)$, $1 \leqslant j \leqslant n$, $\varphi_{in}(v) \to \varphi_i(v)$ in $L_1(0,1)$, $1 \leqslant i \leqslant l$, and $\psi_{in}(u) \to 1$ or 0 if $u \in (u_{i-1}, u_i)$ or $u \in (0,1) - [u_{i-1}, u_i]$, $1 \leqslant i \leqslant l$, where $0 = u_0 < u_1 = \rho_1 < u_2 = \rho_1 + \rho_2 < \ldots < u_l = \sum_{i=1}^{l} u_i = 1$. In view of Theorem 4, α_n satisfies Conditions 1 and 2 with

$$\alpha(u, v, w) = \alpha_{(i)}(v, w) \text{ if } u \in [u_{i-1}, u_i),\quad 1 \leqslant i \leqslant l, \tag{49}$$

where

$$\alpha_{(i)}(v, w) = -\varphi_i(v) \text{ or } \varphi_i(v) \text{ if } w \in \Delta w_1, \text{ or } w \in \Delta w_2 .$$

Thus $r_0(\alpha)$ defined in (25), $R_0(\alpha)$ in (26), equations (34) and (35) after exchanging roles of u and v and putting

$\beta(u) = \beta_i$ if $u \in [u_{i-1}, u_i)$, $1 \leqslant i \leqslant l$, and K *in* (36) are of the form

$$r_0(\alpha) = 0 \tag{50}$$

$$R_0(\alpha) = \sup \left\{ \frac{1}{2} \sum_{i=1}^{l} \rho_i \int \varphi_i(v)\ [f_i^{(2)} - f_i^{(1)}]\ dv \,\middle|\, f_i^{(k)}(v) \geqslant 0,\ \frac{1}{2} \sum_{i=1}^{l} \rho_i\ (f_i^{(2)} + f_i^{(1)}) = 1, \right.$$

$$\left. \frac{1}{2} \int (f_i^{(2)} + f_i^{(1)})\ dv = 1,\ 1 \leqslant i \leqslant l,\ k = 1, 2 \right\}, \tag{51}$$

$$\int \left[\exp(-\lambda\beta_j)\ \cosh(\lambda\varphi_j) \Big/ \sum_{i=1}^{l} \rho_i \exp(-\lambda\beta_i)\ \cosh(\lambda\varphi_i) \right] dv = 1,$$

$$1 \leqslant j \leqslant l, \tag{52}$$

$$\int \Big[\sum_{i=1}^{l} \rho_i \varphi_i \exp(-\lambda\beta_i) \sinh(\lambda\varphi_i) \Big/ \sum_{i=1}^{l} \rho_i \exp(-\lambda\beta_i) \cosh(\lambda\varphi_i) \Big] dv = r, \tag{53}$$

and

$$K(r;\alpha) = \lambda \Big(r - \sum_{i=1}^{l} \rho_i \beta_i\Big) - \int \log \sum_{i=1}^{l} \rho_i \exp(-\lambda\beta_i) \cosh(\lambda\varphi_i)\, dv. \tag{54}$$

THEOREM 5. *If at least one of $\varphi_i(v)$ is not a. e. zero, $R_0(\alpha)$ defined in (51) is positive, and for $0 < r < R_0(\alpha)$, there exists a solution $(\lambda, \beta_1, \dots, \beta_l)$ with $\lambda > 0$ of equations (52) - (53). All the other solutions are of the form $(\lambda, \beta_1 + c, \dots, \beta_l + c)$ with c arbitrary.*

$K(r;\alpha)$ is evaluated uniquely from (54) for $0 < r < R_0(\alpha)$. In particular $K = 0$ if $r \leqslant 0$.

