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AN ASYMPTOTIC EXPANSION OF A WEAK SOLUTION FOR A NONLINEAR WAVE EQUATION

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Abstract. In this paper, we consider a nonlinear wave equation associated with the Dirichlet boundary condition. First, the existence and uniqueness of a weak solution are proved by using the Faedo-Galerkin method. Next, we present an asymptotic expansion of high order in many small parameters of a weak solution. This extends recent corresponding results where an asymptotic expansion of a weak solution in two or three small parameters is established.

1. Introduction

In this paper, we consider the following initial and boundary value problem:

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(u)u_x\right) = f(x,t,u,u_x,u_t),\ 0<x<1,\ 0<t<T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \tag{1.3}$$

where $\widetilde{u}_0$, $\widetilde{u}_1$, μ, f are given functions satisfying conditions specified later.

Equation (1.1) constitutes a relatively simple case of a more general equation as follows:

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t,u)u_x\right) = f(x,t,u,u_x,u_t),\ 0<x<1,\ 0<t<T. \tag{1.4}$$

In the special cases that the function $\mu(x,t,u)$ is independent of u, $\mu(x,t,u)\equiv 1$ or $\mu(x,t,u)=\mu(x,t)$, and the nonlinear term f has the simple forms, problem (1.4) with various initial-boundary conditions has been studied by many authors, for example Ortiz, Dinh [18], Long, Dinh [2,3,6,8], Long, Diem [9], Long, Dinh, Diem [10–12], Long, Truong [13,14], Long, Ngoc [15], Ngoc, Hang, Long [16] and the references therein.

In [4], Ficken and Fleishman established the unique global existence and stability of solutions for the equation

$$u_{xx} - u_{tt} - 2\alpha u_t - \beta u = \varepsilon u^3 + \gamma,\ \varepsilon > 0. \tag{1.5}$$

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Rabinowitz [19] proved the existence of periodic solutions for

$$u_{xx} - u_{tt} - 2\alpha u_t = \varepsilon f(x,t,u,u_x,u_t), \tag{1.6}$$

where ε is a small parameter and f is periodic in time.

In a paper of Caughey and Ellison [1], a unified approach to the previous cases was presented discussing the existence and uniqueness and asymptotic stability of classical solutions for a class of nonlinear continuous dynamical systems.

In [11], Long, Dinh and Diem have studied the linear recursive schemes and asymptotic expansion for the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t) + \varepsilon g(x,t,u,u_x,u_t), \tag{1.7}$$

with the mixed nonhomogeneous conditions

$$u_x(0,t) - h_0 u(0,t) = g_0(t),\ u(1,t) + h_1 u(1,t) = g_1(t). \tag{1.8}$$

In the case of $g_0, g_1 \in C^3(\mathbb{R}_+)$, $f \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $g \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, and some other conditions, an asymptotic expansion of the weak solution u_ε of order $N+1$ in ε is considered.

However, by the fact that it is difficult to consider problem (1.4) with some initial-boundary conditions in the case that $\mu(x,t,u)$ depends on u, few works were done as far as we know. In order to solve this problem, the linearization method for nonlinear term is usually used. Let us present this technique as follows.

First, we note that for each $v = v(x,t)$ belonging to X, a suitable space of function, we can give some suitable assumptions to obtain a unique solution $u \in X$ of the problem with respect to $\mu = \mu(x,t,v(x,t)) = \widetilde{\mu}(x,t)$ and $f = f(x,t,v,v_x,v_t) = \widetilde{f}(x,t)$. It is obvious that u depends on v, so we can suppose that $u = A(v)$. Therefore, the above problem can be reduced to a fixed point problem for the operator $A : X \to X$. Based on these ideas, with a chosen first term u_0, the usual iteration $u_m = A(u_{m-1})$, $m = 1,2,...$, is applied to establish a sequence $\{u_m\}$ that converges to the solution of the problem, and hence the existence results follow.

Without loss of generality we need only to consider the problem (1.1)-(1.3) instead of the problem (1.2)-(1.4) in order to avoid making the treatment too complicated.

The paper consists of four sections. First, some preliminaries are assembled in Section 2. We begin Section 3 by establishing a sequence of approximate solutions of the problem (1.1)-(1.3) based on the Faedo-Galerkin method. Thanks to a priori estimates, this sequence is bounded in an appropriate space, from which, using compact embedding theorems and Gronwall's lemma, we deduce the existence of a unique weak solution of problems (1.1) – (1.3). In Sections 4, an asymptotic expansion of a weak solution $u = u_{\varepsilon_1,\varepsilon_2,...,\varepsilon_p}(x,t)$ of order $N+1$ in p small parameters ε_1, ε_2, ..., ε_p for the equation

$$u_{tt} - \frac{\partial}{\partial x}\left(\left[\mu(u) + \sum_{i=1}^p \varepsilon_i \mu_i(u)\right] u_x\right) = f(x,t,u,u_x,u_t) + \sum_{i=1}^p \varepsilon_i f_i(x,t,u,u_x,u_t), \tag{1.9}$$

associated to $(1.1)_{2,3}$, with $\mu \in C^{N+2}(\mathbb{R})$, $\mu_i \in C^{N+1}(\mathbb{R})$, $\mu(z) \geq \mu_0 > 0$, $\mu_i(z) \geq 0$ for all $z \in \mathbb{R}$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ and $f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $i = 1, 2, ..., p$ is established. This result is a relative generalization of [12–14], where an asymptotic expansion of a solution in two or three small parameters is obtained.

2. Preliminaries

Let $\Omega = (0,1)$. We denote the function spaces used in this paper by the usual notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$, $H_0^m = H_0^m(\Omega)$.

Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $||\cdot||$ stands for the norm in L^2 and we denote by $||\cdot||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of real functions $u : (0,T) \to X$ measurable, such that $||u||_{L^p(0,T;X)} < +\infty$, with

$$||u||_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p dt \right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} ||u(t)||_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. For $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f = f(x,t,u,v,w)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial v}$, $D_5 f = \frac{\partial f}{\partial w}$ and $D^\alpha f = D_1^{\alpha_1} ... D_5^{\alpha_5} f$, $\alpha = (\alpha_1, ..., \alpha_5) \in \mathbb{Z}_+^5$, $|\alpha| = \alpha_1 + ... + \alpha_5 = k$, $D^{(0,0,...,0)} f = f$.

On H^1 we shall use the norm

$$||v||_{H^1} = \left(||v||^2 + ||v_x||^2 \right)^{1/2}. \tag{2.1}$$

Then the following lemma is known as a standard one.

Lemma 2.1. *The embedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \quad \textit{for all } v \in H^1. \tag{2.2}$$

Remark 2.1. On H_0^1, the two norms $v \longmapsto ||v||_{H^1}$ and $v \longmapsto ||v_x||$ are equivalent. Furthermore,

$$||v||_{C^0(\overline{\Omega})} \leq ||v_x|| \text{ for all } v \in H_0^1. \tag{2.3}$$

3. Existence and uniqueness of a weak solution

We make the following assumptions:

$$\begin{aligned} &(\mathrm{H}_1) \qquad \widetilde{u}_0 \in H_0^1 \cap H^2, \widetilde{u}_1 \in H_0^1, \\ &(\mathrm{H}_2) \qquad \mu \in C^2(\mathbb{R}), \mu(z) \geq \mu_0 > 0 \ \forall z \in \mathbb{R}, \\ &(\mathrm{H}_3) \qquad f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3). \end{aligned}$$

With μ and f satisfying the assumptions (H_2) and (H_3) respectively, for each $T^* > 0$ and $M > 0$ we put

$$\widetilde{K}_M(\mu) = \|\mu\|_{C^2([-M,M])}\,, \; K_M(f) = \|f\|_{C^1(D^*(M))}\,, \tag{3.1}$$

where $D^*(M) = \{(x,t,u,v,w) \in [0,1] \times [0,T^*] \times \mathbb{R}^3 : |u|\,, |v|\,, |w| \leq M\}$.

Also for each $T \in (0,T^*]$ and $M > 0$, we set

$$\begin{aligned} W(M,T) = \big\{ & v \in L^\infty(0,T;H_0^1 \cap H^2) : v_t \in L^\infty(0,T;H_0^1) \text{ and } v_{tt} \in L^2(Q_T), \\ & \text{with } \; ||v||_{L^\infty(0,T;H_0^1\cap H^2)},\; ||v_t||_{L^\infty(0,T;H_0^1)},\; ||v_{tt}||_{L^2(Q_T)} \leq M \Big\}, \end{aligned} \tag{3.2}$$

$$W_1(M,T) = \{v \in W(M,T) :\, v_{tt} \in L^\infty(0,T;L^2)\}, \tag{3.3}$$

with $Q_T = \Omega \times (0,T)$.

We choose the first term $u_0 \equiv \widetilde{u}_0 \in W_1(M,T)$, suppose that

$$u_{m-1} \in W_1(M,T),\; m \geq 1, \tag{3.4}$$

and associate with the problem (1.1)-(1.3) the following variational problem:

Find $u_m \in W_1(M,T)$ such that

$$\langle u_m''(t), v\rangle + \langle \mu_m(t)\nabla u_m(t), \nabla v\rangle = \langle F_m(t), v\rangle \;\; \forall v \in H_0^1, \tag{3.5}$$

$$u_m(0) = \widetilde{u}_0,\;\; u_m'(0) = \widetilde{u}_1, \tag{3.6}$$

where

$$\mu_m(x,t) = \mu(u_{m-1}(x,t)), \quad F_m(x,t) = f(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t)). \tag{3.7}$$

Then we have the following theorem.

Theorem 3.1. *Suppose that* (H_1)-(H_3) *hold. Then, there exist constants* $M > 0$, $T > 0$ *such that the problem* (3.5)-(3.7) *has a unique solution* $u_m \in W_1(M,T)$.

Proof. The proof consists of three steps.

Step 1: *The Faedo-Galerkin Approximation* (introduced by Lions [5]). Consider a special basis $\{w_j\}$ on H_0^1 : $w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$.

Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.8}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \langle \mu_m(t)\nabla u_m^{(k)}(t), \nabla w_j\rangle = \langle F_m(t), w_j\rangle, \quad 1 \leq j \leq k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k}, \quad \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \end{cases} \tag{3.9}$$

in which

$$\begin{cases} \widetilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \widetilde{u}_0 \text{ strongly in } H_0^1 \cap H^2, \\ \widetilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \widetilde{u}_1 \text{ strongly in } H_0^1. \end{cases} \tag{3.10}$$

Then the system (3.9) can be rewritten in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \sum_{i=1}^{k} b_{mij}^{(k)}(t) c_{mi}^{(k)}(t) = f_{mj}(t), \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \ 1 \le j \le k, \end{cases} \tag{3.11}$$

where

$$b_{mij}^{(k)}(t) = \langle \mu_m(t) \nabla w_i, \nabla w_j \rangle, \ f_{mj}(t) = \langle F_m(t), w_j \rangle, 1 \le i, \ j \le k. \tag{3.12}$$

Note that by (3.4) it is not difficult to prove that the system (3.11) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on $[0, T]$. We omit the details.

