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ON THE FOUNDATION OF THE MAXIMUM PRINCIPLE

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Introduction.

It is now common knowledge that the maximum principle, like many other optimality necessary conditions, is based upon the separation theorem for convex sets. Nevertheless, the specific features of optimal control problems make the application of the separation theorem a rather involved process. For this reason, although elementary proofs are known for cases in which more or less stringent conditions are imposed upon the data, the foundation of the maximum principle under the most general hypotheses usually requires a fairly elaborate machinery (see e.g. the proofs given in [1] and [4].)

The purpose of the present paper is to provide an approach which would enable us to simplify substantially the derivation of the maximum principle and at the same time, to relax, without extra cost, some assumptions generally made in standard optimal control problems. In section 1 we shall establish a general multiplier rule which seems to be well adapted to control problems. In section 2 we shall show that the previous theorem can be applied to every control system whose dynamics is described by an equation of the form

$$\dot{x}(t) = f(x(t), t), f \in F,$$

provided some very general conditions upon the family F are satisfied. As a consequence of this result we shall obtain a new simple proof of the main theorem in [2] concerning variational sets of differential equations. The last section 3 is devoted to the derivation of the maximum principle for problems with boundary conditions more general than in the usual setting.

The paper as a whole could be considered a simple, self-contained presentation of the maximum principle under general hypotheses. Apart from Gronwall's lemma* which will be used in section 2, we do not assume, as a prerequisite for reading the paper, any special knowledge of the theory of differential equations. In contrast, theorems on the dependence of solutions of differential equations with respect to initial conditions and parameters will be obtained as by-products in the course of the proof.

1. An abstract maximum principle.

Consider the system

$$(x, q) \in A \times C, \quad K(x, q) = 0, \quad G(x) \in M \tag{1}$$

where A is an open subset of a Banach space X, C a subset of a linear normed space Q, $K: A \times Q \to X$, $G: A \to R^k$ two mappings and M is a closed convex cone in R^k.

Let $(\bar{x}, \bar{q})$ be a given solution of (1) and assume that:

(I) For every finite-dimensional linear manifold L in Q passing through $\bar{q}$ the restriction of K to $A \times L$ is Fréchet differentiable at $(\bar{x}, \bar{q})$ and its partial derivative $\bar{K}_x : X \to X$ at this point, with respect to x, is a bijection (with inverse $\bar{K}_x^{-1}$).

(II) There exist a convex set V in Q, containing $\bar{q}$ as an internal point, and a continuous mapping $H: V \to A$ such that

$$\bar{x} = H(\bar{q}), \quad (\forall q \in V) \quad K(H(q), q) = 0 \tag{2}$$

(III) There exists a convex set C' in Q, containing 0 and having the following property: to every triple (S, ε, η), where $S = [q^0, q', \ldots, q^h] \subset C'$ $(h \leqslant k)$**, ε and η are two positive numbers such that $0 < \varepsilon < \delta$, with δ depending possibly on S one can associate a continuous mapping $\xi: S \to C \cap V$ satisfying

$$(\forall q \in S) \quad \| \xi(q) - \bar{q} - \varepsilon q \| < \varepsilon \eta. \tag{3}$$

(IV) G is continuous and there exists a continuous mapping $G': X \to R^k$ which is an *M-derivative* of G at $\bar{x}$ in the following sense: G' is M-convex*** and to every pair (x, W), where $x \in X$ and W is a ball in R^k (centered at 0), one can associate a number $\delta > 0$ such that

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*) This lemma (which is easy to prove) reads as follows: if $y, z: [0, 1] \to R^1$ are continuous, if $\varphi: [0, 1] \to R^1$ is integrable, and if

$$(\forall t \in [0, 1]) \quad y(t) \leqslant \int_0^t |\varphi(s)|\, y(s)\, ds + |z(t)|,$$

then $(\forall t \in [0, 1])$ $y(t) \leqslant |z(t)| + c \times \int_0^t |\varphi(s)| \, |z(s)|\, ds$ with $c = \exp \int_0^1 |\varphi(s)|\, ds$,

**) By [E] we denote the convex hull of the set E.

***) G' is said to be M-convex if for all $x^1, x^2 \in X$ and $t \in (0, 1)$:

$$G'(tx^1 + (1-t)x^2) \in tG'(x^1) + (1-t)G'(x^2) + M$$

$$\| z - x \| < \delta, \; 0 < \varepsilon < \delta \Rightarrow G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon (G'(x) + W) + M \quad (4)$$

Let D denote the set of all $x \in A$ such that $K(x, q) = 0$ for some $q \in C$ (which may depend on x). We say that the system (1) is *critical* if

$$0 \notin \operatorname{int} (G(D) - M) \quad (5)$$

THEOREM 1. *Under assumptions* (I) *through* (IV), *if the system* (1) *is critical, then there exists a nonzero vector* $\wedge \in M^*$ *such that*

$$(\forall q \in - C') \qquad \wedge G' \bar{K}_x^{-1} \; \bar{K}_q . q \leqslant 0 \quad (5)$$

$$\wedge G(\bar{x}) = 0, \quad (6)$$

where M^* is the conjugate of M, i.e. the cone formed by all vectors $\wedge \in R^k$ such that $\langle \wedge, y \rangle \geqslant 0$ for all $y \in M$, $\bar{K}_x$ and $\bar{K}_q$ are the partial derivatives of K at $(\bar{x}, \bar{q})$.

We shall first prove a number of auxiliary propositions.

Lemma 1.1. If G' is an M-derivative of G at $\bar{x}$ in the sense defined in (IV) and if G' is continuous, then to every pair (Σ, W) where Σ is a compact set in X and W is a ball in R^k, one can associate a number $\delta > 0$ such that (4) holds for all $x \in \Sigma$.

Proof. By definition of an M-derivative, for every $x \in \Sigma$ there is a number $\delta(x) > 0$ such that $G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon (G'(x) + {}^1/_2 \; W) + M$ whenever $\|z - x\| < \delta(x)$; $0 < \varepsilon < \delta(x)$. Because of the continuity of G' we can suppose $\delta(x)$ to be so small that $G'(z) \in G'(x) + 1/2 \; W$ whenever $\|z - x\| < {}^1/_2 \, \delta(x)$. But Σ being compact there exists a finite set $\{x^i, i \in I\}$ such that Σ can be covered by the balls of centers x^i and radii ${}^1/_2 \, \delta(x^i)$ $i \in I$. Let $\delta = \min \{{}^1/_2 \, \delta(x^i) : i \in I\}$. Then for every $x \in \Sigma$ there is $i \in I$ such that $\| x - x^i \| < {}^1/_2 \, \delta(x^i)$ and hence $G'(x) \in G'(x^i) + {}^1/_2 \; W$. So if $\| z - x \| < \delta, 0 < \varepsilon < \delta$, then $\| z - x^i \| \leqslant \| z - x \| + \| x - x^i \| < \delta(x^i)$ and we have $G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon (G'(x^i) + {}^1/_2 \; W) + M \subset \varepsilon (G'(x) + W) + M$.

Lemma 1.2. If G' is an M-derivative of G at $\bar{x} = H(\bar{q})$ and if H' is a $\{0\}$-derivative of H at $\bar{q}$ then $G'H'$ is an M-derivative of GH at $\bar{q}$.

Proof. Since H' is a $\{0\}$-derivative of H, H' is an affine mapping and hence $G'H'$ is an M-convex mapping. Now, if $q \in Q$ and W is a ball in R^k, then there is $\delta > 0$ such that $G(\bar{x} + \varepsilon z) - G(\bar{x}) \in \varepsilon (G'(H'(q) + W) + M$ whenever $\| z - H'(q) \| < \delta, 0 < \varepsilon < \delta$. On the other hand, there is $\eta > 0$ such that $\| H(\bar{q} + \varepsilon p) - H(\bar{q}) - \varepsilon H'(q) \| < \varepsilon \, \delta$ whenever $\| p - q \| < \eta$, $0 < \varepsilon < \eta$, and we may always suppose $\eta < \delta$. Then, for $\| p - q \| < \eta$, $0 < \varepsilon < \eta$, we have $GH(\bar{q} + \varepsilon p) - GH(\bar{q}) \in \varepsilon (G'H'(q) + W) + M$.

Given a set E in R^k, we shall say that a convex set E' in R^k is a *convex approximation* of E (at 0) if to every triple (P, ε, W), where P is a h-simplex in $E'(h \leqslant k)$, $0 < \varepsilon < \delta$, with δ depending possibly on P, and W is a ball in R^k one can associate an upper semi-continuous mapping $\zeta: P \to 2^E$ such that for every $y \in P$ the set $\zeta(y) \cap \varepsilon(y + W)$ is convex and nonempty.

Lemma 1. 3. Let $\widetilde{G}: V \to R^k$, $\widetilde{G}': Q \to R^k$ be two mappings. If (III) holds, if $\widetilde{G}$ is continuous, $\widetilde{G}(\overline{q}) \in M$, and if for every finite-dimensional subspace L' of Q the restriction of $\widetilde{G}'$ to L' is continuous and is an M-derivative of the restriction of $\widetilde{G}$ to $L = L' + \overline{q}$, then the set $E' = \widetilde{G}(\overline{q}) + \widetilde{G}'(C') - M$ is a convex approximation of $E = \widetilde{G}(C \cap V) - M$.