Proof. In view of Theorem 3, it remains to prove that there exists a solution $(\lambda, \beta_1, \dots, \beta_l)$ with $\lambda > 0$ of (52) - (53). The other assertions of the theorem are verified easily. The solvability of (52) - (53) is proved quite the same as in Woodworth's paper (Theorem 4 § 3, [9]), i. e., by six Lemmas successively. Let us suggest these Lemmas with only a little interpretation if necessary. The first three Lemmas are those of Woodworth (cf. Appendix in [9]).

LEMMA 1. *Let $K_j(v)$, $1 \leqslant j \leqslant l$, be a. e. positive functions on (0,1). Define*

$$K.(v) = \sum_{i=1}^{l} \rho_i K_i(v)$$

where

$$\rho_i > 0, \sum_{i=1}^{l} \rho_i = 1.$$

Let $K_j(v) / K.(v)$ be bounded a.e. away from zero, say

$$K_j / K. > a > 0 \text{ a.e. on } (0,1), \quad 1 \leqslant j \leqslant l.$$

Then there exist constants $h_1, h_2, \dots, h_l$ such that

$$\sum_{i=1}^{l} \rho_i h_i = 1, \quad \textit{with } a \leqslant h_j \leqslant \max_{1 \leqslant i \leqslant l} (1/\rho_i) = b, \textit{ say,}$$

and

$$h_j = \int_0^1 \frac{K_j(v)}{\sum_i \rho_i K_i(v) / h_i} \, dv, \quad 1 \leqslant j \leqslant l.$$

Proof. By means of the Brouwer fixed point Theorem (see. e. g., Kakutani, 1941, [4]) •

LEMMA 2. *Lemma 1 remains true with $a = 0$, moreover*

$$h_j > 0 \ , \ 1 \leqslant j \leqslant l.$$

LEMMA 3. *There is only one solution $h = (h_1, \ldots, h_l)$ for the general case $a = 0$, such that*

$$0 < h_j \leqslant b, \quad 1 \leqslant j \leqslant l$$

and

$$\sum_i \rho_i h_i = 1.$$

COROLLARY. *For each $\lambda \geqslant 0$ there exists a solution $(\beta_1(\lambda), \ldots, \beta_l(\lambda))$ to (52). Any other solution is of the form $(\beta_1(\lambda) + c, \ldots, \beta_l(\lambda) + c)$ where c is constant. The solution satisfying $\sum_i \rho_i \exp(\lambda \beta_i(\lambda)) = 1$ is unique.*

Proof. Put $K_i(v) = \cosh(\lambda \varphi_i(v))$ and

$$h_i = \exp(\lambda \beta_i(\lambda)) \ , \ 1 \leqslant i \leqslant l.$$

LEMMA 4. *Let $m(\lambda)$ stand for the left member of (53), where $(\beta_1, \ldots, \beta_l) = (\beta_1(\lambda), \ldots, \beta_l(\lambda))$, a solution of (52) for $\lambda \geqslant 0$. Then $m(0) = 0 = r_0(\alpha)$, and $m(\lambda)$ is continuous in $\lambda \geqslant 0$.*

Proof. Rewrite $m(\lambda) = \iiint \alpha(u, v, w) \, f_\lambda(u, v, w) \, du \, dv \, dw$, where α is defined by (49) and $f_\lambda = f_{j\lambda}(v, w)$, say, if $u \in [u_{j-1}, u_j)$, $1 \leqslant j \leqslant l$, with

$$f_{j\lambda} = \exp\{\lambda(\alpha_{(j)} - \beta_j(\lambda))\} \Big/ \sum_{i=1}^{l} \rho_i \exp(-\lambda \beta_i(\lambda)) \cosh(\lambda \varphi_i(v)).$$

Check that $f_\lambda \in \mathcal{F}$, and go on to verify the continuity of f_λ in $\lambda > 0$.