Step 2: *A Priori Estimates.* Put

$$s_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds, \tag{3.13}$$

where

$$p_m^{(k)}(t) = ||\dot{u}_m^{(k)}(t)||^2 + ||\sqrt{\mu_m(t)} \nabla u_m^{(k)}(t)||^2, \tag{3.14}$$

$$q_m^{(k)}(t) = ||\nabla \dot{u}_m^{(k)}(t)||^2 + ||\sqrt{\mu_m(t)} \Delta u_m^{(k)}(t)||^2. \tag{3.15}$$

Then, it follows from (3.8), (3.9), (3.13) – (3.15) that

$$\begin{aligned} s_m^{(k)}(t) =& s_m^{(k)}(0) + 2\langle \nabla \mu_m(0) \nabla \widetilde{u}_{0k}, \Delta \widetilde{u}_{0k} \rangle + 2\langle F_m(0), \Delta \widetilde{u}_{0k} \rangle \\ &+ \int_0^t ds \int_0^1 \mu_m'(x, s) \left(|\nabla u_m^{(k)}(x, s)|^2 + |\Delta u_m^{(k)}(x, s)|^2 \right) dx \\ &+ 2 \int_0^t \langle \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right), \Delta u_m^{(k)}(s) \rangle ds \\ &- 2\langle \nabla \mu_m(t) \nabla u_m^{(k)}(t), \Delta u_m^{(k)}(t) \rangle \\ &+ 2 \int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds - 2\langle F_m(t), \Delta u_m^{(k)}(t) \rangle \\ &+ 2 \int_0^t \langle \frac{\partial F_m}{\partial t}(s), \Delta u_m^{(k)}(s) \rangle ds + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\ =& q_m^{(k)}(0) + 2\langle \nabla \mu_m(0) \nabla \widetilde{u}_{0k}, \Delta \widetilde{u}_{0k} \rangle + 2\langle F_m(0), \Delta \widetilde{u}_{0k} \rangle + \sum_{j=1}^{7} I_j. \end{aligned} \tag{3.16}$$

We shall estimate the terms I_j, $j = 1, 2, ..., 7$ on the right hand side of (3.16) as follows.

First term. From (3.1), (3.4), and (3.7) we have

$$\left| \mu_m'(x, t) \right| \le M \widetilde{K}_M(\mu). \tag{3.17}$$

Hence,

$$
\begin{aligned}
I_1 &= \int_0^t ds \int_0^1 \mu_m'(x,s) \left(|\nabla u_m^{(k)}(x,s)|^2 + |\Delta u_m^{(k)}(x,s)|^2 \right) dx \\
&\leq \frac{1}{\mu_0} M \widetilde{K}_M(\mu) \int_0^t s_m^{(k)}(s) ds.
\end{aligned} \tag{3.18}
$$

Second term. The Cauchy-Schwarz inequality leads to

$$
|I_2| = 2 \left| \int_0^t \langle \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right), \Delta u_m^{(k)}(s) \rangle ds \right| \leq \frac{2}{\sqrt{\mu_0}} \int_0^t \widetilde{I}_2(s) \sqrt{q_m^{(k)}(s)} ds, \tag{3.19}
$$

where $\widetilde{I}_2(s) = \left\| \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right) \right\|$ and so

$$
\begin{aligned}
\widetilde{I}_2(s) &= \left\| \nabla \mu_m(s) \nabla \dot{u}_m^{(k)}(s) + \frac{\partial}{\partial s} (\nabla \mu_m(s)) \nabla u_m^{(k)}(s) \right\| \\
&\leq \| \nabla \mu_m(s) \|_{C^0(\overline{\Omega})} \left\| \nabla \dot{u}_m^{(k)}(s) \right\| + \left\| \frac{\partial}{\partial s} \nabla \mu_m(s) \right\| \left\| \nabla u_m^{(k)}(s) \right\|_{C^0(\overline{\Omega})} \\
&\leq \left(\| \nabla \mu_m(s) \|_{C^0(\overline{\Omega})} + \frac{1}{\sqrt{\mu_0}} \left\| \frac{\partial}{\partial s} \nabla \mu_m(s) \right\| \right) \sqrt{s_m^{(k)}(s)}.
\end{aligned} \tag{3.20}
$$

On the other hand, by $\nabla \mu_m(x,s) = \mu'(u_{m-1}(x,s)) \nabla u_{m-1}(x,s)$, we get

$$
\begin{aligned}
\| \nabla \mu_m(s) \|_{C^0(\overline{\Omega})} &\leq \widetilde{K}_M(\mu) \| \nabla u_{m-1}(s) \|_{C^0(\overline{\Omega})} \leq \widetilde{K}_M(\mu) \sqrt{2} \| \nabla u_{m-1}(s) \|_{H^1} \\
&= \widetilde{K}_M(\mu) \sqrt{2} \sqrt{ \| \nabla u_{m-1}(s) \|^2 + \| \Delta u_{m-1}(s) \|^2 } \leq 2M \widetilde{K}_M(\mu).
\end{aligned} \tag{3.21}
$$

Similarly, from the equality

$$
\begin{aligned}
\frac{\partial}{\partial s} \nabla \mu_m(x,s) =& \mu''(u_{m-1}(x,s)) u_{m-1}'(x,s) \nabla u_{m-1}(x,s) \\
&+ \mu'(u_{m-1}(x,s)) \nabla u_{m-1}'(x,s)
\end{aligned} \tag{3.22}
$$

we obtain

$$
\begin{aligned}
\left\| \frac{\partial}{\partial s} \nabla \mu_m(s) \right\| &\leq \widetilde{K}_M(\mu) \left[\left\| u_{m-1}'(s) \right\|_{C^0(\overline{\Omega})} \| \nabla u_{m-1}(s) \| + || \nabla u_{m-1}'(s) || \right] \\
&\leq (1+M) M \widetilde{K}_M(\mu).
\end{aligned} \tag{3.23}
$$

This inequality and (3.20), (3.21) imply

$$
\widetilde{I}_2(s) = \left\| \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right) \right\| \leq \left(2 + \frac{1+M}{\sqrt{\mu_0}} \right) M \widetilde{K}_M(\mu) \sqrt{s_m^{(k)}(s)}. \tag{3.24}
$$

Consequently,

$$
|I_2| \leq \frac{2}{\sqrt{\mu_0}} \left(2 + \frac{1+M}{\sqrt{\mu_0}} \right) M \widetilde{K}_M(\mu) \int_0^t s_m^{(k)}(s) ds. \tag{3.25}
$$

Third term. Applying again the Cauchy-Schwarz inequality, we infer

$$|I_3| = \left| -2\langle \nabla \mu_m(t) \nabla u_m^{(k)}(t), \Delta u_m^{(k)}(t) \rangle \right| \le \frac{1}{\beta} \left\| \nabla \mu_m(t) \nabla u_m^{(k)}(t) \right\|^2 + \beta \left\| \Delta u_m^{(k)}(t) \right\|^2 \tag{3.26}$$

for all $\beta > 0$. On the other hand,

$$\begin{aligned} \left\| \nabla \mu_m(t) \nabla u_m^{(k)}(t) \right\| &= \left\| \nabla \mu_m(0) \nabla \widetilde{u}_{0k} + \int_0^t \frac{\partial}{\partial s} \left(\nabla \mu_m(s) \nabla u_m^{(k)}(s) \right) ds \right\| \\ &\le \| \nabla \mu_m(0) \|_{C^0(\overline{\Omega})} \| \nabla \widetilde{u}_{0k} \| + \int_0^t \widetilde{I}_2(s) ds. \end{aligned} \tag{3.27}$$

Thus,

$$\begin{aligned} |I_3| \le & \frac{\beta}{\mu_0} q_m^{(k)}(t) + \frac{2}{\beta} ||\nabla \mu_m(0)||^2_{C^0(\overline{\Omega})} ||\nabla \widetilde{u}_{0k}||^2 \\ & + \frac{2}{\beta} T \left(2 + \frac{M+1}{\sqrt{\mu_0}} \right) M^2 \widetilde{K}_M^2(\mu) \int_0^t s_m^{(k)}(s) ds \end{aligned} \tag{3.28}$$

for all $\beta > 0$.

Fourth term. Using (H_3) we obtain from (3.1), (3.4) and (3.14)

$$\begin{aligned} I_4 =& 2 \int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \le 2K_M(f) \int_0^t ||\dot{u}_m^{(k)}(s)|| ds \\ \le & T K_M^2(f) + \int_0^t p_m^{(k)}(s) ds. \end{aligned} \tag{3.29}$$

Fifth term. Combining (3.4), (3.7) and (3.13)-(3.15), we get

$$\begin{aligned} |I_5| =& \left| -2\langle F_m(t), \Delta u_m^{(k)}(t) \rangle \right| \le \frac{1}{\beta} \|F_m(t)\|^2 + \beta \left\| \Delta u_m^{(k)}(t) \right\|^2 \\ \le & \frac{1}{\beta} \left\| F_m(0) + \int_0^t \frac{\partial F_m}{\partial s}(s) ds \right\|^2 + \frac{\beta}{\mu_0} q_m^{(k)}(t) \\ \le & \frac{2}{\beta} \|F_m(0)\|^2 + \frac{2}{\beta} T \int_0^T \left\| \frac{\partial F_m}{\partial s}(s) \right\|^2 ds + \frac{\beta}{\mu_0} s_m^{(k)}(t) \text{ for all } \beta > 0. \end{aligned} \tag{3.30}$$

Note that

$$\begin{aligned} \frac{\partial F_m}{\partial t}(t) =& D_2 f[u_{m-1}] + D_3 f[u_{m-1}] u'_{m-1}(t) + D_4 f[u_{m-1}] \nabla u'_{m-1}(t) \\ & + D_5 f[u_{m-1}] u''_{m-1}(t), \end{aligned} \tag{3.31}$$

where $D_i f[u_{m-1}] = D_i f(x, t, u_{m-1}(x,t), \nabla u_{m-1}(x,t), u'_{m-1}(x,t))$, $i = 2, ..., 5$. So, from (3.1), (3.4) and (3.31) we obtain

$$\begin{aligned} \left\| \frac{\partial F_m}{\partial t}(t) \right\| \le & K_M(f) \left(1 + \left\| u'_{m-1}(t) \right\| + \left\| \nabla u'_{m-1}(t) \right\| + \left\| u''_{m-1}(t) \right\| \right) \\ \le & K_M(f) \left(1 + 2M + \left\| u''_{m-1}(t) \right\| \right). \end{aligned} \tag{3.32}$$

Hence,

(3.33)

$$\begin{aligned}|I_5| \leq &\frac{2}{\beta}||F_m(0)||^2 + \frac{4}{\beta}TK_M^2(f)\int_0^T \left[(1+2M)^2 + \left\|u''_{m-1}(s)\right\|^2\right] ds + \frac{\beta}{\mu_0}s_m^{(k)}(t)\\ \leq &\frac{2}{\beta}||F_m(0)||^2 + \frac{4}{\beta}TK_M^2(f)\left[(1+2M)^2T + M^2\right] + \frac{\beta}{\mu_0}s_m^{(k)}(t) \text{ for all } \beta > 0.\end{aligned}$$

Sixth term. By (3.1), (3.4), (3.15), (3.32) we obtain

$$\begin{aligned}|I_6| =&2\left|\int_0^t \langle\frac{\partial F_m}{\partial t}(s), \Delta u_m^{(k)}(s)\rangle ds\right|\\ \leq&\int_0^t \left\|\frac{\partial F_m}{\partial t}(s)\right\| ds + \int_0^t \left\|\frac{\partial F_m}{\partial t}(s)\right\| \left\|\Delta u_m^{(k)}(s)\right\|^2 ds\\ \leq&K_M(f)\left[(1+2M)T + \sqrt{T}\left(\int_0^T \left\|u''_{m-1}(s)\right\|^2 ds\right)^{1/2}\right]\\ &+\frac{1}{\mu_0}K_M(f)\int_0^t \left(1+2M+\left\|u''_{m-1}(s)\right\|\right) q_m^{(k)}(s)ds\\ \leq&K_M(f)\left[(1+2M)T + \sqrt{T}M\right]\\ &+\frac{1}{\mu_0}K_M(f)\int_0^t \left(1+2M+\left\|u''_{m-1}(s)\right\|\right) q_m^{(k)}(s)ds.\end{aligned} \tag{3.34}$$

Seventh term. Equation (3.9) can be rewritten as

$$\langle\ddot{u}_m^{(k)}(t), w_j\rangle - \langle\frac{\partial}{\partial x}\left(\mu_m(t)\nabla u_m^{(k)}(t)\right), w_j\rangle = \langle F_m(t), w_j\rangle, 1 \leq j \leq k. \tag{3.35}$$

Hence, by replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating we obtain

(3.36)

$$\begin{aligned}I_7 = &\int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \leq 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\|^2 ds + 2\int_0^t \|F_m(s)\|^2 ds\\ \leq&2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\|^2 ds + 2TK_M^2(f).\end{aligned}$$

We estimate the term $\left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla v_m^{(k)}(s)\right)\right\|$.