Proof. Consider a h-simplex $P = [\overset{0}{y}, \ldots, y^h]$ in E' $(h \leqslant k)$ and a ball W in R^k. Select $q^i \in C'$ such that $y^i \in \widetilde{G}(\overline{q}) + G'(\widetilde{q}^i) - M$ and let S be the convex hull of $q^o, \ldots, q^h$. From Lemma 1. 1 (applied to the resrictions of $\widetilde{G}$ and $\widetilde{G}'$ to L, $L' = L - \overline{q}$ resp., where L is the linear manifold through $\overline{q}$ and S), there is a number $\delta > 0$ such that

$$q \in S,\ 0 < \varepsilon < \delta \Rightarrow \widetilde{G}(\overline{q} + \varepsilon\, q) \in \widetilde{G}'(\overline{q}) + \varepsilon\, \widetilde{G}'(q) + {}^1/_2\, W) + M.$$

Since $\overline{q}$ is an internal point of V, one can assume δ to be so small that $\overline{q} + \delta S \subset V$. Then, using the continuity of $\widetilde{G}$ and the compactness of $\overline{q} + \delta S$, one can find for every ε a number $\eta = \eta(\varepsilon) > 0$ such that

$$q \in S,\ q' \in V,\ 0 < \varepsilon < \delta,\ \| q' - (\overline{q} + \varepsilon\, q) \| < \varepsilon\, \eta \Rightarrow \widetilde{G}(q') \in \widetilde{G}(\overline{q} + \varepsilon q) + \frac{\varepsilon}{2} W.$$

On the other hand, using condition (III) and assuming, accordingly, δ to be sufficiently small, one can associate to ε, η a continuous mapping $\xi: S \to C \cap V$ satisfying (3). Then for every $q \in S$ we have

$$\widetilde{G}(\xi(q)) \in \widetilde{G}(\overline{q} + \varepsilon q) + \frac{\varepsilon}{2}\, W \subset \widetilde{G}(\overline{q}) + \varepsilon\, (\widetilde{G}'(q) + W) + M.$$

Let us define for every $y \in P$ with $y = \Sigma\, t_i y^i$, $t_i \geqslant 0$, $\Sigma\, t_i = 1$:

$$\zeta(y) = = \widetilde{G}(\xi(\Sigma\, t_i q^i)) - M \subset E. \tag{7}$$

Then, noting that $\widetilde{G}'(\Sigma\, t_i q^i) \in \Sigma\, t_i \widetilde{G}'(q^i) + M \subset \Sigma\, t_i y^i - \widetilde{G}(\overline{q}) + M = y - \widetilde{G}(\overline{q}) + M$ (because $\widetilde{G}'$ is M-convex), we get from the above

$\widetilde{G}(\xi(\Sigma t_i q^i)) \in \varepsilon\,(y + W) + (1 - \varepsilon)\,\widetilde{G}\,(\overline{q}) + M \subset \varepsilon\,(y + W) + M$, and hence, $\zeta(y) \cap \varepsilon\,(y + W) \neq \emptyset$. This proves the Lemma, since the upper semi-continuity of ζ is ensured by the continuity of G and ξ.

Lemma 1.4. If E' is a convex approximation of E, then $0 \in \operatorname{int} E'$ implies $0 \in \operatorname{int} E$.

Proof. Assume $0 \in \operatorname{int} E'$ and let P be a k-simplex in E' containing 0 in its interior, W a ball such that $W - W \subset P$. Then there exist $\varepsilon > 0$ and a mapping $\zeta : P \to 2^E$ with the properties specified in the definition of a convex approximation. Replacing if necessary $\zeta\,(y)$ by $\zeta(y) \cap \varepsilon(y + W)$, we can always assume the set $\zeta\,(y)$ to be convex for every $y \in P$. Let us consider now an arbitrary element w of W and let us define a mapping $\pi : P \to 2^P$ by setting $\pi\,(y) = P \cap (y + w - \frac{1}{\varepsilon}\zeta\,(y))$. Then $\pi\,(y)$ is nonempty because from $\zeta\,(y) \cap \varepsilon\,(y + W) \neq \emptyset$ we have $y + w' \in \frac{1}{\varepsilon}\,\zeta\,(y)$ for some $w' \in W$ and, consequently, $u \in y + w - \frac{1}{\varepsilon}\,\zeta\,(y)$ with $u = w - w' \in W - W \subset P$. Furthermore, $\pi\,(y)$ is convex and π is upper semi-continuous, because if $u^n \in \pi\,(y^n), u^n \to u^0,\ y^n \to y^0$, then $y^n + w - u^n \in \frac{1}{\varepsilon}\ \zeta\,(y^n)$ and so, by the upper semi-continuity of ζ, $y^0 + w - u^0 \in \frac{1}{\varepsilon}\ \zeta\,(y^0)$, which means $u^0 \in \pi\,(y^0)$. Hence, by Kakutani's fixed point theorem, one can find an element $y \in P$ such that $y + w - y \in \frac{1}{\varepsilon}\ \zeta\,(y)$, i. e. $\varepsilon w \in \zeta\,(y) \subset E$. Thus, $\varepsilon\,W \subset E$, proving that $0 \in \operatorname{int} E$.

Proof of Theorem 1. If (I), (II) hold, then, according to the implicit function theorem, for every finite-dimensional linear manifold L in Q passing through $\overline{q}$, the restriction of H to $V \cap L$ has a derivative equal to

$$H' = -\overline{K}_x^{-1}\,\overline{K}_q : L - \overline{q} \to X.$$

Morever, if L_1, L_2 are two such manifolds and H'_1, H'_2 are the corresponding derivatives, then H'_1, H'_2 must agree on $(L_1 - \overline{q}) \cap (L_2 - \overline{q})$, because of the uniqueness of the Frechet derivative. Therefore, one can speak of a linear mapping $H' : Q \to X$ defined on all of Q.

But it is obvious that the restriction of H' to any $L - \overline{q}$ is a $\{0\}$ - derivative of the restriction of H to L, at point $\overline{q}$, and hence, according to Lemma 1.2, the restriction of $\widetilde{G}' = G'H'$ to $L - \overline{q}$ is an M - derivative of the restriction of $\widetilde{G} = GH$ to L at $\overline{q}$. Since M' is linear, it is continuous on $L - \overline{q}$. Therefore, by Lemma 1.3 the set $G(\bar{x}) + G' H' (C') - M$ is a convex approximation of $GH(C \cap V) - M$. Now, if system (1) is critical, then $0 \bar{\in} \text{int} (GH (C \cap V) - M)$, because $H(C \cap V) \subset D$. Hence, by Lemma 1.4

$$0 \bar{\in} \text{int}(G(\bar{x}) + G'H'(C') - M),$$

and, since the latter set is convex (because $G'H'$ is M-convex), there is by the separation theorem a nonzero vector Λ such that

$$(\forall q \in C')(\forall y \in M) \qquad \langle \Lambda, G(\bar{x}) + G'H'(q) - y \rangle \leqslant 0.$$

This implies by a standard argument $\Lambda \in M^*$ and $\langle \Lambda, G(\bar{x}) + G'H'(q) \rangle \leqslant 0$ for all $q \in C'$. Setting $q = 0$, we have $\Lambda G(\bar{x}) \leqslant 0$, hence (6) follows, because the converse inequality is implied by $G(\bar{x}) \in M$. Since $H' = -\bar{K}_x^{-1} \bar{K}_q$, we thus obtain (5). Q. E. D.

2. Variational sets of differential equations.

We now show that the basic assumptions (I), (II), (III) of section 1 are satisfied for a large class of optimal control problems.

Let Ω be a bounded open set in R^n. We shall denote by $\mathcal{B}(\Omega, R^n)$ the linear space of all mappings $f(\sigma, t)$ from $\bar{\Omega} \times [0,1]$ into R^n suchthat $f(\sigma, t)$ is continuously differentiable with respect to σ, $f(\sigma, t)$ and the matrix $f_\sigma(\sigma, t)$ of first partial derivatives of $f(\sigma, t)$ with respect to σ are measurable with respect to t, and there exists for each f an integrable function $m(t)$ satisfying

$$(\forall \sigma \in \bar{\Omega})(\forall t \in [0,1]) \quad |f(\sigma, t)| + |f_\sigma(\sigma, t)| \leqslant m(t). \tag{8}$$

In this space two elements f^1, f^2 are regarded as being equivalent if for every $\sigma \in \bar{\Omega}$, $f^1(\sigma, t) = f^2(\sigma, t)$ for almost all $t \in [0,1]$. It is easily seen that $\mathcal{B}(\Omega, R^n)$is a normed space, with

$$\| f \| = \sup \left\{ \left| \int_{t'}^{t''} f(\sigma, t)\, dt \right| : \sigma \in \Omega,\ t', t'' \in [0,1] \right\}. \tag{9}$$

Let $Q = R^n \times \mathcal{B}(\Omega, R^n)$, $X = \mathcal{C}^{(n)}$, the Banach space of continuous mappings $x(t) : [0,1] \to R^n$, with the usual norm $\|x\| = \max \{ |x(t)| : 0 \leqslant t \leqslant 1 \}$

Let F be a given set in $\mathcal{B}(\Omega, R^n)$. We shall be concerned with the system

$$(x, q) \in A \times C,\ K(x, q) \overset{def}{=} x(t) - \omega - \int_0^t f(x(s), s)\, ds = 0. \tag{10}$$

where A is the set of all $x(.) \in \mathcal{C}^{(n)}$ such that $(\forall t)\ x(t) \in \Omega$, $C = \Omega \times F$, the set of all $q = (\omega, f)$ with $\omega \in \Omega$, $f \in F$.