LEMMA 5. *If at least one of $\varphi_i(v)$ is not a. e. zero, $m(\lambda)$ is strictly increasing in $\lambda \geqslant 0$.*

Proof. Let $0 \leqslant \lambda_0 < \lambda_1$. Then $\log(f_{\lambda_1}/f_{\lambda_0})$ is of the form $(\lambda_1 - \lambda_0)\alpha + a_1(u) + a_2(v)$. By means of the information inequality it is easy to verify

$$0 < \iiint f_{\lambda_1} \log(f_{\lambda_1}/f_{\lambda_0})\, du\, dv\, dw - \iiint f_{\lambda_0} \log(f_{\lambda_1}/f_{\lambda_0})\, du\, dv\, dw =$$

$$= (\lambda_1 - \lambda_0)\ (m(\lambda_1) - m(\lambda_0)).$$

LEMMA 6. $m(\lambda) \to R_0(\alpha)$ *as* $\lambda \to \infty$.

Proof. Rewrite

$$m(\lambda) = \frac{1}{2} \sum_{i=1}^{l} \rho_i \int \varphi_i(v) \left(f_{i\lambda}^{(2)}(v) - f_{i\lambda}^{(1)}(v) \right) dv,$$

where $f_i^{(k)}(v) = f_i(v, w)$ for $w \in \Delta w_k$, $k = 1, 2$. Next verify that there exists a sequence $\lambda_n \to \infty$ such that

$$f_{i\lambda_n}^{(k)}(v) \to \tilde{f}_i^{(k)}(v) \qquad 1 \leqslant i \leqslant l,$$

and

$\tilde{g}_i^{(k)} = \frac{1}{2} \rho_i \tilde{f}_i^{(k)}$ $1 \leqslant i \leqslant l$, $k = 1,2$, satisfy the condition of Lemma (additional). Now apply this Lemma (additional), Lemma 4 and the dominated convergence Theorem.

LEMMA (ADDITIONAL) : *Let $\varphi_1(x), \ldots, \varphi_l(x)$ be integrable with respect to a measure μ on a set $\mathcal{X}$. Let any $g_i^{(k)}(x) \geqslant 0$, $1 \leqslant i \leqslant l$, $k = 1, 2$, be such that*

$$\sum_{i=1}^{l} \left(g_i^{(2)}(x) + g_i^{(1)}(x) \right) = 1 \text{ on } \mathcal{X}.$$

Define $\{ \tilde{g}_i^{(k)}(x) \}$ satisfying $\sum_{i=1}^{l} \left(\tilde{g}_i^{(2)}(x) + \tilde{g}_i^{(1)}(x) \right) = 1$ as follows : at a point

$x \in \mathscr{X}$, if $|\varphi_{i_1}| = \ldots = |\varphi_{i_r}| > |\varphi_{i_{r+1}}| \geqslant \ldots \geqslant |\varphi_{i_l}|$. Let us put

$$\sum_{j=1}^{r} \tilde{g}_{i_j}^{(k_j)}(x) = 1,$$

where for $1 \leqslant j \leqslant r$, $\tilde{g}_{i_j}^{(k_j)}(x) \geqslant 0$ with $k_j = 1$ or 2 if $\varphi_{i_j} < 0$ or > 0, and put all the remaining $\tilde{g}_i^{(k)} = 0$. Then

$$\sum_{i=1}^{l} \int_{x} \varphi_i\left(\tilde{g}_i^{(2)} - \tilde{g}_i^{(1)}\right) d\mu \geqslant \sum_{i=1}^{l} \varphi_i\left(g_i^{(2)} - g_i^{(1)}\right) d\mu .$$

Proof. We have always

$$\sum_{i=1}^{l} \varphi_i\left(\tilde{g}_i^{(2)} - \tilde{g}_i^{(1)}\right) = \sum_{j=1}^{r} |\varphi_{i_j}| \, \tilde{g}_{i_j}^{(k_j)} = |\varphi_{j_1}| \geqslant$$

$$\geqslant \sum_{i=1}^{l} |\varphi_i| \left(g_i^{(2)} + g_i^{(1)}\right) \geqslant \sum_{i=1}^{l} \varphi_i \left(g_i^{(2)} - g_i^{(1)}\right).$$

REMARK 4. One can establish a similar result as Theorem 5 for the statistic (40*) mentioned in Remark 3.

