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H-PROPERTY, NORMAL STRUCTURE AND FIXED POINTS OF NONEXPANSIVE MAPPINGS IN METRIC LINEAR SPACES

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Let E be a vector space over the scalar field R and d an invariant linear metric on E. The metric space (E, d) is said to be strictly convex [1] if whenever $r > 0$, x, $y \in E$, $x \neq y$, $d(x, 0) \leq r$ and $d(y, 0) \leq r$, then

$$d\Big(\frac{x+y}{2}, 0\Big) < r.$$

The metric linear space (E, d) is said to satisfy:

U.C. I: If given $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, 0) < r + \delta$, $d(y, 0) < r + \delta$ and $d(x, y) \geq \varepsilon$ imply

$$d\Big(\frac{x+y}{2}, 0\Big) < r.$$

U.C. II: If given $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, 0) \leq r$, $d(y, 0) \leq r$ and $d(x, y) \geq \varepsilon$ imply

$$d\Big(\frac{x+y}{2}, 0\Big) \leq r - \delta.$$

U.C. III: If given $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, 0) = r$, $d(y, 0) = r$ and $d(x, y) \geq 0$ imply

$$d\Big(\frac{x+y}{2}, 0\Big) \leq r - \delta.$$

Ahuja, Narang and Trehan [1] introduced the notions of strict convexity and U.C. I (uniform convexity) in metric linear spaces which are generalizations of the corresponding concepts in normed linear spaces. Sastry

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and Naidu [2] introduced the notions of U.C. II and U.C. III in metric linear spaces and showed that the three forms of uniform convexity in metric linear spaces are not always equivalent. Wu Junde and others [3], [4] showed that if (E, d) is a complete U.C. I (U.C. II or U.C. III) metric linear space, then (E, d) is reflexive. Thus each bounded closed convex subset of (E, d) is weakly compact and weakly sequentially compact [5, p. 318]. In this paper we will consider the H-property, normal structure and fixed points of nonexpansive mappings in metric linear spaces. We show that U.C. I (U.C. II or U.C. III) metric linear space has the H-property, U.C. II metric linear space has the normal structure and if K is a nonempty bounded closed convex subset of a complete U.C. II metric linear space, then each nonexpansive mapping $T : K \to K$ has a fixed point.

The following lemmas will be used in the sequel:

Lemma 1 [2]. *Let (E, d) be a strictly convex metric linear space. Then for each $\varepsilon > 0$, $\{x \in E \big| d(x, 0) \leq \varepsilon\}$ is an absolutely convex absorbing neighbourhood of 0 in (E, d).*

Lemma 2 [2]. *Let (E, d) be a U.C. I (U.C. II or U.C. III) metric linear space, then (E, d) is strictly convex.*

Lemma 3]3], [4]. *Let (E, d) be a strictly convex metric linear space. If $\{x \in E \big| d(x, 0) \leq r\} \neq E$, then the Minkowski gauge P_r of $\{x \in E \big| d(x, 0) \leq r\}$ is a strictly convex norm and $P_r(x) = 1$ if and only if $d(x, 0) = r$.*

The metric linear space (E, d) is said to have the H-property if for each $r > 0$, whenever $d(x_n, 0) = r$, $d(x_0, 0) = r$, $(n = 1, 2, \dots)$, and $x_n \to x_0$ (weakly), we must have $x_n \to x_0$ (strongly).

A point x_0 of a bounded closed convex subset K of (E, d) is said to be diametral whenever

$$\operatorname{diam} K = \sup \{d(x, y) \big| x, y \in K\} = \sup \{d(x_0, y) \big| y \in K\}.$$

A bounded closed convex subset K of (E, d) is said to have the normal structure whenever given any bounded closed convex subset C of K containing more than one point, there exists a non-diametral point $x \in C$. If each bounded closed convex subset K of (E, d) has the normal structure, then (E, d) is said to have the normal structure.

Let C be a subset of (E, d). A mapping $T : C \to E$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.

The following result deals with the H-property in metric spaces:

Theorem 1. *If (E, d) satisfies U.C. I or U.C. II or U.C. III, then (E, d) has the H-property.*

Proof. At first, we prove Theorem 1 for the case when (E, d) satisfies U.C. I. Let $\{x_n\} \subseteq E$, $x_0 \in E$ and $d(x_n, 0) = r$ $(n = 1, 2, \dots)$, $d(x_0, 0) = r$ and $x_n \to x_0$ (weakly). If $x_n \to x_0$ (strongly) does not hold, then there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x_0) \geq \varepsilon_0$. Without loss of generality we may assume that for each $n \in N$, $d(x_n, 0) \geq \varepsilon_0$. Since (E, d) satisfies U.C. I, there exists $\delta > 0$ such that whenever $d(x, 0) < r + \delta$, $d(y, 0) < r + \delta$ and $d(x, y) \geq \varepsilon_0$ we have $d\Big(\dfrac{x+y}{2}, 0\Big) < r$. Let $V = \{x \big| d(x, 0) \leq \delta/2\}$. Then for each $y \in V$ we have