Following Halkin and Neustadt [2], we shall say that the set F is *convex-under-switching* if: whenever $f^1, f^2 \in F$ and $\theta \in (0, 1)$, then $\chi_{[0,\theta]} f^1 + \chi_{(\theta,1]} f^2 \in F$, where χ_E is the characteristic function of E, i. e. $\chi_E(t) = 1$ if $t \in E$ and $\chi_E(t) = 0$ if $t \bar{\in} E$.

Let $(\overline{x,q}) = (\bar{x}, (\bar{\omega}, \bar{f}))$ be a solution of (10), so that

$$\frac{d\bar{x}(t)}{dt} = \bar{f}(\bar{x}(t), t) \text{ for almost all } t \in [0,1], \qquad \bar{x}(0) = \bar{\omega}.$$

THEOREM 2. *Whatever may be* F, *conditions* (I) *and* (II) *are always satisfied for the system* (10). *If* F *is convex-under-switching, condition* (III) *is satisfied as well.*

The proof of this theorem will proceed in several steps.

I. First let us show that condition (I) is satisfied. Consider an arbitrary finite-dimensional linear manifold L in Q, passing through $\bar{q} = (\bar{\omega}, \bar{f})$. Since $K(x, q)$ is linear in q, the restriction of K to $X \times L$ has at each point (x, q) a partial derivative, with respect to q, equal to

$$K_q(x(.), q) : \delta q = (\delta\omega, \delta f) \mapsto -\delta\omega - \int_0^t \delta f(x(s), s)\, ds. \tag{11}$$

Further, a short computation yields

$$K_x(x(.), q) : \delta x(.) \mapsto \delta x(t) - \int_0^t f_\sigma(x(s), s)\, \delta x(s)\, ds. \tag{12}$$

It is not hard to see that both $K_q(x, q)$ and $K_x(x, q)$ depend continuously on $(x, q) \in \mathcal{C}^{(n)} \times L$. Indeed, if $\{q^i = (\omega^i, f^i), i = 1,\dots, m\}$ is a basis of $L - \bar{q}$, then we have, $q = \Sigma\lambda_i q^i$,

$$\| K_q(\hat{x}, \hat{q}) . q - K_q(\tilde{x}, \tilde{q}) . q \| \leqslant \Sigma \mid \lambda_i \mid \int_0^1 \mid f^i(\hat{x}(s), s) - f^i(\tilde{x}(s), s) \mid ds$$

and an analogous argument holds for K_x. It follows then from a well known theorem of analysis (see e. g. [5]), that the restriction of K to $A \times L$ is continuously differentiable. To complete the proof of (I) it remains thus to establish the following

Lemma 2. 1. The mapping

$$\bar{K}_x = K_x(\bar{x}(.), \bar{q}) : y(.) \mapsto y(t) - \int_0^t \bar{f}_\sigma(\bar{x}(s), s) . y(s)\, ds \tag{13}$$

is a bijection from $\mathcal{C}^{(n)}$ onto itself and we have

$$\bar{K}_x^{-1} : z(.) \mapsto z(t) + \Gamma(t) \int_0^t \Gamma^{-1}(s) . \bar{f}_\sigma(\bar{x}(s), s) . z(s)\, ds, \tag{14}$$

where $\Gamma(t)$ is the matrix-valued function that satisfies the system

$$\dot{\Gamma}(t) = \bar{f}_\sigma(\bar{x}(t), t) . \Gamma(t), \qquad \Gamma(0) = \text{the identity}. \tag{15}$$

Proof. For every given $z(.) \in \mathcal{C}^{(n)}$, there exists an $y(.)$ satisfying the equation $\overline{K}_x . y(.) = z(.)$, namely

$$y(t) = z(t) + \Gamma(t)\int_0^t \Gamma^{-1}(s) . \overline{f}_\sigma(\overline{x}(s), s) . z(s)\, ds.$$

Indeed, we have

$$\int_0^t \overline{f}_\sigma(\overline{x}(s), s) . \Gamma(s)\left(\int_0^s \Gamma^{-1}(\tau) . \overline{f}_\sigma(\overline{x}(\tau), \tau) . z(\tau)\, d\tau\right) ds =$$

$$= \int_0^t \Gamma'(s)\left(\int_0^s \Gamma^{-1}(\tau) . \overline{f}_\sigma(\overline{x}(\tau)) . z(\tau)\, d\tau\right) ds =$$

$$= \Gamma(t)\int_0^t \Gamma^{-1}(s) . \overline{f}_\sigma(\overline{x}(s), s) . z(s)\, ds - \int_0^t \Gamma(s)\,[\Gamma^{-1}(s)\overline{f}_\sigma(\overline{x}(s), s)]\, z(s)\, ds$$

$$= \Gamma(t)\int_0^t \Gamma^{-1}(s) . \overline{f}\ (\overline{x}(s), s) . z(s)\, ds - \int_0^t \overline{f}_\sigma(\overline{x}(s), s) . z(s)\, ds,$$

hence

$$\int_0^t f_\sigma\ (x(s), s)\ y(s)\, ds = y(t) - z(t),$$

i.e. $y(.)$ is a solution of the equation $\overline{K}_x . y(.) = z(.)$. This solution is unique, since by Gronwall's lemma the equality $\overline{K}_x . y(.) = 0$, i.e.

$$y(t) - \int_0^t \overline{f}_\sigma(\overline{x}(s), s)\ y(s)\, ds = 0$$

implies $y(t) \equiv 0$. Thus, Lemma 2.1, and hence, Condition (I), holds.

II. Let us now proceed with *Condition* (II). To prove this condition is satisfied, we need only establish the following proposition on the dependence on parameters of solutions of differential equations.

For every $\alpha > 0$ let Q_α denote the set of all $q = (\omega, f) \in \Omega \times \mathcal{B}\ (\Omega, R^n)$ such that

$$|\ \omega - \overline{\omega}\ | \leqslant \alpha, \tag{16}$$

$$(\forall x \in \Omega) \quad \int_0^t (|\ f(x, t) - \overline{f}\ (x, t)\ | + |\ f_\sigma(x, t) - \overline{f}_\sigma(x, t)\ |)\ dt \leqslant \alpha \tag{17}$$

Lemma 2. 2. If α is sufficiently small, then for every $q \in Q_\alpha$ the equation (10) has an unique solution $x(.) = H(q)$, and the mapping $N: Q_\alpha \to A$ is continuous.

Proof. Consider the mapping $J : \mathcal{C}^{(n)} \times Q \to \mathcal{C}^{(n)}$ defined by

$$J(x(.), q) = \Gamma(t) \int_0^t \Gamma^{-1}(s) \left[x(s) - \omega - \int_0^s f(x(\tau), \tau), d\tau \right] ds +$$

$$+ \omega + \int_0^t f(x(s), s)\, ds \tag{18}$$

It is easy to see that for every fixed q:

$$K(x(.), q) = 0 \Leftrightarrow x(.) = J(x(.), q).$$

Indeed, if $x(.) = J(x(.), q)$, then

$$\Gamma(t) \int_0^t \Gamma^{-1}(s)\, y(s)\, ds - y(t) = 0$$

with $y(t) = x(t) - \omega - \int_0^t f(x(s), s)\, ds$. Applying Gronwall's lemma yields

$|\Gamma^{-1}(t) y(t)| = 0$, hence $y(t) = 0$ for all t. Conversely, if $K(x(.), q) = 0$, then obviously, $x(.) = J(x(.), q)$.

Also observe that J can be rewritten as

$$J(x(.), q) = \Gamma(t) \int^t \Gamma^{-1}(s)\, f(x(s), s) - \bar{f}_\sigma(\bar{x}(s), s)\, (x(s) - \omega)\, ds + \omega \tag{19}$$

Indeed, integrating by parts yields

$$\int_0^t \Gamma^{-1}(s) \left(\int_0^s f(x(\tau), \tau)\, d\tau \right) ds = - \int_0^t \Gamma^{-1}(s) f(x(s), s)\, ds +$$

$$+ \Gamma^{-1}(t) \int_0^t f(x(s), s)\, ds,$$

hence

$$J(x(.), q) = \Gamma(t) \int_0^t \Gamma^{-1}(s)\, (x(s) - \omega)\, ds + [\Gamma(t) \int_0^t \Gamma^{-1}(s)\, f(x(s), s)\, ds$$

Hence, for every t:

$$|J(x(t), q) - \bar{x}(t)| = |J(x(t), q) - J(\bar{x}(t), \bar{q})| \leqslant$$
$$\leqslant |J(x(t), q) - J(\bar{x}(t), q)| + |J(\bar{x}(t), q) - J(\bar{x}(t), \bar{q})| \leqslant$$
$$\leqslant \gamma \int_0^t [r_1(x, \bar{x}, q, s) + r_2(x, \bar{x}, q, s)]\, ds + \alpha(\gamma + \gamma\beta + 1). \qquad (25)$$

Now we observe that, because of (8) and the continuity of $\bar{f}_x(x, t)$ with respect to x,

$$\int_0^1 |\bar{f}_\sigma(\bar{x}(s) + h, s) - \bar{f}_\sigma(x(s), s)|\, ds \to 0 \text{ as } h \to 0.$$

Consequently, one can take $\varepsilon \in (0,1)$ to be so small that whenever $\| x - \bar{x} \| = < \varepsilon$ we have:

$$x(.) \in A, \int_0^1 |\bar{f}_\sigma(x(s), s) - \bar{f}_\sigma(\bar{x}(s), s)|\, ds < \frac{1}{8\gamma}. \qquad (26)$$

Let $\alpha > 0$ be sufficiently small to ensure $8\alpha(\gamma + \gamma\beta + 1) < \varepsilon$. Then for every $x(.) \in A_\varepsilon$, $q \in Q_\alpha$ we can write, according to (22) (23) (24) (25):

$$\| J(x(.), q) - \bar{x} \| \leqslant \gamma \| x - \bar{x} \| \left(3\alpha + \frac{1}{8\gamma}\right) + \alpha(\gamma + \gamma\beta + 1) <$$
$$< \gamma\varepsilon\left(\frac{3}{8\gamma} + \frac{1}{8\gamma} + \frac{\varepsilon}{8}\right) < \varepsilon,$$

which means that J maps $A_\varepsilon \times Q_\alpha$ into A_ε. On the other hand, from (22) (23) (24) we get for any $x^1(.)$, $x(.) \in A_\varepsilon$ and $q \in Q_\alpha$:

$$\| J(x^1, q) - J(x, q) \| \leqslant \gamma \| x^1 - x \| \left(3\alpha + \frac{3}{8\gamma}\right) < \frac{3}{4} \| x^1 - x \|,$$

which is the desired relation (21).