REMARK 5. For the statistic defined in (40) with $c_n(1 + [nu])$ approaching a step function, say $\psi(u) = c_i$ if $u \in [u_{i-1}, u_i)$, $1 \leqslant i \leqslant l$, and $a_n(1 + [nv])$ converges to $\varphi(v)$ in $L_1(0,1)$, Theorem 5 is employed, putting $\varphi_i(v) = c_i \varphi(v)$.

REMARK 6. In case $l = 1$, Theorem 5 reduces to a simple result for the signed rank test without regression constants. Since its simplicity but importance it shall be presented separately below.

APPLICATION TO THE SIGNED RANK TESTS WITHOUT REGRESSION CONSTANTS.

COROLLARY OF THEOREM 5.

Let

$$S_n = \sum_{i=1}^{n} a_n(R_i^{+}) \; sgn \; X_i \tag{55}$$

with the scores a_{ni} satisfying

$$\int^{1} |\, a_n(1 + [nv]) - \varphi(v) \,|\, dv \to 0 \text{ for some } \varphi \in L_1(0,1). \tag{56}$$

Then under the hypothesis $\mathcal{H}$

$$\lim_{n \to \infty} -\frac{1}{n} \log P(S_n \geq nr_n) = K(r; \varphi) \tag{57}$$

for $r_n \to r < R_0(\varphi) = \int_0^1 |\varphi(v)| dv$. *For* $0 < r < R_0(\varphi)$ *the constant* $K(r; \varphi)$ *is evaluated from*

$$K(r; \varphi) = \lambda r - \int_0^1 \log \cosh(\lambda \varphi(v))\, dv, \tag{58}$$

where $\lambda > 0$ *is a unique solution of*

$$\int_0^1 \varphi(v) \operatorname{tgh}(\lambda \varphi(v))\, dv = r. \tag{59}$$

For $r \leq 0$, $K(r; \varphi) = 0$.

Proof. In case $l = 1$, equation (52) gets automatically an equality. Consequently we may put $\beta = 0$, for convenience, into (53) and (54).

Remark 7. In a special case $\varphi = R^{-1}$, the Corollary reduces to Klotz's result mentioned in the Introduction, moreover with only a weaker assumption on $R : \int_0^\infty x dR < \infty$.

Example 2. Consider the sign test

$$S_n = \sum_{i=1}^{n} \operatorname{sgn} X_i$$

Let us be in testing the symmetry hypothesis $\mathcal{H}$ against $\mathcal{A}_1 = \{ \mathcal{G}_\theta : 0 < \theta < 1 \}$ where $\mathcal{G}_\theta$ is a family of all densities such that

$$\mathcal{G}_\theta = \left\{ g(x) : \int_0^\infty g(x)\, dx = \frac{1}{2} + \frac{1}{2}\theta \right\}$$

Applying the Corollary to S_n, we get $\varphi = 1$. Hence, by (59), for $0 < r < 1$,

$$\lambda = \operatorname{arctgh}(r) = \frac{1}{2} \log \frac{1+r}{1-r}.$$

Then (58) follows

$$K(r) = \lambda r - \log \cosh(\lambda) = \frac{1}{2}\Big((1+r)\log(1+r) + (1-r)\log(1-r)\Big).$$

By the law of large numbers

$\frac{1}{n} S_n \to \theta$ as $n \to \infty$ with probability 1 under any $g \in \mathcal{G}_\theta$. Consequently, by (1), the exact slope of the sign test is

$$c(\theta) = (1+\theta)\log(1+\theta) + (1-\theta)\log(1-\theta)$$

with respect to any density $g \in \mathcal{G}_\theta$.