$$d(x_n + y, 0) \leq r + \delta/2,$$

$$d(x_0 + y, 0) \leq r + \delta/2,$$

$$d(x_n + y, x_0 + y) \geq \varepsilon_0.$$

Thus, from the uniform convexity of (E, d), it follows that

$$d\Big(\frac{x_n + x_0}{2} + y, 0\Big) < r.$$

Therefore

$$\frac{x_n + x_0}{2} + V \subseteq \{x \big| d(x, 0) \leq r\}.$$

It is obvious that $\{x \big| d(x, 0) \leq r\} \neq E$. Let P_r be the Minkowski gauge of $\{x \big| d(x, 0) \leq r\}$. From Lemma 3 we know that (E, P_r) is a normed space and $P_r(x_0) = 1$. By the Hahn-Banach theorem there exists $f \in (E, P_r)'$ such that $\|f\|_{P_r} = 1$ and $f(x_0) = P_r(x_0) = 1$. Then $f \in (E, d)'$. Since $x_n \to x_0$ (weakly), we have

$$\lim_{n \to \infty} f(x_n + x_0) = f(x_0 + x_0) = 2.$$

Note that $\{x \big| d(x, 0) \leq \delta/2\}$ is an absorbing balanced subset of E. There exists $y_0 \in \{x \big| d(x, 0) \leq \delta/2\}$ such that $f(y_0) = t_0 > 0$. Since $d\Big(\dfrac{x_n + x_0}{2} + y_0, 0\Big) \leq r$,

$$P_r\Big(\frac{x_n + x_0}{2} + y_0\Big) \leq 1.$$

Thus we have

$$\Big|f\Big(\frac{x_n+x_0}{2}+y_0\Big)\Big| \le \|f\|_{P_r}\cdot P_r\Big(\frac{x_n+x_0}{2}+y_0\Big) \le 1.$$

Letting $n\to\infty$ it follows that $f(x_0+y_0)=1+t_0\le 1$. This contradicts $t_0>0$.

The U.C. II or U.C. III case can be proved by the same method. □

The next two result deal with the normal structure in metric linear spaces.

Theorem 2. *Let (E,d) be a strictly convex metric linear space. Then each compact convex subset of (E,d) has the normal structure.*

Proof. If K is a compact convex subset of (E,d) and K does not have non-diametral points ($\operatorname{diam} K>0$), then for any $x_1\in K$ there exists $x_2\in K$ such that $d(x_1,x_2)=\operatorname{diam} K$. Since K is a convex subset, $\frac{x_1+x_2}{2}\in K$. Thus, there exists $x_3\in K$ such that $d\big(\frac{x_1+x_2}{2},x_3\big)=\operatorname{diam} K$. In this way we get a sequence $\{x_n\}\subseteq K$ such that

$$d\Big(\frac{x_1+x_2+\cdots+x_n}{n},x_{n-1}\Big)=\operatorname{diam} K.$$

Let $r=\operatorname{diam} K$ and P_r be the Minkowski gauge of $\big\{x\big|d(x,0)\le r\big\}$. It is obvious that $\big\{x|d(x,0)\le r\big\}\ne E$. From Lemma 3 we have $P_r(x)=1$ if and only if $d(x,0)=r$. Since

$$d\Big(\frac{x_1+x_2+\cdots+x_n}{n}-x_{n+1},0\Big)=\operatorname{diam} K=r,$$

we get

$$P_r\Big(x_{n+1}-\frac{x_1+\cdots+x_n}{n}\Big)=1.$$

Note that for any x_m and x_k, $d(x_m,x_k)\le \operatorname{diam} K=r$. Then we have $P_r(x_m-x_k)\le 1$. From

$$\begin{aligned}P_r\Big(\frac{x_{n+1}-x_1+x_{n+1}-x_2+\cdots+x_{n+1}-x_n}{n}\Big)&=1\\ &\le \frac{1}{n}\sum_{i=1}^{n}P_r(x_{n+1}-x_i)\le 1\end{aligned}$$

it follows that $P_r(x_{n+1} - x_k) = 1$ for $k = 1, 2, \dots, n$. So $d(x_{n+1}, x_k) = r$. This shows that the sequence $\{x_n\}$ has no Cauchy subsequences, i.e., K is not compact. □

Theorem 3. *If (E, d) satisfies U.C. II, then each bounded closed convex subset of (E, d) has the normal structure.*

Proof. Let K be a bounded closed convex subset of (E, d) and $\operatorname{diam} K = r > 0$. Take $x_1, x_2 \in K$ satisfying $d(x_1, x_2) \geq r/2$. Then for each $y \in K$, $d(x_1, y) \leq r$, $d(x_2, y) \leq r$. Since (E, d) satisfies U.C. II, there exists $\delta > 0$ such that for each $y \in K$,

$$d\Big(\frac{x_1 + x_2}{2} - y, 0\Big) \leq r - \delta.$$

This show that

$$\sup\Big\{d\Big(\frac{x_1 + x_2}{2}, y\Big)\Big| y \in K\Big\} \leq r - \delta < r.$$

So $\dfrac{x_1 + x_2}{2}$ is a non-diametral point. □

We do not know whether U.C. I or U.C. III metric linear spaces also have the normal structure.

Finally we consider fixed points of nonexpancive mappings in metric linear spaces.

Theorem 4. *Let (E, d) be a complete U.C. II metric linear space and K a bounded closed convex subset of (E, d). If $T : K \to K$ is a nonexpansive mapping, then T has a fixed point.*

Proof. From [4, Th. 2] and Theorem 3 it follows that K is a weakly compact subset of (E, d) and has the normal structure. Now proceeding as in Theorem 3 of [6, p. 39], we can get the result. □

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