Thus, provided α and ε be small enough, $J(A_\varepsilon \times Q_\alpha) \subset A_\varepsilon$ and (21) holds for every fixed $q \in Q_\alpha$. If now we can show that for every fixed $x \in A_\varepsilon$ the mapping $q \mapsto J(x, q)$ is continuous on Q_α, then by a well known result (see e. g. [5], theorem 46_2), the mapping J has for each $q \in Q_\alpha$ an unique fixed point $x \in A_\varepsilon$, such that $x(.) = H(q)$, where $H: Q_\alpha \to A_\varepsilon$ is a continuous mapping, and the Lemma will be proved.

Therefore, the only thing that remains to be shown is that for every fixed $x(.) \in A_\varepsilon$ the mapping $J(x, .) : Q_\alpha \mapsto \mathcal{C}^{(n)}$ is continuous. But we have obviously, for any $q, q^1 \in Q_\alpha$:

$$\| K(x, q) - K(x, q^1) \| \leqslant |\omega - \omega^1| + \| f - f^1 \|,$$

so that the mapping $q \mapsto K(x, q)$ is continuous. Since from (18) (14) (20) we can easily deduce

$$J(x(.), q) = x(.) - \bar{K}_x^{-1} \cdot K(x(.), q),$$

the continuity of the mapping $q \mapsto J(x, q)$ follows.

Lemma 2.2 shows that Condition (II) holds with $V = Q_\alpha$, for $\alpha > 0$ sufficiently small (the fact that Q_α is convex and contains $\bar{q} = (\bar{\omega}, \bar{f})$ as an internal point is obvious).

III. Finally, let us examine when Condition (III) is satisfied.

We shall say that the set F is *quasiconvex* if for every positive number α and for every integer $k \geqslant 1$ Condition (III) holds with $V = Q_\alpha$, $C' = [\Omega \times F] - \bar{q}$, where $[\Omega \times F]$ is the convex hull of $\Omega \times F$.

The following property is an immediate consequence of the definition of quasiconvexity and the hypothesis that Ω is open in R^n.

The set F is quasiconvex if and only if for every $\alpha > 0$ and every simplex $T = [f^1, \ldots, f^h]$ with vertices $f^1, \ldots, f^h$ in $F - \bar{f}$ there exists a positive number δ such that to every pair (ε, η), where $0 < \varepsilon < \delta$, $\eta > 0$ one can associate a continuous mapping $\theta : T \to F_\alpha$ (the set of all elements of F satisfying (17)) such that

$$(\forall f \in T) \qquad \| \theta(f) - \bar{f} - \varepsilon f \| < \varepsilon\eta. \tag{27}$$

Thus, to prove the second part of Theorem 2, it suffices to show that every convex - under - switching set F is quasiconvex.

Before doing this, we shall establish a fact, the geometric meaning of which is rather simple.

Let $g^0, \ldots, g^h$ be given functions of t, integrable on $[0, 1]$ and let $P^h = \{\lambda = (\lambda_0, \ldots, \lambda_h) : \lambda_i \geqslant 0, \Sigma\lambda_i = 1\}$. With every $\lambda \in P^h$ and every partition π of the interval $[0, 1]$ into subintervals $\Delta_1, \ldots, \Delta_m$ $(m = m(\pi))$ we associate a function g_π^λ in the following way. We divide each Δ_s $(s = 1, \ldots, m)$ into $h + 1$ small intervals Δ_{si} $(i = 0, \ldots, h)$ in such a manner that $|\Delta_{si}| = \lambda_i |\Delta_s|$ ($|\Delta|$ is the length of Δ) and we define

$$g_\pi^\lambda(t) = \sum_{s,i} \chi_{\Delta_{si}}(t) \cdot g^i(t).$$

For any two partitions $\pi = \{\Delta_s\}$ and $\pi' = \{\Delta_{s'}\}$ we write $\pi < \pi'$ whenever each Δ_s is contained in some $\Delta_{s'}$.

Lemma 2.3. Given any system $g^0, \ldots, g^h$ and any $\varepsilon > 0$ there exists a partition π' such that for all $\pi < \pi'$ we have

$$(\forall \lambda \in P^h) \sup_{0 \leqslant t_1, t_2 \leqslant 1} \left| \int_{t_1}^{t_2} \sum_{i=0}^{h} (\lambda_i g^i(t) - g_\pi^\lambda(t))\, dt \right| < \varepsilon \tag{28}$$

Proof. Assume first that each function $g^i(t)$ is piecewise constant and select a partition $\pi' = \{\Delta_{s'}\}$ so small that for all i, s':

$$(\forall t \in \text{int } \Delta_{s'}) \qquad g^i(t) = \text{const} = \gamma_{is'} \tag{29}$$

$$\int_{\Delta_{s'}} | g^i(t) | \, dt = | \gamma_{is'} | \, | \Delta_{s'} | < \frac{\varepsilon}{4(h+1)} \tag{30}$$

If $\pi = \{\Delta_s\}$ is a partition such that $\pi < \pi'$, then, since every Δ_s is contained in some $\Delta_{s'}$, we have, from (29) and (30), for every $\lambda \in P^h$:

$$\int_{\Delta_s} (\sum_{i=0}^{h} \lambda_i g^i(t) - g_\pi^\lambda(t) \, dt = 0,$$

$$\int_{\Delta_s} | \sum \lambda_i g^i(t) - g_\pi^\lambda(t) | \, dt \leqslant 2 \sum \lambda_i \int_{\Delta_s} | g^i(t) | \, dt < \frac{\varepsilon}{2},$$

whence (28) follows.

In the general case, since each g^i is integrable, we can approximate it by a piecewise constant function $\tilde{g}^i$ such that

$$\int_0^1 | g^i(t) - \tilde{g}^i(t) | \, dt < \frac{\varepsilon}{3}.$$

If π' is a partition such that for all $\pi < \pi'$:

$$\sup_{0 \leqslant t_1, t_2 \leqslant 1} \left| \int_{t_1}^{t_2} (\sum_{i=0}^{h} \lambda_i \tilde{g}^i(t) - \tilde{g}_\pi^\lambda(t)) \, dt \right| < \frac{\varepsilon}{3},$$

then it will be the desired partition for $g^0, \ldots, g^h$, as can be easily verified.

Note that the previous lemma is in fact a simplified version of the Approximation Lemma in [1] and relies on the simple idea that $\lambda \, a.b = a.\lambda \, b.$

We now come to the closing part of the proof of Theorem 2.

Lemma 2.4. If the set F is convex-under-switching, it is quasiconvex.

Proof. Let $\alpha > 0$ and $T = [f^1, \ldots, f^h]$ be given ($f^i \in F - \bar{f}$). From the definition of $\mathfrak{B}(\Omega, R^n)$ there exists an integrable function $m(t)$ such that for all $\sigma \in \Omega$, $t \in [0, 1]$:

$$| \bar{f}(\sigma, t) | + | \bar{f}_\sigma(\sigma, t) | \leqslant m(t),$$

$$| f^i(\sigma, t) - \bar{f}(\sigma, t) | + | f^i_\sigma(\sigma, t) - \bar{f}_\sigma(\sigma, t) | \leqslant m(t), \; i = 1, \ldots, h.$$

Let $\delta > 0$ be so small that

$$\text{mes } E < \delta \Rightarrow \int_E m(t) \, dt < \alpha \tag{31}$$

and consider any pair (ε, η), where $0 < \varepsilon < \delta$, $\eta > 0$.

