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STRONG REGULARITY OF $SF-$RINGS

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Abstract. In this paper, we study the strong regularity of left $SF-$rings and obtain the following results: Let R be a left $SF-$ring. If R satisfies one of the following conditions, then R is a strongly regular ring: 1) R is a left WZI ring; 2) R is a right WZI ring; 3) R is a left PZI ring; 4) R is a right PZI ring; 5) R is a semicommutative ring; 6) R is a reversible ring.

1. Introduction

All rings considered in this paper are associative rings with identity, and all modules are unitary. The symbols $Soc(_RR)$ ($Soc(R_R)$, resp.), $Z_l(R)$ ($Z_r(R)$, resp.), ($J(R)$, $P(R)$ and $N(R)$, resp.) will stand respectively for the left (right) socle, the left (right) singular ideal, the Jacobson radical, the prime radical and the set of all nilpotent elements of R. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X, respectively. Especially, if $X = a$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

A ring R is called a (von Neumann) regular ring (cf. Goodearl [2]) if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring R is strongly regular (cf. Rege [8]) if for every $a \in R$ there exists $b \in R$ such that $a = a^2b$. A ring R is called reduced (cf. Ramamurthi [7]) if $N(R) = 0$. It is well known that R is a strongly regular ring if and only if R is a reduced regular ring. A ring R is called left (resp., right) quasi-duo ring if every maximal left (resp., right) ideal of R is an ideal. A ring R is called MELT (MERT, resp.) ring if every maximal essential left (resp., right) ideal of R is an ideal. According to Ramamurthi [7], a ring R is called a left (right, resp.) $SF-$ ring if each simple left (resp., right) $R-$module is flat. It is known that regular rings are left and right $SF-$rings. Ramamurthi [7] initiated the study of $SF-$rings and the question whether an $SF-$ring is necessarily regular. In recent years, $SF-$rings have been studied by many authors and the regularity of $SF-$rings which satisfy certain additional conditions have been shown (cf. Ramamurthi [7]; Rege [8]; Yue Chi

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Ming [11–13];Zhang [14, 15]; Zhang and Du [16, 17]; Zhou and Wang [19, 20]; Zhou [21]. But the question remains open. Yue Chi Ming [13] proved the strong regularity of right $SF-$rings whose complement left ideals are ideals. He also proposed the following question: Is R strongly regular if R is a left $SF-$ring whose complement left ideals are ideals? Zhang and Du [16] affirmatively answered the question. Zhou and Wang [19] proved that if R is a right $SF-$ring whose maximal essential right ideals are all an $GW-$ideals then R is a regular ring. Zhang [15] proved that if R is both MELT and a right $SF-$ring, then R is a regular ring. Zhou [21] proved that if R is a left $SF-$ring whose complement left (right) ideals are $W-$ideals, then R is a strongly regular ring. The main purpose of this paper is to study the (strong) regularity of left $SF-$rings in terms of WZI rings and PZI rings.

Following Zhou and Wang [19], a left ideal L of a ring R is called a $GW-$ideal, if for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. Clearly, ideal is $GW-$ideal, but the converse is not true, in general, Zhou and Wang ([19, Example 1.2]).

Following Lednid and Vaserst [5], an additive subgroup L of a ring R is said to be a quasi-ideal if $xrx \in L$ and $rxr \in L$ for $x \in L$ and $r \in R$. Obviously, every ideal of R is a quasi-ideal. But there exists an example of a four-dimensional Banach algebra A whose quasi-ideal Y is not an ideal. Note that $A = A * Y$ is the exterior (Drassmann) algebra on a two dimensional real vector space Y.

According to Zhou [21], a left ideal L of a ring R is called a weak ideal ($W-$ideal), if for any $0 \neq a \in L$, there exists $n \geq 1$ such that $a^n \neq 0$ and $a^nR \subseteq L$. A right ideal K of a ring R is defined similarly to be a weak ideal. Clearly, ideals are $W-$ideals and $W-$ideals are $GW-$ideals, but the converses are not true in general, Zhou [21].

According to Cohn [1], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be semicommutative if $ab = 0$ implies $aRb = 0.1$

A ring R is called a left (right, resp.) WZI ring if for any $a \in R$, $l(a)$ (resp., $r(a)$) is a $W-$ideal of R.