By (3.1), (3.4) and (3.13)-(3.15)

(3.37)
$$\begin{aligned}\left\|\frac{\partial}{\partial x}\left(\mu_m(s)\nabla u_m^{(k)}(s)\right)\right\| &= \left\|\nabla\mu_m(s)\nabla u_m^{(k)}(s)+\mu_m(s)\Delta u_m^{(k)}(s)\right\|\\ &\le \|\nabla\mu_m(s)\|_{C^0(\overline{\Omega})}\left\|\nabla u_m^{(k)}(s)\right\|+\|\mu_m(s)\|_{C^0(\overline{\Omega})}\left\|\Delta u_m^{(k)}(s)\right\|\\ &\le \frac{1}{\sqrt{\mu_0}}M\widetilde{K}_M(\mu)\sqrt{p_m^{(k)}(s)}+\frac{1}{\sqrt{\mu_0}}\widetilde{K}_M(\mu)\sqrt{q_m^{(k)}(s)}\\ &\le \frac{1}{\sqrt{\mu_0}}(1+M)\widetilde{K}_M(\mu)\sqrt{s_m^{(k)}(s)}.\end{aligned}$$

Therefore, from (3.36) and (3.37) we obtain

$$I_7 \le 2TK_M^2(f)+\frac{2}{\mu_0}(1+M)^2\widetilde{K}_M^2(\mu)\int_0^t s_m^{(k)}(s)ds. \tag{3.38}$$

Choosing $\beta>0$ such that $\frac{2\beta}{\mu_0}\le\frac{1}{2}$, from (3.13)-(3.16) and seven above we get the estimations

$$\begin{aligned} s_m^{(k)}(t) \le& \widetilde{C}_{0k}(\beta,f,\mu,\widetilde{u}_0,\widetilde{u}_1,\widetilde{u}_{0k},\widetilde{u}_{1k})+\widetilde{C}_1(\beta,f,M,T)\\ &+\int_0^t\left(\widetilde{C}_2(\beta,f,\mu,M,T)+\frac{2}{\mu_0}K_M(f)\left\|u''_{m-1}(s)\right\|\right)s_m^{(k)}(s)ds,\end{aligned} \tag{3.39}$$

where

(3.40)
$$\begin{cases}\widetilde{C}_{0k}(\beta,f,\mu,\widetilde{u}_0,\widetilde{u}_1,\widetilde{u}_{0k},\widetilde{u}_{1k}) = 2s_m^{(k)}(0)+4\langle\nabla\mu_m(0)\nabla\widetilde{u}_{0k},\Delta\widetilde{u}_{0k}\rangle\\ \quad +4\langle F_m(0),\Delta\widetilde{u}_{0k}\rangle+\frac{4}{\beta}||\nabla\mu_m(0)||^2_{C^0(\overline{\Omega})}||\nabla\widetilde{u}_{0k}||^2+\frac{4}{\beta}||F_m(0)||^2,\\ \widetilde{C}_1(\beta,f,M,T) = 2\left(3+\frac{4}{\beta}\left[(1+2M)^2T+M^2\right]\right)TK_M^2(f)\\ \quad +2\left[(1+2M)\sqrt{T}+M\right]\sqrt{T}K_M(f),\\ \widetilde{C}_2(\beta,f,\mu,M,T) = 2+\frac{2}{\mu_0}\left[1+2\sqrt{\mu_0}\left(2+\frac{M+1}{\sqrt{\mu_0}}\right)\right]M\widetilde{K}_M(\mu)\\ \quad +\frac{2}{\mu_0}(1+2M)K_M(f)+4\left[\frac{2}{\beta}T\left(2+\frac{M+1}{\sqrt{\mu_0}}\right)M^2+\frac{1}{\mu_0}(1+M)^2\right]\widetilde{K}_M^2(\mu).\end{cases}$$

By (H$_1$) we can deduce from (3.10), $(3.40)_1$ that there exists $M>0$, independent of m and k, such that

$$\widetilde{C}_{0k}(\beta,f,\mu,\widetilde{u}_0,\widetilde{u}_1,\widetilde{u}_{0k},\widetilde{u}_{1k}) \le \frac{1}{2}M^2. \tag{3.41}$$

Notice that, by the assumptions (H_2), (H_3), we deduce from $(3.40)_{2,3}$ that

$$\lim_{T\to 0_+}\widetilde{C}_1(\beta,f,M,T)=\lim_{T\to 0_+}T\widetilde{C}_2(\beta,f,\mu,M,T)=0. \tag{3.42}$$

So, by (3.40) and (3.42), we can choose $T>0$ such that

(3.43)
$$\left(\frac{1}{2}M^2+\widetilde{C}_1(\beta,f,M,T)\right)\exp\left(T\widetilde{C}_2(\beta,f,\mu,M,T)+\frac{2}{\mu_0}K_M(f)\sqrt{T}M\right)\leq M^2,$$

and

(3.44)
$$k_T=\left(1+\frac{1}{\sqrt{\mu_0}}\right)\sqrt{T}\sqrt{4K_M^2(f)+(3+2M)^2M^2\widetilde{K}_M^2(\mu)}e^{T\left[1+\frac{1}{2\mu_0}M\widetilde{K}_M(\mu)\right]}<1.$$

Finally, it follows from (3.39), (3.41) and (3.43) that

$$\begin{aligned}s_m^{(k)}(t)\leq&M^2\exp\left(-T\widetilde{C}_2(\beta,f,\mu,M,T)-\frac{2}{\mu_0}K_M(f)\sqrt{T}M\right)\\&+\int_0^t\left(\widetilde{C}_2(\beta,f,\mu,M,T)+\frac{2}{\mu_0}K_M(f)\left\|u''_{m-1}(s)\right\|\right)s_m^{(k)}(s)ds.\end{aligned}\tag{3.45}$$

By using Gronwall's lemma we deduce from (3.4), (3.43), (3.45) that

(3.46)
$$\begin{aligned}s_m^{(k)}(t)\leq&M^2\exp\left(-T\widetilde{C}_2(\beta,f,\mu,M,T)-\frac{2}{\mu_0}K_M(f)\sqrt{T}M\right)\\&\times\exp\left[\int_0^T\left(\widetilde{C}_2(\beta,f,\mu,M,T)+\frac{2}{\mu_0}K_M(f)\left\|u''_{m-1}(s)\right\|\right)ds\right]\\\leq&M^2\exp\left(-T\widetilde{C}_2(\beta,f,\mu,M,T)-\frac{2}{\mu_0}K_M(f)\sqrt{T}M\right)\\&\times\exp\left[T\widetilde{C}_2(\beta,f,\mu,M,T)+\frac{2}{\mu_0}K_M(f)\sqrt{T}\left\|u''_{m-1}\right\|_{L^2(Q_T)}\right]\leq M^2.\end{aligned}$$

Therefore,

$$u_m^{(k)}\in W(M,T)\ \forall m,k\in\mathbb{N}.\tag{3.47}$$

Step 3. *Limiting Process.*

By (3.47) we can extract from $\{u_m^{(k)}\}$ a subsequence, still denoted by $\{u_m^{(k)}\}$, such that

$$\begin{cases}u_m^{(k)}\to u_m\text{ in }L^\infty(0,T;H_0^1\cap H^2)\text{ weakly*},\\\dot{u}_m^{(k)}\to u'_m\text{ in }L^\infty(0,T;H_0^1)\text{ weakly*},\\\ddot{u}_m^{(k)}\to u''_m\text{ in }L^2(Q_T)\text{ weakly},\end{cases}\tag{3.48}$$

as $k\to\infty$, and

$$u_m\in W(M,T).\tag{3.49}$$

Based on (3.48), passing to the limit as $k\to\infty$ in (3.9)-(3.10), we have u_m satisfying (3.5) – (3.7). On the other hand, it follows from (3.5) and $(3.48)_1$ that

(3.50)
$$u''_m=\mu'(u_{m-1})\nabla u_{m-1}\nabla u_m+\mu_m\Delta u_m+f(x,t,u_{m-1},\nabla u_{m-1},u'_{m-1})\in L^\infty(0,T;L^2).$$

Consequently, $u_m \in W_1(M,T)$, and the proof of Theorem 3.1 is complete. $\square$

Theorem 3.2. *Suppose that* (H_1)-(H_3) *hold. Then, there exist* $M>0$ *and* $T>0$ *satisfying* (3.41), (3.43), (3.44) *such that the problem* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$. *Furthermore, the linear recurrent sequence* $\{u_m\}$ *defined by* (3.5)-(3.7) *converges to the solution* u *strongly in the space*

$$W_1(T)=\{w\in L^\infty(0,T;H_0^1): w'\in L^\infty(0,T;L^2)\}, \tag{3.51}$$

with the estimation

$$\|u_{mx}-u_x\|_{L^\infty(0,T;L^2)}+\left\|u_m'-u'\right\|_{L^\infty(0,T;L^2)}\le Ck_T^m \text{ for all } m\in\mathbb{N}, \tag{3.52}$$

where C *is a constant depending only on* T, $\widetilde{u}_0$, $\widetilde{u}_1$ *and* k_T.

Proof. **(i)** *Existence.* First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [5])

$$\|w\|_{W_1(T)}=\|w_x\|_{L^\infty(0,T;L^2)}+\left\|w'\right\|_{L^\infty(0,T;L^2)}. \tag{3.53}$$

Next, we prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m=u_{m+1}-u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t),w\rangle+\langle \mu_{m+1}(t)\nabla v_m(t),\nabla w\rangle = \langle \frac{\partial}{\partial x}\left[(\mu_{m+1}(t)-\mu_m(t))\nabla u_m(t)\right],w\rangle \\ \qquad\qquad\qquad\qquad + \langle F_{m+1}(t)-F_m(t),w\rangle,\ \forall w\in H_0^1, \\ v_m(0)=v_m'(0)=0. \end{cases} \tag{3.54}$$

Taking $w=v_m'$ in $(3.54)_1$, after integrating in t, we get

$$\begin{aligned} z_m(t) &= \textstyle\int_0^t ds\int_0^1 \mu_{m+1}'(x,s)\left|\nabla v_m(s)\right|^2dx+2\int_0^t\langle F_{m+1}(s)-F_m(s),v_m'(s)\rangle ds \\ &\quad +2\textstyle\int_0^t\langle\frac{\partial}{\partial x}\left[(\mu_{m+1}(s)-\mu_m(s))\nabla u_m(s)\right],v_m'(s)\rangle ds=\sum_{i=1}^3 J_i, \end{aligned} \tag{3.55}$$

in which

$$z_m(t)=\|v_m'(t)\|^2+\left\|\sqrt{\mu_{m+1}(t)}\nabla v_m(t)\right\|^2, \tag{3.56}$$

and all integrals on the right hand side of (3.55) are estimated as follows.