From (8) and the continuity of each function $f^i(\sigma, t)$ with respect to σ, it follows that each integral

$$\int_0^1 f^i(\sigma, t) \, dt, \; i = 1, \ldots, h$$

is a continuous function of σ. Since $\bar{\Omega}$ is a compact set, there exists a finite set $\sigma^1, \ldots, \sigma^r \in \bar{\Omega}$ and a number $\rho > 0$ such that

$$\sigma \in \overline{\Omega}\,,\ |\sigma - \sigma^j| < \rho \Rightarrow \int_0^1 |f^i(\sigma, t) - f^i(\sigma^j, t)|\, dt < \frac{7}{3h} \tag{32}$$

Let $f^0 = 0$. By Lemma 2.3 there exists a partition $\pi = \{\Delta_s\}$ of the interval $[0, 1]$ such that we have for all $j = 1, \dots, r$:

$$(\forall \lambda \in P^h) \quad \sup_{0 \leqslant t', t'' \leqslant 1} \left| \int_{t'}^{t''} (\sum_{i=0}^{h} \lambda_i f^i(\sigma^j, t) - f_\pi^\lambda(\sigma^j, t)\, dt \right| < \frac{\eta}{3}\,, \tag{33}$$

where

$$f_\pi^\lambda(\sigma, t) = \sum_{s,i} \chi_{\Delta_{si}}(t) \cdot f^i(\sigma, t)\,.$$

If $\sigma \in \Omega$, then $|\sigma - \sigma^j| < \rho$ for some j and by (32) we have

$$\int_0^1 | f_\pi^\lambda(\sigma, t) - f_\pi^\lambda(\sigma^j, t) |\, dt \leqslant \sum_{i=1}^{h} \int_0^1 \left| f^i(\sigma, t) - f^i(\sigma^j, t) \right| dt < \frac{\eta}{3}\,,$$

so that, taking (32) and (33) into account, we can write for any t', t'':

$$\left| \int_{t'}^{t''} \left(\sum_{i=0}^{h} \lambda_i f^i(\sigma, t) - f_\pi^\lambda(\sigma, t) \right) dt \right| \leqslant \sum_{i=1}^{h} \lambda_i \int_{t'}^{t''} \left| f^i(\sigma, t) - f^i(\sigma^j, t) \right| dt +$$

$$+ \left| \int_{t'}^{t''} (\Sigma \lambda_i f^i(\sigma^j, t) - f_\pi^\lambda(\sigma^j, t))\, dt \right| + \int_{t'}^{t''} \left| f_\pi^\lambda(\sigma^j, t) - f_\pi^\lambda(\sigma, t) \right| dt < \eta \tag{34}$$

Define now the mapping θ by setting for every $f = \sum_{i=1}^{h} \mu_i f^i(\sigma, t)$ with $\mu_i \geqslant 0$, $\Sigma \mu_i = 1$:*

$$\theta(f) = \overline{f}(\sigma, t) + f_\pi^\lambda(\sigma, t)\,, \text{ where } \lambda = (1 - \varepsilon, \varepsilon\mu_1, \dots, \varepsilon\mu_h).$$

It is easily seen that $\theta : T \rightarrow F_\alpha$. Indeed, $\theta(f) \in F$ because F is convex-under-switching; furthermore, if we denote by E the set of all t for which

$$\theta(f)[\sigma, t] - \overline{f}(\sigma, t) \neq 0 \text{ at least at one } \sigma \in \Omega\,,$$

* If T is not a $(h-1)$ — simplex, we can triangulate it in such a way that all vertices of the triangulation are among $f^1, \dots, f^h$. Then every $f \in T$ belongs to the relative interior of an uniquely determined subsimplex $[f^i, i \in I]$ with $I = I(f) \subset \{1, \dots, h\}$; hence $f = \sum_i \mu_i f^i$ with $\mu_i \geqslant 0$, $\sum_i \mu_i = 1$, $\mu_i = 0\ (i \,\overline{\in}\, I)$ and μ_i are uniquely determined. These are just the μ_i we have in mind here.

then mes $E = \varepsilon < \delta$, and so by (31)

$$\int_0^1 (| \theta(f)[\sigma, t] - \bar{f}(\sigma, t) | + | \theta(f)_\sigma [\sigma, t] \bar{f}_\sigma(\sigma, t) |) \, dt \leqslant \alpha,$$

which means that $\theta(f) \in F_\alpha$.

The condition (27) is easy to check, too. Indeed, we have for all $\sigma \in \Omega$, $t', t'' \in [0, 1]$:

$$\left| \int_{t'}^{t''} (f_\pi^\lambda(\sigma, t) - \varepsilon f(\sigma, t)) \, dt \right| =$$

$$\left| \int_{t'}^{t''} (f_\pi^\lambda - (1-\varepsilon) f^0 - \varepsilon \sum_{i=1}^{h} \mu_i f^i) \, dt \right| = \left| \int_{t'}^{t''} (f_\pi^\lambda - \sum_{i=0}^{h} \lambda_i f^i) \, dt \right| < \eta$$

(see (34)), hence (27) follows.

Thus, it remains to prove the continuity of θ. To do this, we observe that for $f = \sum_{i=1}^{h} \mu_i f^i$, $f' = \sum_{i=1}^{h} \mu'_i f^i$, with $\mu, \mu' \in P^{h-1}$, if we denote by Δ_σ the set of all t such that $\theta(f)[\sigma, t] \neq \theta(f')[\sigma, t]$, then $\Delta_\sigma \subset \Delta$ for all $\sigma \in \Omega$, where Δ is some set with mes $\Delta \to 0$ when $|\mu - \mu'| \to 0$. But from the choice of $m(t)$ it follows that

$$\| \theta(f) - \theta(f') \| \leqslant | \theta(f) - \bar{f}(x, t) | + | \bar{f}(x, t) - \theta(f') | \leqslant 2m(t)$$

for all $\sigma \in \Omega$, $0 \leqslant t \leqslant 1$. Therefore,

$$\| \theta(f) - \theta(f') \| = \sup \left\{ \left| \int_{t_1}^{t_2} [\theta(f) - \theta(f')] \, dt \right| : \sigma \in \Omega, \ 0 \leqslant t_1, t_2 \leqslant 1 \right\} \leqslant 2 \int_\Delta m(t) \, dt \to 0$$

when mes $\Delta \to 0$. Consequently, $\| \theta(f) - \theta(f') \| \to 0$ when $|\mu - \mu'| \to 0$, as was to be shown.

The proof of Theorem 2 is complete.

Remark. By an argument almost identical to that used for the proof of lemma 1.3, one could prove the following proposition: if assumptions (I) (II) (III) of section 1 are satisfied, then for every h-simplex S in $H'(C')$ ($h \leqslant k$) and for every $\varepsilon \in [0,1]$ there exists a continuous mapping $\zeta : S \to D - \bar{x}$ such that

$$(\forall \, x \in S) \qquad \zeta(x) - \varepsilon x = o(\varepsilon),$$

where $o(\varepsilon)/\varepsilon \to 0$ uniformly with respect to all $x \in S$.

From this property and Theorem 2 it follows that: a set F in $\mathcal{B}(\Omega, R^n)$ which is convex-under-switching is a *variational set* in the sense introduced in [2] (the sets D and $H'(\Omega \times F - q)$ can be easily identified with L and $N - \bar{x}$ resp. in [2]. We thus obtain a new simple proof of the basic result in [2], under somewhat weaker assumptions than in [2]

3. The maximum principle for optimal control problems.

In this section we shall derive from the previous results the classical Pontryagin maximum principle for optimal control problems.

We continue using the same notations and assumptions as before.

Consider the system (10) and let there be given, in addition, a continuously differentiable mapping $p: \Omega \times \Omega \to R^{m+1}$ and a closed convex cone N in R^m. Let $p = (p_0, \ldots, p_m)$. A typical optimal control problem is the following: find an absolutely continuous n-vector function $x(t)$ such that:

1) for some $q = (\omega, f) \in C = \Omega \times F$ we have (10), i. e.

$\dot{x}(t) = f(x(t), t)$ for almost all $t \in [0,1]$; $x(0) = \omega$;

2) $(p_1(x(0), x(1)), \ldots, p_m(x(0), x(1)) \in N$;

3) $p_0(x(0), x(1))$ achieves a minimum.

THEOREM 3. *Assume F to be a quasiconvex family. If $\bar{x}(t)$ is an optimal solution, corresponding to $\bar{q} = (\bar{\omega}, \bar{f})$, then there exist an absolutely continuous n-vector valued function $\bar{\psi}(t)$ satisfying the maximum principle*

$$\int_0^1 \bar{\psi}(t)\, \bar{f}(\bar{x}(t), t)\, dt = \max_{f \in F} \int_0^1 \bar{\psi}(t)\, f(\bar{x}(t), t)\, dt \tag{34}$$

and a nonzero $(m+1)$-vector $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, \ldots, \lambda_m) \in N^*$ and

$$\frac{d\bar{\psi}}{dt} = -\bar{\psi}(t)\, \bar{f}(\bar{x}(t), t), \tag{35}$$

$$\bar{\psi}(0) = -\Lambda \bar{p}_{\sigma_0}, \quad \bar{\psi}(1) = \Lambda \bar{p}_{\sigma_1}, \tag{36}$$

$$\sum_{i=1}^{m} \lambda_i\, p_i(\bar{x}(0), \bar{x}(1)) = 0, \tag{37}$$

where $\bar{p}_{\sigma_0}, \bar{p}_{\sigma_1}$ denote the partial derivatives of $p(\sigma_0, \sigma_1): \Omega \times \Omega \to R^{m+1}$ at point $(\bar{x}(0), \bar{x}(1))$, with respect to σ_0 and σ_1 respectively.

Proof. Before proceeding to the proof, it is worthwhile noticing that the present theorem differs from the standard formulation of the maximum principle by the presence of the conditions (37) and $(\lambda_1, \ldots, \lambda_m) \in N^*$. In fact, in the standard formulation, $N = \{0\} \subset R^m$, so that these conditions are automatically fulfilled.