A ring R is called a left (right, resp.) PZI ring if for any $0 \neq a \in R$, there exists $n \geq 1$ such that $l(a^n)$ ($r(a^n)$, resp.) is an ideal of R.

Clearly, semicommutative rings are left and right WZI rings and left and right PZI rings.

2. Some properties of WZI rings and PZI rings

According to Hwang [3], a ring R is called NCI if $N(R) = 0$ or if there exists a nonzero ideal of R contained in $N(R)$. Clearly, NI rings are NCI, but the converse is not true, in general, by Hwang [3].

Following Wei and Chen [10], left $R-$module M is called $nil-$injective if for any $a \in N(R)$, every left $R-$homomorphism Ra to R extends to R. Evidently, $YJ-$injective modules are $nil-$injective, but the converse is not true, in general, by Wei and Chen [10].

Proposition 2.1. (1) *Left or right WZI rings are Abelian.*

(2) *Left or right PZI rings are Abelian.*

(3) *Left or right WZI rings are NCI.*

(4) *Let R be a left WZI ring. If every simple singular left $R-$module is $nil-$injective, then R is a reduced ring.*

(5) *If R is a left WZI ring, then $N_2(R) = \{a \in R | a^2 = 0\} \subseteq P(R)$.*

Proof. (1) Let R be a left WZI ring and $e \in E(R)$. Since $1 - e \in l(e)$, there exists $n \geq 1$ such that $(1-e)^n \neq 0$ and $(1-e)^n R \subseteq l(e)$. Therefore we obtain $(1-e)Re = 0$ for each $e \in E(R)$, so R is an Abelian ring.

Similarly, we can show that right WZI rings are Abelian.

(2) It is trivial.

(3) If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$. Let $n \geq 1$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Since R is a left WZI ring, $l(a)$ is a $W-$ideal. Since $(a^{n-1})^2 = 0$ and $0 \neq a^{n-1} \in l(a)$, $a^{n-1}R \subseteq l(a)$. Hence $Ra^{n-1}R \subseteq N(R)$ is a nonzero ideal of R. This shows that R is a NCI ring.

Similarly, we can show that right WZI rings are NCI.

(4) Let $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $l(a) \subseteq M$. If M is not essential in ${}_RR$, then $M = l(e)$ for some $e \in E(R)$. Thus $ae = 0$ because $a \in l(a) \subseteq M$. By (1), R is an Abelian ring, so $ea = 0$. This gives $e \in l(a) \subseteq l(e)$, a contradiction. Hence M is an essential left ideal of R, so R/M is a simple singular left $R-$module. By hypothesis, R/M is a $nil-$injective left $R-$module. Let $f : Ra \longrightarrow R/M$ defined by $f(ra) = r + M$. Then f is a well defined left $R-$homomorphism, so there exists a left $R-$homomorphism $g : R \longrightarrow R/M$ such that $g(a) = f(a)$. Hence there exists $c \in R$ such that $1 + M = f(a) = g(a) = ag(1) = ac + M$. Since R is a left WZI ring, $aR \subseteq l(a)$, so $ac \in l(a) \subseteq M$. This leads to $1 \in M$, which is a contradiction. Hence $a = 0$.

(5) It follows from the proof of (4). □

A ring R is called left NV if every simple singular left $R-$module is $nil-$injective. Clearly, left $V-$rings and reduced rings are left NV. By Proposition 2.1, we have the following corollary.

Corollary 2.2. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*
(2) *R is a reversible ring and left NV ring;*
(3) *R is a semicommutative ring and left NV ring;*
(4) *R is a left WZI ring and left NV ring.*

Kim, Nam and Kim [4, Theorem 4] proved that if R is a left WZI ring whose every simple singular left module is YJ-injective, then R is a reduced weakly regular ring. Hence, by Corollary 2.2, we have the following corollary.

Corollary 2.3. *Let R be a left WZI ring. If every simple singular left $R-$module is $YJ-$injective, then R is a reduced weakly regular ring.*

Wei [9, Theorem 16] proved that a ring R is a strongly regular ring if and only if R is a semicommutative MELT ring whose simple singular left modules are $YJ-$injective. Hence, by Corollary 2.2, we have the following corollary.

Corollary 2.4. *A ring R is a strongly regular ring if and only if R is a MELT left WZI ring whose every simple singular left module is $YJ-$injective.*

It is well known that a ring R is a reduced ring if and only if R is a semiprime ring and semicommutative ring. On the other hand, semiprime left (right) WZI rings or semiprime left (right) PZI rings are semicommutative, so we have the following proposition.