First integral. By (H_2),

$$|J_1|\le\left|\textstyle\int_0^t ds\int_0^1\mu_{m+1}'(x,s)\left|\nabla v_m(s)\right|^2dx\right|\le\frac{1}{\mu_0}M\widetilde{K}_M(\mu)\textstyle\int_0^t z_m(s)ds. \tag{3.57}$$

Second integral. Also by (H_3),

$$\begin{aligned}\|F_{m+1}(t)-F_m(t)\| &\le 2K_M(f)\left[\|\nabla v_{m-1}(t)\|+\left\|v_{m-1}'(t)\right\|\right] \\ &\le 2K_M(f)\left\|v_{m-1}\right\|_{W_1(T)},\end{aligned} \tag{3.58}$$

so

$$\begin{aligned}|J_2| &\le 2\left|\int_0^t\left\langle F_{m+1}(s)-F_m(s),v_m'(s)\right\rangle ds\right| \\ &\le 4TK_M^2(f)\left\|w_{m-1}\right\|_{W_1(T)}^2+\int_0^t z_m(s)ds.\end{aligned} \tag{3.59}$$

Third integral. Using (H_2),

$$(3.60)\quad \begin{aligned} |J_3| &= 2\left|\int_0^t \langle \tfrac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right], v_m'(s)\rangle ds\right| \\ &\le 2\int_0^t \left\|\tfrac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]\right\| \|v_m'(s)\|\, ds \\ &\le \int_0^t \left\|\tfrac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]\right\|^2 ds + \int_0^t z_m(s)ds. \end{aligned}$$

Note that

$$(3.61)\quad \begin{aligned} &\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(t)-\mu_m(t)\right)\nabla u_m(t)\right] \\ &= \left(\mu_{m+1}(t)-\mu_m(t)\right)\Delta u_m(t) + \mu'(u_m(t))\nabla v_{m-1}(t)\nabla u_m(t) \\ &\quad + \left(\mu'(u_m(t))-\mu'(u_{m-1}(t))\right)\nabla u_{m-1}(t)\nabla u_m(t). \end{aligned}$$

Hence

$$(3.62)\quad \begin{aligned} &\left\|\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(t)-\mu_m(t)\right)\nabla u_m(t)\right]\right\| \\ &\le \|\mu_{m+1}(t)-\mu_m(t)\|_{C^0(\overline{\Omega})}\|\Delta u_m(t)\| \\ &\quad + \left\|\mu'(u_m(t))\right\|_{C^0(\overline{\Omega})}\|\nabla v_{m-1}(t)\|\,\|\nabla u_m(t)\|_{C^0(\overline{\Omega})} \\ &\quad + \left\|\mu'(u_m(t))-\mu'(u_{m-1}(t))\right\|_{C^0(\overline{\Omega})}\|\nabla u_{m-1}(t)\|\,\|\nabla u_m(t)\|_{C^0(\overline{\Omega})}. \end{aligned}$$

On the other hand,

$$(3.63)\quad \begin{aligned} &||\nabla u_m(t)||_{C^0(\overline{\Omega})} \le \sqrt{2}\,\|\nabla u_m(t)\|_{H^1} \le \sqrt{2}\sqrt{||\nabla u_m(t)||^2+\|\Delta u_m(t)\|^2} \le 2M, \\ &\|\mu'(u_m(t))\|_{C^0(\overline{\Omega})} \le \widetilde{K}_M(\mu), \\ &\|\mu_{m+1}(t)-\mu_m(t)\|_{C^0(\overline{\Omega})} \le \widetilde{K}_M(\mu)\,\|\nabla v_{m-1}(t)\| \le \widetilde{K}_M(\mu)||v_{m-1}||_{W(T)_1}, \\ &\|\mu'(u_m(t))-\mu'(u_{m-1}(t))\|_{C^0(\overline{\Omega})} \le \widetilde{K}_M(\mu)\,\|\nabla v_{m-1}(t)\| \le \widetilde{K}_M(\mu)||v_{m-1}||_{W_1(T)}. \end{aligned}$$

Therefore, we deduce from (3.62) and (3.63) that

$$(3.64)\quad \left\|\tfrac{\partial}{\partial x}\left[\left(\mu_{m+1}(t)-\mu_m(t)\right)\nabla u_m(t)\right]\right\| \le (3+2M)\,M\widetilde{K}_M(\mu)||v_{m-1}||_{W_1(T)}.$$

Hence

$$(3.65)\quad |J_3| \le (3+2M)^2 M^2 T\widetilde{K}_M^2(\mu)||v_{m-1}||^2_{W_1(T)} + \int_0^t z_m(s)ds.$$

A combination of (3.55), (3.56), (3.57), (3.59) and (3.65) yields

$$(3.66)\quad \begin{aligned} z_m(t) &\le T\left[4K_M^2(f) + (3+2M)^2 M^2\widetilde{K}_M^2(\mu)\right]||v_{m-1}||^2_{W_1(T)} \\ &\quad + \left(2+\tfrac{1}{\mu_0}M\widetilde{K}_M(\mu)\right)\int_0^t z_m(s)ds. \end{aligned}$$

Using Gronwall's lemma, this inequality leads to

$$(3.67)\quad \|v_m\|_{W_1(T)} \le k_T\,\|v_{m-1}\|_{W_1(T)}\ \forall m\in\mathbb{N},$$

consequently

$$\|u_{m+p} - u_m\|_{W_1(T)} \leq \tfrac{k_T^m}{1-k_T} \|u_1 - u_0\|_{W_1(T)} \quad \forall m,\, p \in \mathbb{N}, \tag{3.68}$$

where k_T is as in (3.44).

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.69}$$

Therefore, a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ can be found such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in } L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in } L^2(Q_T) \text{ weakly}, \end{cases} \tag{3.70}$$

and

$$u \in W(M,T). \tag{3.71}$$

Note that

$$\begin{cases} \|\mu(u_{m-1}) - \mu(u)\|_{L^\infty(Q_T)} \leq \widetilde{K}_M(\mu) \|u_{m-1} - v\|_{W_1(T)}, \\ \|F_m - f(\cdot,\cdot,u,u_x,u')\|_{L^\infty(0,T;L^2)} \leq 2K_M(f) \|u_{m-1} - u\|_{W_1(T)}. \end{cases} \tag{3.72}$$

Hence, from (3.69) and (3.72) we get

$$\begin{cases} \mu(u_m) \to \mu(u) & \text{strongly in} \quad L^\infty(Q_T), \\ F_m \to f(\cdot,\cdot,u,u_x,u') & \text{strongly in} \quad L^\infty(0,T;L^2). \end{cases} \tag{3.73}$$

Finally, passing to the limit in (3.5) – (3.7) as $m = m_j \to \infty$, it follows from (3.69), (3.70) and (3.73) that there exists $u \in W(M,T)$ satisfying the equation

$$\begin{cases} \langle u''(t), w \rangle + \langle \mu(u(t)) u_x(t), w_x \rangle = \left\langle f(\cdot,t,u(t),u_x(t),u'(t)), w \right\rangle, \forall w \in H_0^1, \\ \qquad u(0) = \widetilde{u}_0, \ u'(0) = \widetilde{u}_1. \end{cases} \tag{3.74}$$

Moreover, by (H_2), (H_3) we obtain from (3.71), $(3.73)_2$ and $(3.74)_1$ that

$$u'' = \mu'(u) u_x^2 + \mu(u) u_{xx} + f(x,t,u,u_x,u') \in L^\infty(0,T;L^2), \tag{3.75}$$

thus $u \in W_1(M,T)$ and Step 1 follows.

(ii) *Uniqueness of a weak solution.*

Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of the problem (1.1)−(1.3). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t) \quad , w \rangle + \langle \mu_1(t) u_x(t), w_x \rangle = \langle \frac{\partial}{\partial x} \left([\mu_1(t) - \mu_2(t)] \, u_{2x}(t) \right), w \rangle \\ \qquad\qquad + \langle F_2(t) - F_1(t), w \rangle \ \ \forall w \in H_0^1, \\ u(0) = \quad u'(0) = 0, \\ \mu_i(t) = \quad \mu(u_i(t)), \ F_i(t) = f(x,t,u_i(t),u_{ix}(t),u'_i(t)), \ i = 1,2. \end{cases} \tag{3.76}$$

We take $w = u'$ in $(3.76)_1$ and integrate in t to get

$$\begin{aligned}\rho(t) = &\int_0^t ds \int_0^1 \mu_1'(x,s) u_x^2(x,s) dx + 2\int_0^t \left\langle F_1(s) - F_2(s), u'(s)\right\rangle ds \\ &+ 2\int_0^t \left\langle \frac{\partial}{\partial x}\left([\mu_1(s) - \mu_2(s)]\, u_{2x}(s)\right), u' \right\rangle ds,\end{aligned} \tag{3.77}$$

where

$$\rho(t) = \left\| u'(t) \right\|^2 + \left\| \sqrt{\mu_1(t)} u_x(t) \right\|^2. \tag{3.78}$$

It follows from (3.77), (3.78) that

$$\rho(t) \le \overline{K}_M \int_0^t \rho(s) ds, \tag{3.79}$$

in which

$$\overline{K}_M = 4\left(1 + \frac{1}{\sqrt{\mu_0}}\right) K_M(f) + \left[\frac{1}{\mu_0} + \frac{2}{\sqrt{\mu_0}}(2+M)M\right] \widetilde{K}_M(\mu). \tag{3.80}$$

Using Gronwall's lemma it follows from (3.79) that $\rho \equiv 0$, i.e., $u_1 \equiv u_2$. Theorem 3.2 is proved completely. $\square$

Remark 3.1. (i) In the case that $\mu \equiv 1$, $f = f(t,u,u_t)$ with $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$ and $f(t,0,0) = 0\ \forall t \ge 0$, some results in [3] have been obtained here.

(ii) In the case that $\mu \equiv 1$, $f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$ and the boundary condition in [9] standing for (1.2), we have also obtained the results concerning the ones in the paper [9].

4. Asymptotic expansion of the solution with respect to many small parameters

In this section, suppose that (H_1)-(H_3) hold. We also make the assumptions:

(H_4) $\mu_i \in C^2(\mathbb{R}),\ \mu_i \ge 0, i = 1,2,...,p,$

(H_5) $f_i \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3), i = 1,2,...,p.$

We consider the following perturbed problem, where $\varepsilon_1, ..., \varepsilon_p$ are p small parameters such that $0 \le \varepsilon_i \le \varepsilon_{i*} < 1$, $i = 1,2,...,p$:

$$(P_{\vec{\varepsilon}}) \begin{cases} u_{tt} - \frac{\partial}{\partial x}\left(\mu_{\vec{\varepsilon}}(u) u_x\right) = F_{\vec{\varepsilon}}(x,t,u,u_x,u_t),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \\ \mu_{\vec{\varepsilon}}(u) = \mu(u) + \sum_{i=1}^p \varepsilon_i \mu_i(u), \\ F_{\vec{\varepsilon}}(x,t,u,u_x,u_t) = f(x,t,u,u_x,u_t) + \sum_{i=1}^p \varepsilon_i f_i(x,t,u,u_x,u_t). \end{cases}$$

By Theorem 3.2, the problem $(P_{\vec{\varepsilon}})$ has a unique local solution u depending on $\vec{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p) : u_{\vec{\varepsilon}} = u(\varepsilon_1, ..., \varepsilon_p)$. When $\vec{\varepsilon} = (0,...,0)$, $(P_{\vec{\varepsilon}})$ is denoted by (P_0). We shall study the asymptotic expansion of the solution of $(P_{\vec{\varepsilon}})$ with respect to p small parameters $\varepsilon_1, \varepsilon_2, ..., \varepsilon_p$.