2. THE SIGNED RANK TESTS WITH THE BEST ASYMPTOTIC EFFICIENCY IN THE BAHADUR SENSE

Let us have at our disposal the statistic defined in (55). Let a density $g \in \mathcal{A}$ obtain and let $G(x) = \int_{-\infty}^{x} g(x)\, dx$ satisfy

$$0 < \beta < 1, \text{ where, say, } \beta = 1 - G(0) \tag{60}$$

Put

a. $H(x) = G(x) - G(-x), \ x \geqslant 0,$

b. $G_o(x) = \frac{1}{\beta}\,(G(x) - G(0)), \ x \geqslant 0,$

c. $G_o^*(v) = G_o(H^{-1}(v)), \ G^*(v) = G(H^{-1}(v)), \quad 0 \leqslant v \leqslant 1,$

where $H^{-1}(v) = \inf \{x : H(x) \geqslant v\}$,

d. $g_o(v) = \frac{d}{dv} G_o^*(v), \ g^*(v) = \frac{d}{dv} G^*(v).$

$$\text{Clearly} \quad 0 \leqslant g^* = \beta g_o^* \leqslant 1. \tag{61}$$

e. $m = N \{ i \mid X_i > 0, \ 1 \leqslant i \leqslant n \}$.

f. $G_{om}(x)$ $(H_n(x))$ to be the experiment distribution function of only positive observations from $X_1, \ldots, X_n$ (of $|X_1|, \ldots, |X_n|$).

g. $G_{om}^*(v) = G_{om}(H_n^{-1}(v))$.

Then with probabitity 1(g)

a. $m/n \to \beta$ as $n \to \infty$, by Borel's theorem.

b. $G_{om}(x) \to G_o(x), \quad H_n(x) \to H(x), \quad G_{om}^*(v) \to G_o^*(v),$ (62)

by Glivenko's theorem, uniformly in $x \in (-\infty, \infty)$ and in $v \in [0,1]$ respectively.

THEOREM 6. *Let $X_1, \ldots, X_n$ be independent observations having the same density $g(x)$ as above. Let S_n be defined by* (55) *provided scores functions $a_n(1 + [nv])$ have uniformly bounded variations on $(0,1)$ and satisfy* (56). *Then*

$$\frac{1}{n} S_n \to \int_0^1 \left(2 g^* (v) - 1\right) \varphi(v) \, dv \tag{63}$$

as $n \to \infty$ *with probability 1 (g)*

Proof. The following relation is clearly satisfied with probability $1(g)$

$$S_n = \sum_{i=1}^{n} a_n (R_i^+) \operatorname{sgn} X_i = 2 \sum_{X_i > 0} a_n (R_i^+) - \sum_{i=1}^{n} a_{ni}$$

It follows from (56) that

$$\frac{1}{n} \sum_{i=1}^{n} a_{ni} \to \int_0^1 \varphi(v) \, dv, \quad \text{as} \quad n \to \infty.$$

So we have to prove only

$$\frac{1}{n} \sum_{X_i > 0} a_n (R_i^+) \to \int_0^1 g^*(v) \, \varphi(v) \, dv \quad \text{as} \quad n \to \infty \tag{64}$$

with probability $1(g)$. It is plain that

$$\frac{1}{n} \sum_{X_i > 0} a_n (R_i^+) = \frac{m}{n} \int_0^1 a_n (1 + [nv]) \; dG^*_{om}(v)$$

$$= \frac{m}{n} \int_0^1 \varphi(v) \; g^*_o (v) \, dv \; +$$

$$+ \frac{m}{n} \int_0^1 \left(a_n (1 + [nv]) - \varphi(v) \right) g^*_o (v) \, dv \; +$$

$$+ \frac{m}{n} \int_0^1 a_n (1 + [nv]) \; d \left(G^*_{om}(v) - G^*_o (v) \right). \tag{65}$$

In view of (61)d. it follows from (62) a. and (56) that the second term in the right member of (65) converges to zero as $n \to \infty$. Since $a_n (1 + [nu])$ have uniformly bounded variations, one can verify by aid of the formula of partial integration and (62) b. that the last term of the member tends to zero as $n \to \infty$ with probability $1(g)$. Now (61) d., (62) a. and (65) reduce to (64). Q. E. D.