Proposition 2.5. *The following conditions are equivalent for a ring R:*
(1) *R is a reduced ring;*
(2) *R is a semiprime left WZI ring;*
(3) *R is a semiprime right WZI ring;*
(4) *R is a semiprime left PZI ring;*
(5) *R is a semiprime right PZI ring.*

Proposition 2.6. *If R is a left WZI ring, then $R/Z_l(R)$ is a semicommutative ring.*

Proof. Assume that $a, b \in R$ such that $ab \in Z_l(R)$. If $aRb \nsubseteq Z_l(R)$, then there exists $c \in R$ such that $acb \notin Z_l(R)$. Let I be a complement left ideal of $l(acb)$ in R. Then $I \neq 0$. Since $ab \in Z_l(R)$, $I \cap l(ab) \neq 0$. Let $0 \neq x \in I \cap l(ab)$. Then $xab = 0$, so $xa \in l(b)$. If $xa = 0$, then $xacb = 0$, so $x \in I \cap l(acb)$, which is a contradiction. Thus $xa \neq 0$. Since R is a left WZI ring, there exists $n \geq 1$ such that $(xa)^n \neq 0$ and $(xa)^n Rb = 0$, especially, $(xa)^n cb = 0$, so $(xa)^{n-1} xacb = 0$, which implies $(xa)^{n-1} x \in I \cap l(acb)$. Hence $(xa)^{n-1} x = 0$, so we have $(xa)^n = 0$, which is a contradiction. Therefore $aRb \subseteq Z_l(R)$, so $R/Z_l(R)$ is a semicommutative ring. □

Similarly, we can prove the following proposition.

Proposition 2.7. *If R is a right WZI ring, then $R/Z_r(R)$ is a semicommutative ring.*

3. Strong regularity of $SF-$rings

Rege [8, Remark 3.13] pointed out that if R is a reduced left (right) $SF-$ring, then R is a strongly regular ring. We can extend this result to right PZI rings.

Proposition 3.1. *If R is a left $SF-$ring and right PZI ring, then R is a strongly regular ring.*

Proof. Assume that $a \in R$. If $a = 0$, we are done. If $a \neq 0$, then there exists $n \geq 1$ such that $a^n \neq 0$ and $r(a^n)$ is an ideal of R because R is a right PZI ring. If $Ra + r(a^n) \neq R$, then there exists a maximal left ideal M of R containing $Ra + r(a^n)$. Since R is a left $SF-$ring, R/M is a flat left $R-$module, so there exists $b \in M$ such that $a = ab$ because $a \in M$. Hence $1 - b \in r(a^n) \subseteq M$, which implies $1 \in M$, a contradiction. Therefore $Ra + r(a^n) = R$. Let $1 = ca + x$, where $c \in R$ and $x \in r(a^n)$. So $a^n = a^n ca$. Write $b = a^{n-1} - a^{n-1}ca$. Then $b^2 = 0$. If $b \neq 0$, then similar to the proof mentioned above, we have $Rb + r(b) = R$, so there exists $d \in R$ such that $b = bdb$. Hence there exists $x \in R$ such that $a^{n-1} = a^{n-1}xa$. If $b = 0$, then $a^{n-1} = a^{n-1}ca$. Repeating the process above, we can obtain that $a = awa$ for some $w \in R$. So R is a regular ring. By Proposition 2.1(2), R is an Abelian ring, so R is a strongly regular ring. □

Because semicommutative rings are right PZI, we have the following corollary.

Corollary 3.2. *If R is a left $SF-$ring and semicommutative ring, then R is a strongly regular ring.*

Since reversible rings are semicommutative, by Corollary 2.2, we obtain the following corollary:

Corollary 3.3. *If R is a left $SF-$ring and reversible ring, then R is a strongly regular ring.*