We use the following notations. For a multi-index $\alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{Z}_+^p$ and $\overrightarrow{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p) \in \mathbb{R}^p$, we put

$$\begin{cases} |\alpha| = \alpha_1 + ... + \alpha_p, \ \alpha! = \alpha_1!...\alpha_p!, \\ \|\overrightarrow{\varepsilon}\| = \sqrt{\varepsilon_1^2 + ... + \varepsilon_p^2}, \ \overrightarrow{\varepsilon}^\alpha = \varepsilon_1^{\alpha_1}...\varepsilon_p^{\alpha_p}, \\ \alpha, \ \beta \in \mathbb{Z}_+^p, \ \alpha \le \beta \Longleftrightarrow \alpha_i \le \beta_i \ \forall i = 1, ..., p. \end{cases} \tag{4.1}$$

First, we state the following lemma.

Lemma 4.1. *Let m, $N \in \mathbb{N}$ and $u_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le N$. Then*

$$\left(\sum\nolimits_{1 \le |\alpha| \le N} u_\alpha \overrightarrow{\varepsilon}^\alpha\right)^m = \sum\nolimits_{m \le |\alpha| \le mN} T_N^{(m)}[u]_\alpha \overrightarrow{\varepsilon}^\alpha, \tag{4.2}$$

where the coefficients $T_N^{(m)}[u]_\alpha$, $m \le |\alpha| \le mN$ depending on $u = (u_\alpha)$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le N$ are defined by the recurrent formulas

(4.3)
$$\begin{cases} T_N^{(1)}[u]_\alpha = u_\alpha, \ 1 \le |\alpha| \le N, \\ T_N^{(m)}[u]_\alpha = \displaystyle\sum_{\beta \in A_\alpha^{(m)}(N)} u_{\alpha - \beta} T_N^{(m-1)}[u]_\beta, \ m \le |\alpha| \le mN, \ m \ge 2, \\ A_\alpha^{(m)}(N) = \{\beta \in \mathbb{Z}_+^p : \beta \le \alpha, \ 1 \le |\alpha - \beta| \le N, \ m - 1 \le |\beta| \le (m-1)N\}. \end{cases}$$

The proof of Lemma 4.1 can be found in [13]. □

Now we assume

$$(\mathrm{H}_6) \qquad \mu \in C^{N+2}(\mathbb{R}), \ \mu_i \in C^{N+1}(\mathbb{R}), \ \mu \ge \mu_0 > 0, \ \mu_i \ge 0, \ i = 1, 2, ..., p,$$

$$(\mathrm{H}_7) \qquad f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3), \ f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3), \ i = 1, 2, ..., p,$$

and use the notations $f[u] = f(x, t, u, u_x, u_t)$, $\mu[u] = \mu(u)$.

Let u_0 be a unique weak solution of the problem (P_0) (as in Theorem 3.2) corresponding to $\overrightarrow{\varepsilon} = (0, ..., 0)$, i.e.,

$$(P_0) \begin{cases} u_0'' - \dfrac{\partial}{\partial x}(\mu(u_0) u_{0x}) = f(x, t, u_0, u_{0x}, u_0') \equiv f[u_0], \ 0 < x < 1, \ 0 < t < T, \\ u_0(0, t) = u_0(1, t) = 0, \\ u_0(x, 0) = \widetilde{u}_0(x), \ u_0'(x, 0) = \widetilde{u}_1(x), \\ u_0 \in W_1(M, T). \end{cases}$$

Let us consider the sequence of weak solutions u_γ, $\gamma \in \mathbb{Z}_+^p$, $1 \le |\gamma| \le N$, defined by the following problems:

$$(\widetilde{P}_\gamma) \begin{cases} u_\gamma'' - \dfrac{\partial}{\partial x}(\mu(u_0) u_{\gamma x}) = F_\gamma, \ 0 < x < 1, \ 0 < t < T, \\ u_\gamma(0, t) = u_\gamma(1, t) = 0, \\ u_\gamma(x, 0) = u_\gamma'(x, 0) = 0, \\ u_\gamma \in W_1(M, T), \end{cases}$$

where $F_\gamma,\ \gamma\in\mathbb{Z}_+^p,\ 1\le|\gamma|\le N$, are defined by the recurrent formulas

(4.4)
$$F_\gamma=\begin{cases} f[u_0]\equiv f(x,t,u_0,u_{0x},u_0'),\ |\gamma|=0,\\ \pi_\gamma[f]+\sum_{i=1}^{p}\pi_\gamma^{(i)}[f_i]+\sum_{1\le|\nu|\le|\gamma|,\ \nu\le\gamma}\frac{\partial}{\partial x}\left[\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_{\gamma-\nu}\right], \\ \qquad\qquad 1\le|\gamma|\le N,\end{cases}$$

with $\rho_\delta[\mu]=\rho_\delta[\mu;\{u_\gamma\}_{\gamma\le\delta}],\ \rho_\delta^{(i)}[\mu]=\rho_\delta^{(i)}[\mu;\{u_\gamma\}_{\gamma\le\delta}],\ \pi_\delta[f]=\pi_\delta[f;\{u_\gamma\}_{\gamma\le\delta}]$, $\pi_\delta^{(i)}[f]=\pi_\delta^{(i)}[f;\{u_\gamma\}_{\gamma\le\delta}],\ |\delta|\le N$, also defined by the recurrent formulas

$$\rho_\delta[\mu]=\begin{cases}\mu(u_0), & |\delta|=0,\\ \sum_{m=1}^{|\delta|}\frac{1}{m!}\mu^{(m)}(u_0)T_\delta^{(m)}[u], & 1\le|\delta|\le N,\end{cases}\tag{4.5}$$

$$\begin{cases}\delta=(\delta_1,\delta_2,...,\delta_p)\in\mathbb{Z}_+^p,\ \delta^{(i-)}=(\delta_1,...,\delta_{i-1},\delta_i-1,\delta_{i+1},...,\delta_p),\\ \rho_\delta^{(i)}[\mu]=\rho_{\delta^{(i-)}}[\mu]=\rho_{\delta_1,\delta_2,...,\delta_{i-1},\ \delta_i-1,\ \delta_{i+1},...,\delta_p}[\mu],\\ \rho_\delta^{(i)}[\mu]=\rho_{\delta_1,\delta_2,...,\delta_{i-1},-1,\delta_{i+1},...,\delta_p}[\mu]=0,\ \text{if } \delta_i=0,\end{cases}\tag{4.6}$$

(4.7)
$$\pi_\delta[f]=\begin{cases} f[u_0],\ |\delta|=0,\\ \sum_{1\le|m|\le|\delta|}\ \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\delta}}\frac{1}{m!}D^m f[u_0]T_N^{(m_1)}[u]_\alpha T_N^{(m_2)}[\nabla u]_\beta T_N^{(m_3)}[u']_\gamma,\\ \qquad\qquad 1\le|\delta|\le N,\end{cases}$$

where $m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3,\ |m|=m_1+m_2+m_3,\ m!=m_1!m_2!m_3!,\ D^m f=D_3^{m_1}D_4^{m_2}D_5^{m_3}f,\ A(m,N)=\{(\alpha,\beta,\gamma)\in\left(\mathbb{Z}_+^p\right)^3: m_1\le|\alpha|\le m_1N,\ m_2\le|\beta|\le m_2N,\ m_3\le|\gamma|\le m_3N\}$,

$$\begin{cases}\pi_\delta^{(i)}[f]=\pi_{\delta^{(i-)}}[f]=\pi_{\delta_1,\delta_2,...,\delta_{i-1},\delta_i-1,\delta_{i+1},...,\delta_p}[f], i=1,2,...,p,\\ \pi_\delta^{(i)}[f]=\pi_{\delta_1,\delta_2,...,\delta_{i-1},-1,\delta_{i+1},...,\delta_p}[f]=0,\ \text{if } \delta_i=0,\\ \delta=(\delta_1,\delta_2,...,\delta_p)\in\mathbb{Z}_+^p,\ \delta^{(i-)}=(\delta_1,...,\delta_{i-1},\delta_i-1,\delta_{i+1},...,\delta_p).\end{cases}\tag{4.8}$$

Then we have the following lemma.

Lemma 4.2. *Let $\rho_\nu[\mu],\ \pi_\nu[f],\ |\nu|\le N$, be the functions defined by the formulas (4.5) and (4.7). Put $h=\sum_{|\gamma|\le N}u_\gamma\vec{\varepsilon}^{\,\gamma}$. Then we have*

$$\mu(h)=\sum_{|\nu|\le N}\rho_\nu[\mu]\vec{\varepsilon}^{\,\nu}+\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\vec{\varepsilon}],\tag{4.9}$$

$$f[h]=\sum_{|\nu|\le N}\pi_\nu[f]\vec{\varepsilon}^{\,\nu}+\|\vec{\varepsilon}\|^{N+1}R_N^{(1)}[f,\vec{\varepsilon}],\tag{4.10}$$

with $\left\|\widetilde{R}_N^{(1)}[\mu,\vec{\varepsilon}]\right\|_{L^\infty(0,T;L^2)}+\left\|R_N^{(1)}[f,\vec{\varepsilon}]\right\|_{L^\infty(0,T;L^2)}\le C$, where C is a constant depending only on $N,\ T,\ f,\ \mu,\ u_\gamma,\ |\gamma|\le N$.

Proof. **(i)** In the case that $N=1$, the proof of (4.9) is easy, so we only consider the case that $N \geq 2$. We write $h = u_0 + \sum_{1\leq|\gamma|\leq N} u_\gamma \vec{\varepsilon}^{\,\gamma} \equiv u_0 + h_1$.

By using Taylor's expansion of the function $\mu(h) = \mu(u_0+h_1)$ around the point u_0 up to order $N+1$, (4.2) leads to

(4.11)

$$\begin{aligned}
\mu(u_0+h_1) =& \mu(u_0) + \sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}(u_0)h_1^m + \frac{1}{N!}\int_0^1 (1-\theta)^N \mu^{(N+1)}(u_0+\theta h_1)h_1^{N+1}d\theta \\
=& \mu(u_0) + \sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}(u_0) \sum_{m\leq|\nu|\leq mN} T_\nu^{(m)}[u]\vec{\varepsilon}^{\,\nu} + \widetilde{R}_N^{(1)}[\mu, h_1] \\
=& \mu(u_0) + \sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}(u_0) \sum_{m\leq|\nu|\leq N} T_\nu^{(m)}[u]\vec{\varepsilon}^{\,\nu} \\
&+ \sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}(u_0) \sum_{N+1\leq|\nu|\leq mN} T_\nu^{(m)}[u]\vec{\varepsilon}^{\,\nu} + \widetilde{R}_N^{(1)}[\mu, h_1]
\end{aligned}$$

with

$$\widetilde{R}_N^{(1)}[\mu, h_1] = \frac{1}{N!}\int_0^1 (1-\theta)^N \mu^{(N+1)}(u_0+\theta h_1)h_1^{N+1}d\theta. \tag{4.12}$$

We also note that

(4.13)

$$\sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}(u_0) \sum_{m\leq|\nu|\leq N} T_\nu^{(m)}[u]\vec{\varepsilon}^{\,\nu} = \sum_{1\leq|\nu|\leq N} \left(\sum_{m=1}^{|\nu|} \frac{1}{m!}\mu^{(m)}(u_0)T_\nu^{(m)}[u] \right) \vec{\varepsilon}^{\,\nu}.$$

On the other hand, if we put

(4.14)

$$\widetilde{R}_N^{(1)}[\mu, \vec{\varepsilon}] = \|\vec{\varepsilon}\|^{-N-1} \left(\sum_{m=1}^{N} \tfrac{1}{m!}\mu^{(m)}(u_0) \sum_{N+1\leq|\nu|\leq mN} T_\nu^{(m)}[u]\vec{\varepsilon}^{\,\nu} + \widetilde{R}_N^{(1)}[\mu, h_1] \right),$$

by the boundedness of the functions u_γ, ∇u_γ, u'_γ, $|\gamma| \leq N$ in the function space $L^\infty(0,T;H^1)$, we then obtain from (4.3), (4.12), (4.14) that $\left\| \widetilde{R}_N^{(1)}[\mu, \vec{\varepsilon}] \right\|_{L^\infty(0,T;L^2)} \leq C$, where C is a constant depending only on N, T, μ, u_γ, $|\gamma| \leq N$. Therefore, we obtain from (4.5), (4.11), (4.13), (4.14) that

(4.15)

$$\begin{aligned}
\mu(u_0+h_1) =& \mu(u_0) + \sum_{1\leq|\nu|\leq N} \left(\sum_{m=1}^{|\nu|} \frac{1}{m!}\mu^{(m)}(u_0)T_\nu^{(m)}[u] \right) \vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^{N+1}\, \widetilde{R}_N^{(1)}[\mu, \vec{\varepsilon}] \\
=& \sum_{|\nu|\leq N} \rho_\nu[\mu]\vec{\varepsilon}^{\,\nu} + \|\vec{\varepsilon}\|^{N+1}\, \widetilde{R}_N^{(1)}[\mu, \vec{\varepsilon}].
\end{aligned}$$

Hence, part 1 of Lemma 4.2 is proved.