In order to be able to apply Theorem 1, let us verify that all conditions (I) through (IV) in section 1 are satisfied. Indeed, if A, C, K are defined as in section 2, then conditions (I) (II) (III) hold by Theorem 2, with $C' = [\Omega \times F] - \bar{q}$ and $V = Q_\alpha$ for α sufficiently small. If now we let $k = m + 1$, $M = R^1_- \times N$ (R^1_- denoting the set of nonpositive numbers),

$G: x(.) \mapsto p(x(0), x(1)) - (y_0, 0, \ldots, 0)$ with $y_0 = p_0(x(0), x(1))$,
$G': x(.) \mapsto \bar{p}_{\sigma_0} \cdot x(0) + \bar{p}_{\sigma_1} \cdot x(1)$,
then it is easily seen that G' is an M-derivative of G at $\bar{x}(.)$, i. e. condition (IV) holds, too. On the other hand, if $\bar{x}(.)$ is optimal, then the system (1), with A, C, K, G, M, as just indicated, is critical, since otherwise there would exist $x(.)$, $q = (\omega, f)$ verifying (10), such that $G(x(.)) \in \text{int}(R^1_- \times N)$, hence $(p_1(x(0), x(1)), \ldots, p_m(x(0), x(1))) \in N$, $p_0(x(0), x(1)) < \bar{p}_0(\bar{x}(0), \bar{x}(1))$, contrary to the optimality of $\bar{x}$. Consequently, by Theorem 1, there exists a nonzero vector $\Lambda = (\lambda_0, \ldots, \lambda_m)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, \ldots, \lambda_m) \in N^*$ and

$$(\forall q = (\omega, f) \in \Omega \times F) \qquad -\Lambda G' \bar{K}_x^{-1} \bar{K}_q (q - \bar{q}) \leqslant 0 \tag{39}$$

$$\sum_{i=1}^{m} \lambda_i p_i(\bar{x}(0), \bar{x}(1)) = 0. \tag{40}$$

But we have from (14) and (11):

$$-\bar{K}_x^{-1} \bar{K}_q \cdot q = \Gamma(t)\left[\omega + \int_0^t \Gamma^{-1}(s) f(\bar{x}(s), s)\, ds\right],$$

where $\Gamma(t)$ is the absolutely continuous matrix-valued function satisfying (15) Therefore, for all $\omega \in \Omega$, $f \in F$:

$$\Lambda \{\bar{p}_{\sigma_0}(\omega - \bar{\omega}) + \bar{p}_{\sigma_1} \Gamma(1)[\omega - \bar{\omega} + \int_0^t \Gamma^{-1}(s)(f(\bar{x}(s), s) - \bar{f}(\bar{x}(s), s))\, ds]\} \leqslant 0 \tag{41}$$

Letting $\omega = \bar{\omega}$, we get

$$(\forall f \in F) \quad \Lambda \bar{p}_{\sigma_1} \Gamma(1) \int_0^1 \Gamma^{-1}(s)[\bar{f}(\bar{x}(s), s)\, \bar{f}(\bar{x}(s), s)]\, ds \leqslant 0,$$

which means that the maximum relation (34) holds if we set

$$\bar{\psi}(t) = \Lambda \bar{p}_{\sigma_1} \Gamma(1) \Gamma^{-1}(t). \tag{42}$$

Then from (42) and (20) we have (35). On the other hand, letting $f = \bar{f}$ in (41), we can write

$$(\forall \omega \in \Omega) \qquad \Lambda[\bar{p}_{\sigma_0} + \bar{p}_{\sigma_1} \Gamma(1)](\omega - \bar{\omega}) \leqslant ,$$

which yields, since Ω is open,

$$\Lambda[\bar{p}_{\sigma_0} + \bar{p}_{\sigma_1} \Gamma(1)] = 0.$$

Combining this with (4.), where we let $t = 0$ and $t = 1$ resp. we obtain (39). This concludes the proof of our theorem.

In the previous problem the initial and terminal times are fixed. A more general problem in which these are *free* could be formulated as follows.

Let there be given, as before, a bounded open set Ω in R^n and a set F in $\mathcal{B}(\Omega, R^n)$, and, in addition, a closed convex cone N in R^m and a continuously differentiable mapping $p(\sigma_0, \sigma_1, t_0, t_1): \Omega \times \Omega \times I \times I \to R^{m+1}$ where $I = (0,1)$. It is required to find a pair (t_0, t_1) and an absolutely continuous n-vector function $x(t)$: $[0,1] \to \Omega$, such that:

1) for some $q = (\omega, f) \in \Omega \times F$ we have

$$\dot{x}(t) = f(x(t), t) \text{ for almost all } t \in [0,1];\ x(0) = \omega;$$

2) $(p_6(x(t_0), x(t_1), t_0, t_1), \dots, p_m(x(t_0), x(t_1), t_0, t_1)) \in N$;

3) $p_0(x(t_0), x(t_1), t_0, t_1)$ achieves a minimum.

THEOREM 4. *Assume the set F to be quasiconvex. If $\overline{x}(t), \overline{t}_0, \overline{t}_1$, is an optimal solution, corresponding to $\overline{q} = (\overline{\omega}, \overline{f})$, then there exists an absolutely continuous n-vector valued function $\overline{\psi}(t)$ satisfying the maximum principle*

$$\int_{\overline{t}_0}^{\overline{t}_1} \overline{\psi}(t) \overline{f}(\overline{x}(t), t)\, dt = \max_{f \in F} \int_{\overline{t}_0}^{\overline{t}_1} \overline{\psi}(t)\ f(\overline{x}(t), t)\, dt \tag{43}$$

and a nonzero $(m+1)$-vector $\wedge = (\lambda_0, \lambda_1, \dots, \lambda_n)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, \dots, \lambda_m) \in N^*$ and

$$\frac{d\overline{\psi}}{dt} = -\overline{\psi}(t) \overline{f}_\sigma(\overline{x}(t), t), \tag{44}$$

$$\overline{\psi}(\overline{t}_0) = -\wedge \overline{p}_{\sigma_0},\ \overline{\psi}(\overline{t}_1) = \wedge \overline{p}_{\sigma_1}, \tag{45}$$

$$\overline{\psi}(\overline{t}_0) \overline{f}_0 = \wedge \overline{p}_{t_0},\ \overline{\psi}(\overline{t}_1) \overline{f}_1 = -\wedge \overline{p}_{t_1}, \tag{46}$$

$$\sum_{t=1}^{m} \lambda_i p_i(\overline{x}(\overline{t}_0), \overline{x}(\overline{t}_1), \overline{t}_0, \overline{t}_1) = 0. \tag{47}$$

where $\overline{f}_j = \overline{f}(\overline{x}(\overline{t}_j), t_j)$, $j = 0, 1$.

Proof. Let $\Delta = I \times I$ (where $I = (0, 1)$), $\widetilde{x} = (\theta, x) \in \Delta \times \mathcal{E}^{(n)}$, $\widetilde{q} = (\alpha, q) \in \Delta \times Q$, $\widetilde{A} = \Delta \times A$, $\widetilde{C} = \Delta \times C$, $M = R^1_- \times N \subset R^{m+1}$,

$$\widetilde{K} : (\widetilde{x}, \widetilde{q}) = ((\theta, x), (\alpha, q)) \mapsto (\theta - \alpha, K(x, q)) \tag{48}$$

(where $K(x, q)$ is defined by (10)),

$$\widetilde{G} : \widetilde{x} = (t_0, t_1, x) \mapsto p(x(t_0), x(t_1), t_0\ t_1) - (y_0, 0, \dots, 0),$$

with $y_0 = p_0(x(t_0), x(t_1), t_0, t_1)$.

$$\widetilde{G}' : \widetilde{x} = (t_0, t_1, x) \mapsto \overline{p}_{\sigma_0}(t_0 \overline{f}_0 + x(\overline{t}_0)) + \overline{p}_{\sigma_1}(t_1 \overline{f}_1 + x(\overline{t}_1)) + \overline{p}_{t_0} \cdot t_0 + \overline{p}_{t_1} \cdot t_1. \tag{49}$$

It is easy to verify that conditions (I) (II) (III) in section 1 are satisfied for the system

$$(\widetilde{x}, q) \in \widetilde{A} \times C, \quad \widetilde{K}(\widetilde{x}, \widetilde{q}) = 0, \quad \widetilde{G}(\widetilde{x}) \in M. \tag{50}$$

we now show that $\widetilde{G}'$ defined by (49) is an M-derivative of $\widetilde{G}$ at $\overline{\widetilde{x}} = (\overline{t}_0, \overline{t}_1, \overline{x})$. To do this we observe that $\widetilde{G}$ is the composition of two mappings: $\widetilde{G} = \widehat{p} R$, with $R: (t_0, t_1, x) \mapsto (t_0, t_1, x(t_0), x(t_1))$, $\widehat{p}: (t_0, t_1, x(t_0), x(t_1)) \mapsto p(x(t_0), x(t_1), t_0, t_1) - (y_0, 0, \dots, 0)$. For each $j = 0,1$ let $R_j: (t_0, t_1, x) \mapsto x(t_j)$. As $\varepsilon \to 0$ and $\widetilde{z} = (\overset{0}{s_0}, s_1, z) \to \widetilde{x} = (t_0, t_1, x)$ we have

$$\frac{1}{\varepsilon} R_j(\widetilde{\overline{x}} + \varepsilon \widetilde{z}) - R_j(\widetilde{\overline{x}}) = \frac{1}{\varepsilon}(\overline{x}(\overline{t}_j + \varepsilon t_j) - \overline{x}(\overline{t}_j)) + z(\overline{t}_j + \varepsilon t_j)$$

$$\to t_j \frac{d\overline{x}(\overline{t}_j)}{dt} + x(\overline{t}_j) .$$

so that $R'_j : (t_0, t_1, x) \mapsto t_j \overline{f}_j + x(\overline{t}_j)$ $\quad \left(\overline{f}_j = \frac{d\overline{x}(\overline{t}_j)}{dt}\right)$

is a $\{0\}$-derivative of R_j at point $\widetilde{\overline{x}} = (\overline{t}_0, \overline{t}_1, \overline{x})$ in the sense defined in section 1 (note that if $\overline{t}_j$ is a continuity point of $\overline{f}(\overline{x}(t), t)$ then $\overline{f}_j = \overline{f}(\overline{x}(\overline{t}_j), \overline{t}_j))$. Therefore

$$R' : (t_0, t_1, x) \mapsto (t_0, t_1, t_0 \overline{f}_0 + x(\overline{t}_0), t_1 \overline{f}_1 + x(\overline{t}_1))$$

is a $\{0\}$-derivative of R at $\widetilde{\overline{x}}$, and hence, by Lemma 1.2, $\widehat{G}' = \widehat{p}' R'$, as defined by (49), is a $\{0\}$-derivative (a fortiori an M-derivative) of $\widetilde{G}$ at $\widetilde{\overline{x}} = (\overline{t}_0, \overline{t}_1, \overline{x})$, as we have asserted.