Lemma 3.4. *If R is a left $SF-$ring and a left WZI ring, then $Z_l(R) = 0$.*

Proof. If $Z_l(R) \neq 0$, then there exists $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. If $Z_l(R) + r(a) \neq R$, then there exists a maximal right ideal L of R containing $Z_l(R) + r(a)$. Since R is a left WZI ring, $R/Z_l(R)$ is semicommutative by Proposition 2.6. Since R is a left $SF-$ ring, $R/Z_l(R)$ is also a left $SF-$ ring (Rege, 1986, Proposition 3.2). By Corollary 2.2, $R/Z_l(R)$ is a strongly regular ring. Since $L/Z_l(R)$ is a maximal right ideal of $R/Z_l(R)$, $L/Z_l(R)$ is an ideal of $R/Z_l(R)$. Hence L is an ideal of R, so there exists a maximal left ideal M containing L. Since R is a left $SF-$ring and $a \in r(a) \subseteq L \subseteq M$, $a = ab$ for some $b \in M$. Thus $1 - b \in r(a) \subseteq M$. This gives $1 \in M$, which is impossible. Hence $Z_l(R) + r(a) = R$. Write $1 = x + y$ for some $x \in Z_l(R)$ and some $y \in r(a)$. Then $a = ax$, so $a \in l(1 - x)$. Since $x \in Z_l(R)$, $l(1 - x) = 0$. Thus $a = 0$, which is a contradiction. Therefore $Z_l(R) = 0$. □

Proposition 3.5. *Let R be a left $SF-$ring. If R is a left WZI ring, then R is a strongly regular ring.*

Proof. By Lemma 3.4, $Z_l(R) = 0$. By Proposition 2.6, R is a semicommutative ring. By Corollary 3.2, R is a strongly regular ring. □

Lemma 3.6. *If R is a left $SF-$ring and a right WZI ring, then $Z_r(R) = 0$.*

Proof. If $Z_r(R) \neq 0$, then there exists $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. Using Proposition 2.7, similar to the proof of Lemma 3.4, we have $Z_r(R) + r(a) = R$ and $R/Z_r(R)$ is a strongly regular ring. Thus $J(R/Z_r(R)) = 0$, which implies $J(R) \subseteq Z_r(R)$. Since R is a left $SF-$ring, $Z_r(R) \subseteq J(R)$ (Zhou, 2007, Lemma 2.9). Therefore $J(R) = Z_r(R)$, so $J(R) + r(a) = R$, this leads to $r(a) = R$. Hence $a = 0$, which is a contradiction. Thus $Z_r(R) = 0$. □

Hence we also get the following proposition by Lemma 3.6, Proposition 2.7 and Corollary 3.2.

Proposition 3.7. *Let R be a left $SF-$ring. If R is a right WZI ring, then R is a strongly regular ring.*

Lemma 3.8. *Let R be a left PZI ring, then $R/Z_r(R)$ is a semicommutative ring.*

Proof. Assume that $a, b \in R$ such that $ab \in Z_r(R)$. If $aRb \nsubseteq Z_r(R)$, then there exists $c \in R$ such that $acb \notin Z_r(R)$. Let I be a complement right ideal of $r(acb)$ in R. Then $I \neq 0$. Since $ab \in Z_r(R)$, $I \cap r(ab) \neq 0$. Let $0 \neq x \in I \cap r(ab)$. Then $abx = 0$. If $bx = 0$, then $acbx = 0$, so $x \in I \cap r(acb)$, which is a contradiction. Thus $bx \neq 0$. Since R is a left PZI ring, there exists $n \geq 1$ such that $(bx)^n \neq 0$ and $l((bx)^n)$ is an ideal of R. Since $a \in l((bx)^n)$, $aR \subseteq l((bx)^n)$, especially, $ac \in l((bx)^n)$, so $acbx(bx)^{n-1} = 0$, which implies $x(bx)^{n-1} \in I \cap r(acb)$. Hence $x(bx)^{n-1} = 0$, so we have $(bx)^n = 0$, which is a contradiction. Therefore $aRb \subseteq Z_l(R)$, so $R/Z_r(R)$ is a semicommutative ring. □

Proposition 3.9. *Let R be a left $SF-$ring. If R is a left PZI ring, then $R/Z_r(R)$ is a strongly regular ring.*

Proof. By Lemma 3.8, $R/Z_r(R)$ is a semicommutative ring, Rege [8, Proposition 3.2], $R/Z_r(R)$ is a strongly regular ring. If $Z_r(R) \neq 0$, then there exists $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. Then, similar to the proof of Lemma 3.6, we have that $Z_r(R) + r(a) = R$ and $J(R) = Z_r(R)$. Thus $r(a) = R$ and so $a = 0$, which is a contradiction. Hence $Z_r(R) = 0$, therefore R is a strongly regular ring. □

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