(ii) We only prove (4.10) for $N \geq 2$. By using Taylor's expansion of the function $f[u_0 + h_1]$ around the point u_0 up to order $N + 1$, we deduce from (4.2) that

(4.16)
$$\begin{aligned}
&f[u_0 + h_1] \\
=&f[u_0] + D_3 f[u_0]h_1 + D_4 f[u_0]\nabla h_1 + D_5 f[u_0]h_1' \\
&+ \sum_{\substack{2\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \frac{1}{m!} D^m f[u_0] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} + R_N^{(1)}[f, h_1] \\
=&f[u_0] + D_3 f[u_0]h_1 + D_4 f[u_0]\nabla h_1 + D_5 f[u_0]h_1' \\
&+ \sum_{\substack{2\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\leq|\nu|\leq|m|N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
&+ R_N^{(1)}[f, h_1] \\
=&f[u_0] + D_3 f[u_0]h_1 + D_4 f[u_0]\nabla h_1 + D_5 f[u_0]h_1' \\
&+ \sum_{\substack{2\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\leq|\nu|\leq N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
&+ \sum_{\substack{2\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{N+1\leq|\nu|\leq|m|N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
&+ R_N^{(1)}[f, h_1],
\end{aligned}$$

where

(4.17)
$$R_N^{(1)}[f, h_1] = \sum_{\substack{|m|=N+1 \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \frac{N+1}{m!} \int_0^1 (1-\theta)^N D^m f[u_0+\theta h_1] h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} d\theta.$$

We also note that

(4.18)
$$\begin{aligned}
&f[u_0] + D_3 f[u_0]h_1 + D_4 f[u_0]\nabla h_1 + D_5 f[u_0]h_1' \\
&+ \sum_{\substack{2\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\leq|\nu|\leq N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
=&f[u_0] + \sum_{\substack{1\leq|m|\leq N \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{|m|\leq|\nu|\leq N} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
=&f[u_0] + \sum_{1\leq|\nu|\leq N} \sum_{\substack{1\leq|m|\leq|\nu| \\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}} \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N) \\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!} D^m f[u_0] T_\alpha^{(m_1)}[u] T_\beta^{(m_2)}[\nabla u] T_\gamma^{(m_3)}[u'] \overrightarrow{\varepsilon}^\nu \\
=&\sum\nolimits_{|\nu|\leq N} \pi_\nu[f] \overrightarrow{\varepsilon}^\nu.
\end{aligned}$$

Similarly, we have also

(4.19)
$$\sum_{\substack{2\le|m|\le N\\ m=(m_1,m_2,m_3)\in\mathbb{Z}_+^3}}\ \sum_{N+1\le|\nu|\le|m|N}\ \sum_{\substack{(\alpha,\beta,\gamma)\in A(m,N)\\ \alpha+\beta+\gamma=\nu}} \frac{1}{m!}D^m f[u_0]T_\alpha^{(m_1)}[u]T_\beta^{(m_2)}[\nabla u]T_\gamma^{(m_3)}[u']\overrightarrow{\varepsilon}^{\nu}$$
$$+R_N^{(1)}[f,h_1] = \|\overrightarrow{\varepsilon}\|^{N+1}\, R_N^{(1)}[f,\overrightarrow{\varepsilon}],$$

where $\left\|R_N^{(1)}[f,\overrightarrow{\varepsilon}]\right\|_{L^\infty(0,T;L^2)} \le C$, with C a constant depending only on N, T, f, u_γ, $|\gamma|\le N$.

Then, (4.10) holds. Lemma 4.2 is proved. □

Remark 4.1. Lemma 4.2 is a generalization of a formula contained in ([7], p.262, (4.38)), and it is useful for obtaining the following Lemma 4.3. These lemmas are the key to the asymptotic expansion of a weak solution $u = u(\varepsilon_1,...,\varepsilon_p)$ of order $N+1$ in p small parameters $\varepsilon_1,...,\varepsilon_p$ as it will be seen below.

Let $u_{\overrightarrow{\varepsilon}} = u(\varepsilon_1,...,\varepsilon_p) \in W_1(M,T)$ be a unique weak solution of the problem $(P_{\overrightarrow{\varepsilon}})$. Then, $v = u_{\overrightarrow{\varepsilon}} - \sum_{|\gamma|\le N} u_\gamma \overrightarrow{\varepsilon}^{\gamma} \equiv u_{\overrightarrow{\varepsilon}} - h$ satisfies

(4.20)
$$\begin{cases} v'' - \dfrac{\partial}{\partial x}\left(\mu_{\overrightarrow{\varepsilon}}(v+h)v_x\right) = F_{\overrightarrow{\varepsilon}}[v+h] - F_{\overrightarrow{\varepsilon}}[h] + \dfrac{\partial}{\partial x}\left(\left[\mu_{\overrightarrow{\varepsilon}}(v+h) - \mu_{\overrightarrow{\varepsilon}}(h)\right]h_x\right) \\ \qquad\qquad + E_{\overrightarrow{\varepsilon}}(x,t),\ 0<x<1,\ 0<t<T, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \\ \mu_{\overrightarrow{\varepsilon}}(v) = \mu(v) + \sum_{i=1}^p \varepsilon_i\mu_i(v), \\ F_{\overrightarrow{\varepsilon}}[v] = f[v] + \sum_{i=1}^p \varepsilon_i f_i[v] = f(x,t,v,v_x,v_t) + \sum_{i=1}^p \varepsilon_i f_i(x,t,v,v_x,v_t), \end{cases}$$

where

(4.21)
$$E_{\overrightarrow{\varepsilon}}(x,t) = f[h] - f[u_0] + \sum_{i=1}^p \varepsilon_i f_i[h] + \frac{\partial}{\partial x}\left(\left[\mu(h) - \mu(u_0) + \sum_{i=1}^p \varepsilon_i\mu_i(h)\right]h_x\right) - \sum_{1\le|\gamma|\le N} F_\gamma \overrightarrow{\varepsilon}^{\gamma}.$$

Then we have the following lemma.

Lemma 4.3. *Suppose that* (H$_1$), (H$_6$) *and* (H$_7$) *hold. Then*

(4.22)
$$\|E_{\overrightarrow{\varepsilon}}\|_{L^\infty(0,T;L^2)} \le \widehat{K}_*\,\|\overrightarrow{\varepsilon}\|^{N+1},$$

where $\widehat{K}_*$ *is a constant depending only on* N, T, f, f_i, μ, μ_i, u_γ, $|\gamma|\le N$, $i = 1,2,...,p$.

Proof. We consider only the case that $N \geq 2$. By using formulas (4.9), (4.10) for the functions $f_i[h]$ and $\mu_i[h]$, we obtain

$$\begin{cases} \mu_i(h) = \sum_{|\nu| \leq N-1} \rho_\nu[\mu_i] \overrightarrow{\varepsilon}^\nu + \|\overrightarrow{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_i, \overrightarrow{\varepsilon}], \\ f_i[h] = \sum_{|\nu| \leq N-1} \pi_\nu[f_i] \overrightarrow{\varepsilon}^\nu + \|\overrightarrow{\varepsilon}\|^N R_{N-1}^{(1)}[f_i, \overrightarrow{\varepsilon}]. \end{cases} \tag{4.23}$$

By (4.6), (4.8), we write $\varepsilon_i \mu_i(h)$ and $\varepsilon_i f_i[h]$ as follows:

$$\begin{aligned} \varepsilon_i \mu_i(h) &= \sum_{|\nu| \leq N-1} \rho_\nu[\mu_i] \varepsilon_i \overrightarrow{\varepsilon}^\nu + \varepsilon_i \|\overrightarrow{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_i, \overrightarrow{\varepsilon}] \\ &= \sum_{1 \leq |\nu| \leq N,\ \nu_i \geq 1} \rho_{\nu_1, \nu_2, \ldots, \nu_{i-1},\ \nu_i - 1,\ \nu_{i+1}, \ldots, \nu_p}[\mu_i] \overrightarrow{\varepsilon}^\nu + \varepsilon_i \|\overrightarrow{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_i, \overrightarrow{\varepsilon}] \\ &= \sum_{1 \leq |\nu| \leq N} \rho_\nu^{(i)}[\mu_i] \overrightarrow{\varepsilon}^\nu + \varepsilon_i \|\overrightarrow{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[\mu_i, \overrightarrow{\varepsilon}]. \end{aligned} \tag{4.24}$$

Similarly

$$\varepsilon_i f_i[h] = \sum_{1 \leq |\nu| \leq N} \pi_\nu^{(i)}[f_i] \overrightarrow{\varepsilon}^\nu + \varepsilon_i \|\overrightarrow{\varepsilon}\|^N R_{N-1}^{(1)}[f_i, \overrightarrow{\varepsilon}]. \tag{4.25}$$

First, we deduce from $(4.23)_2$ and (4.25) that

$$\begin{aligned} & f[h] - f[u_0] + \sum_{i=1}^p \varepsilon_i f_i[h] \\ &= \sum_{1 \leq |\nu| \leq N} \pi_\nu[f] \overrightarrow{\varepsilon}^\nu + \|\overrightarrow{\varepsilon}\|^{N+1} R_N^{(1)}[f, \overrightarrow{\varepsilon}] \\ &\quad + \sum_{i=1}^p \left[\sum_{1 \leq |\nu| \leq N} \pi_\nu^{(i)}[f_i] \overrightarrow{\varepsilon}^\nu + \varepsilon_i \|\overrightarrow{\varepsilon}\|^N R_{N-1}^{(1)}[f_i, \overrightarrow{\varepsilon}] \right] \\ &= \sum_{1 \leq |\nu| \leq N} \left[\pi_\nu[f] + \sum_{i=1}^p \pi_\nu^{(i)}[f_i] \right] \overrightarrow{\varepsilon}^\nu \\ &\quad + \|\overrightarrow{\varepsilon}\|^{N+1} \left[R_N^{(1)}[f, \overrightarrow{\varepsilon}] + \sum_{i=1}^p \frac{\varepsilon_i}{\|\overrightarrow{\varepsilon}\|} \|\overrightarrow{\varepsilon}\|^N R_{N-1}^{(1)}[f_i, \overrightarrow{\varepsilon}] \right] \\ &= \sum_{1 \leq |\nu| \leq N} \left[\pi_\nu[f] + \sum_{i=1}^p \pi_\nu^{(i)}[f_i] \right] \overrightarrow{\varepsilon}^\nu + \|\overrightarrow{\varepsilon}\|^{N+1} R_N^{(1)}[f, f_1, ..., f_p, \overrightarrow{\varepsilon}], \end{aligned} \tag{4.26}$$

where $R_N^{(1)}[f, f_1, ..., f_p, \overrightarrow{\varepsilon}] = R_N^{(1)}[f, \overrightarrow{\varepsilon}] + \sum_{i=1}^p \frac{\varepsilon_i}{\|\overrightarrow{\varepsilon}\|} \|\overrightarrow{\varepsilon}\|^N R_{N-1}^{(1)}[f_i, \overrightarrow{\varepsilon}]$ is bounded in the function space $L^\infty(0, T; L^2)$ by a constant depending only on N, T, f, f_i, u_γ, $|\gamma| \leq N$, $i = 1, 2, ..., p$.