Thus, all assumptions listed in section 1 are satisfied for the system (50). If now $\widetilde{\overline{x}} = (\overline{t}_0, \overline{t}_1, \overline{x})$ is an optimal solution, then, as can easily be seen, this system is critical. Hence, by Theorem 1, there exists a nonzero vector $\Delta = (\lambda_0, \dots, \lambda_m)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, \dots, \lambda_m) \in N^*$ and, denoting $\widetilde{\overline{q}} = (\overline{\alpha}, \overline{q}) = (\overline{t}_0, \overline{t}_1, \overline{x})$, we have

$$(\forall \widetilde{q} = (\alpha, q) \in \Delta \times [\Omega \times F] - \widetilde{\overline{q}}): \qquad -\Lambda \widetilde{G}' \overline{\overline{K}}_{\widetilde{x}}^{-1} \overline{\overline{K}}_{\widetilde{q}} \cdot \widetilde{q} \leqslant 0, \tag{51}$$

$$\sum_{i=1}^{m} \lambda_i p_i (\overline{x}(\overline{t}_0), \overline{x}(\overline{t}_1), \overline{t}_0, \overline{t}_1) = 0. \tag{52}$$

But it follows from (48) that

$$\overline{\overline{K}}_{\widetilde{x}} : (\theta, x) \mapsto (\theta, \overline{K}_x . x), \; \overline{\overline{K}}_{\widetilde{q}} : (\alpha, q) \mapsto (-\alpha, \overline{K}_q \cdot q),$$

hence

$$\overline{\overline{K}}_{\widetilde{x}}^{-1} \overline{\overline{K}}_{\widetilde{q}} \cdot \widetilde{q} = (-\alpha, \overline{K}_x^{-1} \overline{K}_q \cdot q),$$

and so, in view of (49), relation (51) becomes

$$(\forall (t_0, t_1) \in \Delta - (\overline{t}_0, \overline{t}_1), \omega \in \Omega - \overline{\omega}, f \in F - \overline{f}):$$

$$\Lambda \{\overline{p}_{\sigma_0} (t_0 \overline{f}_0 + y(\overline{t}_0)) + \overline{p}_{\sigma_1} (t_1 \overline{f}_1 + y(\overline{t}_1)) + \overline{p}_{t_0} t_0 + \overline{p}_{t_1} t_1\} \leqslant 0 \tag{53}$$

where, according to (40)

$$y(t) = -\overline{K}_x^{-1} \overline{K}_q \cdot q = \Gamma(t) \left[\omega + \int_0^t \Gamma^{-1}(s) f(\overline{x}(s), s)\, ds\right]. \tag{54}$$

By letting $\omega = 0$, $f = 0$, we get

$$\Lambda(\overline{p}_{\sigma_0} \overline{f}_0 + \overline{p}_{t_0}) = 0, \qquad \Lambda(\overline{p}_{\sigma_1} \overline{f}_1 + \overline{p}_{t_1}) = 0.$$

Also, by letting $t_0 = 0$, $t_1 = 0$, $f = 0$, we have, since Ω is open,

$$\wedge (\overline{p}_{\sigma_0}\,\Gamma(\overline{t}_0) + \overline{p}_{\sigma_1}\,\Gamma(\overline{t}_1)) = 0.$$

Finally, by letting $t_0 = 0$, $t_1 = 0$, $\omega = 0$, and taking into account the just written equality, we obtain

$$(\forall f \in F)\ \wedge \overline{p}_{\sigma_1}\,\Gamma(\overline{t}_1)\int_{\overline{t}_0}^{\overline{t}_1} \Gamma^{-1}(s)\,[f(\overline{x}(s), s) - \overline{f}(\overline{x}(s), s)]\,ds \leqslant 0$$

which becomes (43) if we set

$$\overline{\Psi}(t) = \wedge\ \overline{p}_{\sigma_1}\,\Gamma(\overline{t}_1)\,\Gamma^{-1}(t).$$

It is now a simple matter to check the remaining relations (44) (45) (46) and thereby to complete the proof.

3. In order to show with more evidence the advantage of the approach we have taken, we now close the paper by studying a non-standard problem. Namely, let us consider the problem differing from the previous one, only in that, instead of assuming the function $p(\sigma_0, \sigma_1, t_0, t_1)$ to be differentiable in the usual sense, we assume only that $p(\sigma_0, \sigma_1, t_0, t_1)$ has at point $(\overline{x}(\overline{t}_0), \overline{x}(\overline{t}_1), \overline{t}_0, \overline{t}_1)$ a continuous M-derivative $p'(.,.,.,.)$ such that $p'(0, 0, 0, 0) = 0$, where $M = \{0\} \times N \subset R^1 \times R^m$.

For example, it may happen that $p(.,.,.,.)$ is M-convex but not necessarily differentiable in the usual sense. In that case, as can be easily verified, $p' : (\sigma_0, \sigma_1, t_0, t_1) \mapsto p(\overline{\sigma}_0 + \sigma_0, \overline{\sigma}_1 + \sigma_1, \overline{t}_0 + t_0, \overline{t}_1 + t_1) - p(\sigma_0, \sigma_1, t_0, t_1)$ is an M-derivative of p at the indicated point.

This kind of problem could be encountered in some practical applications. For example, if $x(t)$ describes the state of the economy at time t (say, the stoce of capital and other facilities), then the boundary conditions may consist in requiring that $x(t_0)$, $x(t_1)$ belong to some prescribed convex sets (of the type of the production-technological sets); that is, these quantities must satisfy inequalities of the form $(p_1(x(t_0), x(t_1), t_0, t_1), \ldots, p_m(x(t_0), x(t_1), t_0, t_1) \in N$, where p is simply N-convex and not necessarily differentiable in the usual sense.

In view of the possible non-differentiabitity of p, the problem cannot be handled by standard methods.

We can prove the following

THEOREM 5. *Assume the set F to be quasiconvex. If $\overline{x}(t)$, $\overline{t}_0$, $\overline{t}_1$ is an optimal solution, corresponding to $\overline{q} = (\overline{\omega}, \overline{f})$, then there exist an absolutely continuous n-vector valued function $\overline{\psi}(t)$ satisfying the maixmum principle*

$$\int_{\overline{t}_0}^{\overline{t}_1} \overline{\psi}(t)\,\overline{f}(\overline{x}(t), t)\,dt = \max_{f \in F} \int_{\overline{t}_0}^{\overline{t}_1} \overline{\psi}(t)\,f(\overline{x}(t), t)\,dt \tag{55}$$

and a nonzero $(m+1)$-vector $\wedge = (\lambda_0, \lambda_1, \ldots, \lambda_m)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, \ldots, \lambda_m)$

$\in N^*$, *together with a vector* $\pi = (\pi_{\sigma_0}, \pi_{\sigma_1}, \pi_{t_0}, \pi_{t_1}) \in R^n \times R^n \times R^1 \times R^1$ *such that* $-\pi$ *is a subgradient* of* $-\wedge p'$ *at* 0 *and*

$$\frac{d\overline{\psi}}{dt} = -\overline{\psi}(t)\,\overline{f}_{,\sigma}(\overline{x}(t), t)\,, \tag{56}$$

$$\overline{\psi}(\overline{t}_0) = -\pi_{\sigma_0}, \qquad \overline{\psi}(\overline{t}_1) = \pi_{\sigma_1}, \tag{57}$$

$$\overline{\psi}(\overline{t}_0)\overline{f}_0 = \pi_{t_0}, \qquad \overline{\psi}(\overline{t}_1)\overline{f}_1 = -\pi_{t_1}, \tag{58}$$

$$\sum_{i=1}^{m} \lambda_i p_i(\overline{x}(\overline{t}_0), \overline{x}(\overline{t}_1), t_0, t_1) = 0, \tag{59}$$

where $\overline{f}_j = \frac{d\overline{x}}{dt}(t_j)$ $(= \overline{f}(\overline{x}(\overline{t}_j), \overline{t}_j)$ if $\overline{f}(\overline{x}(t), t)$ is continuous at $\overline{t}_j)$.

Proof. Just as above, it is easily seen that the system (50) satisfies all conditions (I) through (IV) of section 1 with

$$\widetilde{C}' = \Delta \times [\Omega \times F] - (\overline{\theta}, \overline{q}) \qquad (\overline{\theta} = (\overline{t}_0, \overline{t}_1))$$

$$\widetilde{G}' : \widetilde{x} = (t_0, t_1, x) \mapsto p'(t_0\overline{f}_0 + x(\overline{t}_0), t_1\overline{f}_1 + x(\overline{t}_1), t_0, t_1). \tag{60}$$

Therefore, if $\overline{\widetilde{x}} = (\overline{t}_0, \overline{t}_1, \overline{x})$ is optimal, then there exists as previously a nonzero vector $\wedge = (\lambda_0, ..., \lambda_m)$ such that $\lambda_0 \leqslant 0$, $(\lambda_1, ..., \lambda_m) \in N^*$ and (51), (52) hold, with G' given this time by (60) in place of (49).