REMARK 8. Clearly (63) remains true when $\beta = 0$ and $\beta = 1$. The respective limits are $-\int_0^1 \varphi(v)\, dv$ and $\int_0^1 \varphi(v)\, dv$.

THEOREM 7. *Let $g(x) \in \mathcal{A}$ be given such that $g \neq 0$ and $g \neq 1$ a. e. Let*

$$S_n = \sum_{i=1}^{n} a_n(R_i^+)\ sgn X_i$$

with scores satisfying assumptions as in Theorem 6 for

$$\varphi(v) = log \frac{g^*(v)}{1 - g^*(v)} \tag{66}$$

Then the exact slope $c(g)$ of $\{ S_n \}$ is the best one among all tests in testing the symmetry hypothesis $\mathcal{H}$ against g, and

$$c(g) = 2 \int_0^1 g^* \ log\ 2g^*\ dv + 2 \int_0^1 (1 - g^*)\ log\ 2\ (1 - g^*)\ dv \tag{67}$$

Proof. For any $f \in \mathcal{H}$

$$\int_{-\infty}^{\infty} g(x) \log \frac{g(x)}{f(x)}\, dx = \int_{-\infty}^{\infty} g(x) \log \frac{g(x)}{h(x)}\, dx + B,$$

where

$$h(x) = \frac{1}{2}\Big(g(x) + g(-x)\Big) \in \mathcal{H}\ ,$$

and

$$B = \int_{-\infty}^{\infty} g(x) \log \frac{h(x)}{f(x)}\, dx = \int_{-\infty}^{0} + \int_{0}^{\infty} = \int_{0}^{\infty} \Big(g(x) + g(-x)\Big) \log \frac{h(x)}{f(x)}\, dx$$

$= \int_{-\infty}^{\infty} h(x) \log \frac{h(x)}{f(x)}\, dx \geqslant 0$, by the information inequality. Then $J(g)$ defined in (5) is determined by

$$J(g) = \int_{-\infty}^{\infty} g(x) \log \frac{g(x)}{h(x)}\, dx = \int_{-\infty}^{0} + \int_{0}^{\infty} =$$

$$= \int_{0}^{\infty} g(x) \log \frac{g(x)}{h(x)}\, dx + \int_{0}^{\infty} g(-x) \log \frac{g(-x)}{h(x)}\, dx =$$

$$= \int_0^\infty g(x) \log \frac{2\,g(x)}{g(x) + g(-x)}\, dx + \int_0^\infty g(-x) \log \frac{2\,g(-x)}{g(x) + g(-x)}\, dx \quad (68)$$

Note that (61) a-d follow

$$v = H\left(H^{-1}(v)\right) = G\left(H^{-1}(v)\right) - G\left(-H^{-1}(v)\right),$$ by the continuity of H, and

$$1 = g\left(H^{-1}(v)\right) \frac{dH^{-1}(v)}{dv} + g\left(-H^{-1}(v)\right) \frac{dH^{-1}(v)}{dv}$$

$$g^* = g\left(H^{-1}\right) \frac{dH^{-1}}{dv} \quad (69)$$

Thus (68) follows

$$J(g) = \int_0^1 g^* \log 2g^*\, dv + \int_0^1 (1 - g^*) \log 2(1 - g^*)\, dv. \quad (70)$$

In view of (5) and (70) we have to prove only (67).

From Theorem 6 we get

$$\frac{1}{n} S_n \to \int_0^1 (2g^* - 1) \log \frac{g^*}{1 - g^*}\, dv = \rho(g), \quad (71)$$

say, as $n \to \infty$, with probability $1(g)$.