On the other hand, we deduce from $(4.23)_1$ and (4.24), that

$$\begin{aligned}
&\left[\mu(h)-\mu(u_0)+\sum_{i=1}^{p}\varepsilon_i\mu_i(h)\right]h_x\\
&=\left\{\sum_{1\le|\nu|\le N}\left[\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right]\vec{\varepsilon}^{\,\nu}\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha}\\
&\quad+\left\{\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\vec{\varepsilon}]+\sum_{i=1}^{p}\varepsilon_i\|\vec{\varepsilon}\|^{N}\widetilde{R}_{N-1}^{(1)}[\mu_i,\vec{\varepsilon}]\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha}\\
&=\left\{\sum_{1\le|\nu|\le N}\left[\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right]\vec{\varepsilon}^{\,\nu}\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha}\\
&\quad+\|\vec{\varepsilon}\|^{N+1}\left\{\widetilde{R}_N^{(1)}[\mu,\vec{\varepsilon}]+\sum_{i=1}^{p}\frac{\varepsilon_i}{\|\vec{\varepsilon}\|}\widetilde{R}_{N-1}^{(1)}[\mu_i,\vec{\varepsilon}]\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha}\\
(4.27)\quad&=\left\{\sum_{1\le|\nu|\le N}\left[\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right]\vec{\varepsilon}^{\,\nu}\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha}\\
&\quad+\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}]\\
&=\sum_{1\le|\nu|\le N,\ |\alpha|\le N}\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_\alpha\vec{\varepsilon}^{\,\nu+\alpha}\\
&\quad+\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}]\\
&=\sum_{1\le|\gamma|\le 2N}\ \sum_{1\le|\nu|\le N,\ |\gamma-\nu|\le N}\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_{\gamma-\nu}\vec{\varepsilon}^{\,\gamma}\\
&\quad+\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}]\\
&=\sum_{1\le|\gamma|\le N}\ \sum_{1\le|\nu|\le|\gamma|,\ \nu\le\gamma}\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_{\gamma-\nu}\vec{\varepsilon}^{\,\gamma}\\
&\quad+\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(2)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}],
\end{aligned}$$

where

(4.28)

$$\begin{cases}
\widetilde{R}_N^{(1)}[\mu,\mu_1,\vec{\varepsilon}]=\left\{\widetilde{R}_N^{(1)}[\mu,\vec{\varepsilon}]+\sum_{i=1}^{p}\frac{\varepsilon_i}{\|\vec{\varepsilon}\|}\widetilde{R}_{N-1}^{(1)}[\mu_i,\vec{\varepsilon}]\right\}\sum\nolimits_{|\alpha|\le N}\nabla u_\alpha\vec{\varepsilon}^{\,\alpha},\\
\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(2)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}]=\|\vec{\varepsilon}\|^{N+1}\widetilde{R}_N^{(1)}[\mu,\mu_1,...,\mu_p,\vec{\varepsilon}]\\
\qquad\qquad+\sum_{N+1\le|\gamma|\le 2N}\ \sum_{1\le|\nu|\le N,\ |\gamma-\nu|\le N}\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_{\gamma-\nu}\vec{\varepsilon}^{\,\gamma}.
\end{cases}$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\left[\mu(h)-\mu(u_0)+\sum_{i=1}^{p}\varepsilon_i\mu_i(h)\right]h_x\right)\\
&=\sum_{1\le|\gamma|\le N}\;\sum_{1\le|\nu|\le|\gamma|,\;\nu\le\gamma}\frac{\partial}{\partial x}\left[\left(\rho_\nu[\mu]+\sum_{i=1}^{p}\rho_\nu^{(i)}[\mu_i]\right)\nabla u_{\gamma-\nu}\right]\overrightarrow{\varepsilon}^{\gamma}\\
&\quad+\|\overrightarrow{\varepsilon}\|^{N+1}\frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu,\mu_1,...,\mu_p,\overrightarrow{\varepsilon}].
\end{aligned}\tag{4.29}$$

Combining (4.4) – (4.8), (4.21), (4.26) and (4.29), the result is

$$\begin{aligned}
E_{\overrightarrow{\varepsilon}}(x,t)=&f[h]-f[u_0]+\sum_{i=1}^{p}\varepsilon_i f_i[h]\\
&+\frac{\partial}{\partial x}\left(\left[\mu(h)-\mu(u_0)+\sum_{i=1}^{p}\varepsilon_i\mu_i(h)\right]h_x\right)-\sum\nolimits_{1\le|\gamma|\le N}F_\gamma\overrightarrow{\varepsilon}^{\gamma}\\
=&\|\overrightarrow{\varepsilon}\|^{N+1}\left[R_N^{(1)}[f,f_1,...,f_p,\overrightarrow{\varepsilon}]+\frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu,\mu_1,...,\mu_p,\overrightarrow{\varepsilon}]\right].
\end{aligned}\tag{4.30}$$

By boundedness of the functions u_γ, ∇u_γ, u_γ', $|\gamma|\le N$ in the function space $L^\infty(0,T;H^1)$, we obtain from (4.26) and (4.28), that

$$\|E_{\overrightarrow{\varepsilon}}\|_{L^\infty(0,T;L^2)}\le\widehat{K}_*\|\overrightarrow{\varepsilon}\|^{N+1},\tag{4.31}$$

where $\widehat{K}_*$ is a constant depending only on N, T, f, f_i, μ, μ_i, u_γ, $|\gamma|\le N$, $i=1,2,...,p$.

The proof of Lemma 4.3 is complete. □

Now we consider the sequence $\{v_m\}$ defined by

$$\begin{cases}
v_0\equiv 0,\\
v_m''-\frac{\partial}{\partial x}\left(\mu_{\overrightarrow{\varepsilon}}(v_{m-1}+h)v_{mx}\right)=F_{\overrightarrow{\varepsilon}}[v_{m-1}+h]-F_{\overrightarrow{\varepsilon}}[h]\\
\qquad\qquad+\frac{\partial}{\partial x}\left(\left[\mu_{\overrightarrow{\varepsilon}}(v_{m-1}+h)-\mu_{\overrightarrow{\varepsilon}}(h)\right]h_x\right)\\
\qquad\qquad+E_{\overrightarrow{\varepsilon}}(x,t),\;0<x<1,0<t<T,\\
v_m(0,t)=v_m(1,t)=0,\\
v_m(x,0)=v_m'(x,0)=0,\;m\ge1.
\end{cases}\tag{4.32}$$

For $m=1$ we have the problem

$$\begin{cases}
v_1''-\dfrac{\partial}{\partial x}\left(\mu_{\overrightarrow{\varepsilon}}(h)v_{1x}\right)=E_{\overrightarrow{\varepsilon}}(x,t),\;0<x<1,\;0<t<T,\\
v_1(0,t)=v_1(1,t)=0,\\
v_1(x,0)=v_1'(x,0)=0.
\end{cases}\tag{4.33}$$

By multiplying the two sides of $(4.33)_1$ by v_1', we find without difficulty from (4.22) that

$$\begin{aligned}
&||v_1'(t)||^2 + ||\sqrt{\mu_{1,\vec{\varepsilon}}(t)}v_{1x}(t)||^2 \\
&=2\int_0^t \langle E_{\vec{\varepsilon}}(s), v_1'(s)\rangle ds + \int_0^t ds \int_0^1 \mu_{1,\vec{\varepsilon}}'(x,s)v_{1x}^2(x,s)dx \qquad (4.34)\\
&\leq T\widehat{K}_*^2 \, \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_1'(s)\right\|^2 ds + \int_0^t ds \int_0^1 \left|\mu_{1,\vec{\varepsilon}}'(x,s)\right| v_{1x}^2(x,s)dx,
\end{aligned}$$

where $\mu_{1,\vec{\varepsilon}}(x,t) = \mu_{\vec{\varepsilon}}(h(x,t)) = \mu(h(x,t)) + \sum_{i=1}^p \varepsilon_i \mu_i(h(x,t))$.

As $\mu_{1,\vec{\varepsilon}}'(x,t) = \mu_{\vec{\varepsilon}}'(h(x,t))h'(x,t)$, we have

(4.35)
$$\left|\mu_{1,\varepsilon_1}'(x,t)\right| \leq M_* \left(\widetilde{K}_{M_*}(\mu) + \sum_{i=1}^p \widetilde{K}_{M_*}(\mu_i)\right) \equiv \zeta_0, \text{ with } M_* = (N+1)M.$$

It follows from (4.34), (4.35) that

(4.36)
$$||v_1'(t)||^2 + \mu_0 \left\|v_{1x}(t)\right\|^2 \leq T\widehat{K}_*^2 \, \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_1'(s)\right\|^2 ds + \zeta_0 \int_0^t ||v_{1x}(s)||^2 ds.$$

Using Gronwall's lemma this inequality gives

(4.37)
$$||v_1'||_{L^\infty(0,T;L^2)} + \|v_{1x}\|_{L^\infty(0,T;L^2)} \leq (1+\frac{1}{\sqrt{\mu_0}})\sqrt{T}\widehat{K}_* \, \|\vec{\varepsilon}\|^{N+1} \exp\left[\frac{1}{2}T\left(1+\frac{\zeta_0}{\mu_0}\right)\right].$$

We shall prove that there exists a constant C_T, independent of m and $\vec{\varepsilon}$, such that

(4.38)
$$\left\|v_m'\right\|_{L^\infty(0,T;L^2)} + \|v_{mx}\|_{L^\infty(0,T;L^2)} \leq C_T \, \|\vec{\varepsilon}\|^{N+1} \text{ with } \|\vec{\varepsilon}\| \leq \varepsilon^* < 1 \text{ for all } m.$$

By multiplying the two sides of $(4.32)_1$ with v_m' and after integration in t we obtain from (4.22) that

$$\begin{aligned}
&||v_m'(t)||^2 + \mu_0||v_{mx}(t)||^2 \\
&\leq T\widehat{K}_*^2 \, \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_m'(s)\right\|^2 ds + \int_0^t ds \int_0^1 \left|\mu_{m,\vec{\varepsilon}}'(x,s)\right| v_{mx}^2(x,s)dx \\
&\quad + 2\int_0^t \|F_{\vec{\varepsilon}}[v_{m-1}+h] - F_{\vec{\varepsilon}}[h]\| \left\|v_m'(s)\right\| ds \qquad (4.39)\\
&\quad + 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\left[\mu_{\vec{\varepsilon}}(v_{m-1}+h) - \mu_{\vec{\varepsilon}}(h)\right] h_x\right)\right\| \left\|v_m'(s)\right\| ds \\
&= T\widehat{K}_*^2 \, \|\vec{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_m'(s)\right\|^2 ds + \widehat{I}_1(t) + \widehat{I}_2(t) + \widehat{I}_3(t),
\end{aligned}$$

where $\mu_{m,\vec{\varepsilon}}(t) = \mu_{\vec{\varepsilon}}(v_{m-1}+h)$. We now estimate the integrals on the right - hand side of (4.39) as follows.