Noting (48) and remembering (11), (14), we can write (51) in the form

$$(\forall \widetilde{q} = (\alpha, q) \in \Delta \times [\Omega \times F] - \overline{\widetilde{q}}):$$
$$p'(t_0\overline{f}_0 + y(\overline{t}_0), t_1\overline{f}_1 + y(\overline{t}_1), t_0, t_1) \leqslant 0. \tag{61}$$

where $\overline{\widetilde{q}} = (\overline{\theta}, \overline{q}) = (\overline{t}_0, \overline{t}_1, \overline{q})$ and $y(t) - \overline{K}_x^{-1}\overline{K}_q \cdot q$ is given by (54).

Let B denote the set of all $z = (\sigma_0, \sigma_1, t_0, t_1)$ such that

$(t_0, t_1) \in \Delta - (\overline{t}_0, \overline{t}_1)$,

$\sigma_j = (t_j - \overline{t}_j)\overline{f}_j + y(\overline{t}_j)$, $j = 0,1$ for some $q = (\omega, f) \in [\Omega \times F]$.

Then, since $\overline{K}_x^{-1}\overline{K}_q$ is linear, it follows that B is a convex set.

From (61) we have

$$(\forall z \in B) \qquad \wedge p'(z) \leqslant 0. \tag{62}$$

Since $\wedge \in M^*$ and p' is by hypothesis M-convex, the function $-\wedge p'(z)$ is convex (in the usual sense). If the convex set $\{z: \wedge p'(z) > 0\}$ is non-empty

* Given a convex function $\varphi : R^s \to R^1$, a vector $\pi \in R^s$ is said to be a subgradient of φ at 0 if $(\forall z \in R^s)\ \varphi(z) \geqslant \varphi(0) + \langle \pi, s \rangle$. Thus if p' is linear (as in Theorem 4), then $\pi = \wedge p'$.

then (62) means that it does not meet the set B. Hence there exists a non-constant affine function φ such that $\varphi(z) \leqslant 0$ for all $z \in B$ and $\varphi(z) \geqslant 0$ for all z verifying $\wedge\, p'(z) > 0$. The latter fact implies that the system

$$\varphi(z) < 0, \qquad - \wedge\, p'(z) < 0$$

has no solution, and since the inequality $\varphi(z) < 0$ has at least one solution (because φ in not constant), one can find a non-negative number c such that $c\,\varphi(z) - \wedge\, p'(z) \geqslant 0$ for all z. Setting $\pi(z) = c\,\pi(z)$ if $(\exists z)\ \wedge\, p'(z) > 0$, and $\pi(z) = 0$ if $(\forall z)\ \wedge\, p'(z) \leqslant 0$, we obtain an affine function $\pi(z)$ such that

$$(\forall z \in B) \qquad \pi(z) \leqslant 0 \tag{63}$$

$$(\forall z \in Z) \qquad \wedge\, p'(z) \leqslant \pi(z), \tag{64}$$

where $Z = R^n \times R^n \times R^1 \times R^1$. We have from (63) $\pi(0) \leqslant 0$ (because $0 \in B$), and from (64) $\pi(0) \geqslant 0$ (because $p'(0) = 0$), hence $\pi(0) = 0$, i.e. $\pi(z)$ is a linear function: $\pi(z) = \langle \pi, z \rangle$, with $\pi = (\pi_{\sigma_0}, \pi_{\sigma_1}, \pi_{t_0}, \pi_{t_1}) \in R^n \times R^n \times R^1 \times R^1$. Relation (64), together with $p'(0) = 0$, show that $-\pi$ is a subgradient of $-\wedge\, p'$ at 0, and from (63) it follows that

$$(\forall (t_0, t_1) \in \Delta - (\bar{t}_0, \bar{t}_1), \omega \in \bar{\Omega} - \omega, f \in F - \bar{f}):$$

$$\pi_{\sigma_0}(t_0 \bar{f}_0 + y(\bar{t}_0)) + \pi_{\sigma_1}(t_1 \bar{f}_1 + y(\bar{t}_1)) + \pi_{t_0} t_0 + \pi_{t_1} t_1 \leqslant 0$$

which is similar to (53). The proof can now be completed in the same way that we completed the proof of Theorem 4.

Remark 3.1. From (34) (or (43)) it is easy to derive the Pontryagin's form of the maximum principle for control problems in the conventional setting. Indeed, in this setting we are given a function $f(\sigma, t, v): R^n \times R^1 \times R^r \to R^n$, a set U in R^r and a class $\mathcal{U}$ of functions $u(t): [0,1] \to U$. The function $f(\sigma, t, v)$ is assumed to be continuous on $R^n \times R^1 \times R^r$ and continuously differentiable with respect to σ. As for $\mathcal{U}$, it is usually the set of all piecewise continuous functions, or the set of all measurable, essentially bounded functions from $[0,1]$ inte U. Anyway it will suffice to suppose that $\mathcal{U}$ is a set of measurable, essentially bounded functions from $[0,1]$ into U such that every constant function is an element of $\mathcal{U}$ and whenever $u^1, u^2 \in \mathcal{U}$, $0 \leqslant \theta \leqslant 1$, then $\chi_{[0,\theta]} u^1 + \chi_{[\theta,1]} u^2 \in \mathcal{U}$. Under these conditions, if F denotes the family of all functions $f(\sigma, t)$ of the form $f(\sigma, t) = f(\sigma, t, u(t))$ for some $u \in \mathcal{U}$, then F is clearly a convex-under-switching family in $\mathfrak{B}(\Omega, R^n)$ (that F is a subset of $\mathfrak{B}(\Omega, R^n)$ follows from the fact that for every $f \in F$ the function

$m(t) = \sup \{ |f(\sigma, t, u(t))| + |f_\sigma(\sigma, t, u(t))| : \sigma \in \bar{\Omega}, 0 \leqslant t \leqslant 1 \}$ is integrable over $[0,1]$).

The maximum relation (34) reads now as follows:

$$\int_0^1 \bar{\psi}(t) f(\bar{x}(t), t, \bar{u}(t))\, dt = \max_{u \in \mathcal{U}} \int_0^1 \bar{\psi}(t) f(\bar{x}(t), t, u(t))\, dt.$$

if t' in a Lebesgue point of the function $\overline{\psi}(t) f(\overline{x}(t), t, \overline{u}(t))$ and if we take $u(t)$ to be equal to $v \in U$ for $t' \leqslant t \leqslant t' + h$ and equal to $\overline{u}(t)$ for all other values of t, we can write

$$\frac{1}{h}\int_{t'}^{t'+h} \overline{\psi}(t) f(\overline{x}(t), t, v)\, dt \leqslant \frac{1}{h}\int_{t'}^{t'+h} \overline{\psi}(t) f(\overline{x}(t), t, \overline{u}(t))\, dt$$

hence, by letting $h \to 0$ and noting that t' is a continuity point (and, therefore, also a Lebesgue point of $\overline{\psi}(t) f(x(t), t, v)$:

$$\overline{\psi}(t') f(\overline{x}(t'), t', v) \leqslant \overline{\psi}(t') f(\overline{x}(t'), t', \overline{u}(t')).$$

Thus, for almost all $t \in [0,1]$:

$$\overline{\psi}(t) f(\overline{x}(t), t, \overline{u}(t)) = \max_{v \in U} \overline{\psi}(t) f(\overline{x}(t), t, v)$$

Remark 3.2. It can be proved that Theorem 1 still holds if $G : A \to Y_1 \times Y_2$, $M = M_1 \times M_2$ with $M_i \subset Y_i$, $Y_1 = R^k$, Y_2 is an arbitrary normed space and int $M \neq \emptyset$. With this extension, Theorem 1 could be used to derive the maximum principle for control problems with restricted phase coordinates.

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ERRATA CORRIGE

The saddle—point theorem under a weaker assumption than constraint qualification

Hoàng Tụy

With reference to the paper published in ACTA SCIENTIARUM VIETNAMICARUM, Tom IX & X (1974), the definition of a surtangent mapping in page 116 should be modified as follows:

Given a mapping $H : D \to Z$, a mapping $h : D \to Z$ is said to be surtangent to H at x_0 if for every $x \in D$ and for every $\eta > 0$ there exist a neighbourhood U of x and a $\delta > 0$ such that whenever $z \in D \cap U$ and $0 < \varepsilon < \delta$ then $H(x^0 + \varepsilon(z - x^0)) \leqq h(x^0 + \varepsilon(x - x^0)) + u$, with $|u| < \varepsilon . \eta$.

ERRATA

de l'auteur de l'article

« *Sur les formules asymptotiques multiples de l'erreur en méthodes aux différences finies* »

paru dans ce journal, t.IX et X, 1974, 41-52.

On est prié d'ajouter à la page 43 $p = ki$, $k = \text{const} > 0$. De plus, le filet non-uniforme mentionné dans les théorèmes 5, 7, 9 doit être celui que l'auteur a employé à la page 89 de ce journal t. II, 1965, par suite les seconds membres de l'équation (2, 9) et les deux dernières relations de l'article devraient être modifiés de manière convenable.

Tạ văn Đĩnh