In view of (69) it is easy to see that $g^* = \frac{1}{2}$ a. e. if and only if $g \in \mathcal{H}$. Consequently $\rho(g) > 0$ for g in our topic. Since (69) and assumptions $g^* \neq 0$, $g^* \neq 1$ a. e., we have $0 < g^* < 1$ or $-1 < 2g^* - 1 < 1$ a. e. Hence

$$\rho(g) < \int_0^1 \left| \log \frac{g^*}{1 - g^*} \right| dv = \int_0^1 |\varphi|\, dv.$$

Now we may apply (58) and (59) in Corollary of Theorem 5 for $r = \rho(g)$ in order to compute $K(\rho(g);\varphi)$. The solution $\lambda = \frac{1}{2}$ is get from (59) with $r = \rho(g)$ defined in (71). In view of (1), (67) is proved at once since (58) with $r = \rho(g)$ defined by (71), $\varphi(v)$ by (66) and $\lambda = \frac{1}{2}$, gives $K(\rho(g);\varphi) = J(g)$ determined in (70). Q. E. D.

Finally let us suggest an example which shows that the sign test is the best one in the Bahadur sense in testing $\mathcal{H}$ against a large family of asymmetric densities. Moreover the sign test is uniformly optimal with respect to the family of alternatives.

EXAMPLE 3. Consider a family $\mathcal{A}^+$ of alternatives

$$\mathcal{A}^+ = \{\mathcal{A}_\theta : 0 < \theta < 1\},$$

where

$$\mathcal{A}_\theta = \Big\{ g(x) : g(x) = \left(\frac{1}{2} + \frac{\theta}{2}\right) h(x) \text{ or } \left(\frac{1}{2} - \frac{\theta}{2}\right) h(-x)$$

$$\text{if } x \geqslant 0 \quad \text{or} \quad x < 0, \text{ for any density } h \text{ on } (0, \infty) \Big\}.$$

Let any $g \in \mathcal{A}_\theta \subset \mathcal{A}^+$ be given. It is computed from (61) that

$$H(x) = \int_{-x}^{x} g(x)\, dx = \left(\frac{1}{2} + \frac{\theta}{2}\right) \int_0^x h(x)\, dx + \left(\frac{1}{2} - \frac{\theta}{2}\right) \int_{-x}^{0} h(-x)\, dx$$

$$= \int_0^x h(x)\, dx, \quad x \geqslant 0,$$

$$H^{-1}(v) \geqslant 0 \quad , \quad 0 \leqslant v \leqslant 1$$

$$G(x) = \int_{-\infty}^{x} g(x)\, dx = \left(\frac{1}{2} - \frac{\theta}{2}\right) + \left(\frac{1}{2} + \frac{\theta}{2}\right) H(x) \text{ for } x \geqslant 0,\ g^*(v) = \frac{1}{2} + \frac{\theta}{2}.$$

By (66),

$$\varphi(v) = \log\left(\frac{1}{2} + \frac{\theta}{2}\right) \Big/ \left(\frac{1}{2} - \frac{\theta}{2}\right) = a(\theta) = \text{constant} > 0.$$

Consequently, by Theorem 7, the sign test $S_n = \sum_{i=1}^{n} \operatorname{sgn} X_i$ is the best one (in the Bahadur sense) uniform in testing $\mathcal{H}$ against $\mathcal{A}^+$. The exact slope of it is the same for any $g \in \mathcal{A}_\theta$, as calculated from (67) with $g^* = \frac{1}{2} + \frac{\theta}{2}$:

$$c(g) = (1 + \theta) \log (1 + \theta) + (1 - \theta) \log (1 - \theta).$$

This result coincides with that in Example 2, noting $\mathcal{A}_\theta \subset \mathcal{G}_\theta$. Similarly for $\mathcal{A}^- = \{\mathcal{A}_\theta, -1 < \theta < 0\}$, the opposite sign test $S_n = -\sum_{i=1}^{n} \operatorname{sgn} X_i$ is optimal.

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