Estimating $\widehat{I}_1(t)$. We have $\mu'_{m,\overrightarrow{\varepsilon}}(x,t) = \mu'_{\overrightarrow{\varepsilon}}(v_{m-1}+h)(v'_{m-1}+h')$, hence

(4.40)
$$\left|\mu'_{m,\overrightarrow{\varepsilon}}(x,t)\right| \leq M_{**}\left(\widetilde{K}_{M_{**}}(\mu) + \sum_{i=1}^{p}\widetilde{K}_{M_{**}}(\mu_i)\right) \equiv \zeta_1, \text{ with } M_{**} = (N+2)M.$$

It follows from (4.40) that

$$\widehat{I}_1(t) = \int_0^t ds \int_0^1 \left|\mu'_{m,\overrightarrow{\varepsilon}}(x,s)\right| v_{mx}^2(x,s)dx \leq \zeta_1 \int_0^t ||v_{mx}(s)||^2 ds. \tag{4.41}$$

Estimating $\widehat{I}_2(t)$. We also note that $\|f[v_{m-1}+h]-f[h]\| \leq 2K_{M_{**}}(f)\,\|v_{m-1}\|_{W_1(T)}$ and $||\ f_i[v_{m-1}\ +h]-f_i[h]\ || \leq 2K_{M_{**}}(f_i)\,\|v_{m-1}\|_{W_1(T)}$, so

$$\|F_{\overrightarrow{\varepsilon}}[v_{m-1}+h]-F_{\overrightarrow{\varepsilon}}[h]\| \leq \zeta_2\,\|v_{m-1}\|_{W_1(T)}, \tag{4.42}$$

where $\zeta_2 = \zeta_2(M_{**},\ f,\ f_1,...,f_p) = 2K_{M_{**}}(f) + 2\sum_{i=1}^{p} K_{M_{**}}(f_i)$. Therefore, we deduce from (4.42) that

$$\begin{aligned}\widehat{I}_2(t) =& 2\int_0^t \|F_{\overrightarrow{\varepsilon}}[v_{m-1}+h]-F_{\overrightarrow{\varepsilon}}[h]\|\,\left\|v'_m(s)\right\| ds\\ \leq& T\zeta_2^2\,\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\|v'_m(s)\right\|^2 ds.\end{aligned} \tag{4.43}$$

Estimating $\widehat{I}_3(t)$. First, we need an estimation for $\left\|\frac{\partial}{\partial x}\left([\mu(v_{m-1}+h)-\mu(h)]\,h_x\right)\right\|$. From the equation

$$\begin{aligned}\frac{\partial}{\partial x}\left([\mu(v_{m-1}+h)-\mu(h)]\,h_x\right) =& [\mu(v_{m-1}+h)-\mu(h)]\,h_{xx}\\ &+ \frac{\partial}{\partial x}[\mu(v_{m-1}+h)-\mu(h)]\,h_x\end{aligned}$$

it follows that

(4.44)
$$\begin{aligned}&\left\|\frac{\partial}{\partial x}\left([\mu(v_{m-1}+h)-\mu(h)]\,h_x\right)\right\|\\ &\leq \|\mu(v_{m-1}+h)-\mu(h)\|_{C^0(\overline{\Omega})}\,\|h_{xx}(s)\|\\ &\quad+ \left\|\frac{\partial}{\partial x}[\mu(v_{m-1}+h)-\mu(h)]\right\|\,\|h_x(s)\|_{C^0(\overline{\Omega})}\\ &\leq \sqrt{2}\,\|h(s)\|_{H^2}\left[\|\mu(v_{m-1}+h)-\mu(h)\|_{C^0(\overline{\Omega})} + \left\|\frac{\partial}{\partial x}[\mu(v_{m-1}+h)-\mu(h)]\right\|\right]\\ &\equiv \sqrt{2}\,\|h(s)\|_{H^2}\left[\widehat{I}_3^{(1)}(s)+\widehat{I}_3^{(2)}(s)\right].\end{aligned}$$

Concerning $\widehat{I}_3^{(1)}(s)$ we have

$$\widehat{I}_3^{(1)}(s) = \|\mu(v_{m-1}+h)-\mu(h)\|_{C^0(\overline{\Omega})} \leq \widetilde{K}_{M_{**}}(\mu)\,\|v_{m-1}\|_{W_1(T)}. \tag{4.45}$$

Concerning $\widehat{I}_3^{(2)}(s)$ we also obtain

(4.46)

$$\begin{aligned}\widehat{I}_3^{(2)}(s) &= \left\| \frac{\partial}{\partial x} \left[\mu(v_{m-1}+h) - \mu(h) \right] \right\| \\ &\leq \left\| \mu'(v_{m-1}+h) \nabla v_{m-1} \right\| + \left\| \left[\mu'(v_{m-1}+h) - \mu'(h) \right] \nabla h \right\| \\ &\leq (1+\|\nabla h(s)\|) \widetilde{K}_{M_{**}}(\mu) \left\| \nabla v_{m-1}(s) \right\| \leq (1+M_*) \widetilde{K}_{M_{**}}(\mu) \left\| v_{m-1} \right\|_{W_1(T)}.\end{aligned}$$

We deduce from (4.44), (4.45) and (4.46) that

$$\left\| \frac{\partial}{\partial x} \left(\left[\mu(v_{m-1}+h) - \mu(h) \right] h_x \right) \right\| \leq \sqrt{2} M_* (2+M_*) \widetilde{K}_{M_{**}}(\mu) \left\| v_{m-1} \right\|_{W_1(T)}. \tag{4.47}$$

Next, by $\mu_{\vec{\varepsilon}}(v) = \mu(v) + \sum_{i=1}^{p} \varepsilon_i \mu_i(v)$, it follows that

$$\left\| \frac{\partial}{\partial x} \left(\left[\mu_{\vec{\varepsilon}}(v_{m-1}+h) - \mu_{\vec{\varepsilon}}(h) \right] h_x \right) \right\| \leq \zeta_3 \left\| v_{m-1} \right\|_{W_1(T)}, \tag{4.48}$$

where

(4.49)

$$\zeta_3 = \zeta_3(M, N, T, \mu, \mu_1, ..., \mu_p) = \sqrt{2} M_* (2+M_*) \left(\widetilde{K}_{M_{**}}(\mu) + \sum_{i=1}^{p} \widetilde{K}_{M_{**}}(\mu_i) \right).$$

By (4.49)

$$\begin{aligned}\widehat{I}_3(t) =& 2 \int_0^t \left\| \frac{\partial}{\partial x} \left(\left[\mu_{\vec{\varepsilon}}(v_{m-1}+h) - \mu_{\vec{\varepsilon}}(h) \right] h_x \right) \right\| \left\| v_m'(s) \right\| ds \\ \leq& T \zeta_3^2 \left\| v_{m-1} \right\|_{W_1(T)}^2 + \int_0^t \left\| v_m'(s) \right\|^2 ds.\end{aligned} \tag{4.50}$$

Combining (4.39), (4.41), (4.43), (4.50) yields

$$\begin{aligned}||v_m'(t)||^2 + \mu_0 ||v_{mx}(t)||^2 \leq& T \widehat{K}_*^2 \left\| \vec{\varepsilon} \right\|^{2N+2} + T(\zeta_2^2 + \zeta_3^2) \left\| v_{m-1} \right\|_{W_1(T)}^2 \\ &+ 3 \int_0^t \left\| v_m'(s) \right\|^2 ds + \zeta_1 \int_0^t ||v_{mx}(s)||^2 ds.\end{aligned} \tag{4.51}$$

By using Gronwall's lemma, we get

$$\left\| v_m \right\|_{W_1(T)} \leq \sigma_T \left\| v_{m-1} \right\|_{W_1(T)} + \delta \text{ for all } m \geq 1, \tag{4.52}$$

where

$$\begin{aligned}\sigma_T =& \sqrt{\zeta_2^2 + \zeta_3^2} \eta_T, \\ \delta =& \eta_T \widehat{K}_* \left\| \vec{\varepsilon} \right\|^{N+1}, \\ \eta_T =& \left(1 + \frac{1}{\sqrt{\mu_0}} \right) \exp \left[\frac{1}{2} T (1 + \frac{\zeta_1}{\mu_0}) \right] \sqrt{T}.\end{aligned} \tag{4.53}$$

Assume that

$$\sigma_T < 1, \text{ with a suitable constant } T > 0. \tag{4.54}$$

We shall now require the following lemma whose proof is immediate.

Lemma 4.4. *Suppose the sequence* $\{\Psi_m\}$ *satisfies*

$$\Psi_m \leq \sigma\Psi_{m-1} + \delta \text{ for all } m \geq 1, \Psi_0 = 0, \tag{4.55}$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ *are given constants. Then,*

$$\Psi_m \leq \delta/(1-\sigma) \text{ for all } m \geq 1. \tag{4.56}$$

Applying Lemma 4.4 with $\Psi_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T = \sqrt{\zeta_2^2 + \zeta_3^2}\eta_T < 1$, $\delta = \eta_T \widehat{K}_* \|\overrightarrow{\varepsilon}\|^{N+1}$, it follows from (4.56) that

$$\|v_m'\|_{L^\infty(0,T;L^2)} + \|v_{mx}\|_{L^\infty(0,T;L^2)} = \|v_m\|_{W_1(T)} \leq \delta/(1-\sigma_T) = C_T \|\overrightarrow{\varepsilon}\|^{N+1}, \tag{4.57}$$

where $C_T = \frac{\eta_T \widehat{K}_*}{1-\sqrt{\zeta_2^2+\zeta_3^2}\eta_T}$.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.32) converges strongly in the space $W_1(T)$ to the weak solution v of problem (4.20). Hence, letting $m \to +\infty$ in (4.57) gives

$$\|v'\|_{L^\infty(0,T;L^2)} + \|v_x\|_{L^\infty(0,T;L^2)} \leq C_T \|\overrightarrow{\varepsilon}\|^{N+1},$$

or

$$\left\| u' - \sum\nolimits_{|\gamma|\leq N} u_\gamma' \overrightarrow{\varepsilon}^\gamma \right\|_{L^\infty(0,T;L^2)} + \left\| u_x - \sum\nolimits_{|\gamma|\leq N} u_{\gamma x} \overrightarrow{\varepsilon}^\gamma \right\|_{L^\infty(0,T;L^2)} \leq C_T \|\overrightarrow{\varepsilon}\|^{N+1}. \tag{4.58}$$

Thus, we have the following theorem.

Theorem 4.5. *Suppose that* (H_1), (H_2), (H_6) *and* (H_7) *hold. Then, there exist constants* $M > 0$ *and* $T > 0$ *such that for every* $\overrightarrow{\varepsilon}$ *with* $\|\overrightarrow{\varepsilon}\| \leq \varepsilon_* < 1$, *the problem* $(P_{\overrightarrow{\varepsilon}})$ *has a unique weak solution* $u = u_{\overrightarrow{\varepsilon}} \in W_1(M,T)$ *satisfying an asymptotic estimation up to order* $N+1$ *as in (4.58), where the functions* u_γ, $|\gamma| \leq N$ *are the weak solutions of the problems* $(\widetilde{P}_\gamma)$, $|\gamma| \leq N$, *respectively.*

Remark 4.2. Typical examples for asymptotic expansions of a weak solution in a small parameter can be found in the works of many authors, such as [3], [7], [9], [10], [11], [18]. However, to our knowledge, in the case of asymptotic expansion in many small parameters, there are only partial results, for example, [12] – [14], [17], concerning asymptotic expansions of a solution in two or three small parameters.

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