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CHERN NUMBERS OF A SINGULAR FIBER, MODULAR INVARIANTS AND ISOTRIVIAL FAMILIES OF CURVES

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Abstract. We use the modular invariants of a family of curves and the Chern numbers of a singular fiber to compute the Chern numbers of the total space. We also explain some inequalities between the local Chern numbers. As an application, one can classify singular fibers according to their Chern numbers.

Introduction

A family of curves of genus g over C is a fibration $f : X \to C$ whose general fibers F are smooth curves of genus g, where X is a complex smooth projective surface. The family is called *semistable* if all of the singular fibers are reduced nodal curves. If $X = F \times C$ and f is just the second projection to C, then we call f a *trivial* family. If all of the smooth fibers of f are isomorphic to each other, equivalently, f becomes trivial under a finite base change $\widetilde{C} \to C$, then f is called *isotrivial*. We always assume that f is relatively minimal, i.e., there is no (-1)-curve in any singular fiber.

A fundamental problem is to find the relationship between the local properties of the (singular) fibers and the global invariants of the surface X.

If $g = 0$, then X is a geometrically ruled surface, and the global invariants of X are the same as the trivial fibration $X = \mathbb{P}^1 \times C$.

If $g = 1$, then X is called an elliptic surface. Kodaira [5] found the global invariants from the singular fibers. The first Chern number $c_1^2(X)$ is always zero, the second Chern number $c_2(X)$ is equal to $12\chi(\mathcal{O}_X)$ by Noether's formula, and

$$c_2(X) = j+6\nu(\mathrm{I}^*)+2\nu(\mathrm{II})+10\nu(\mathrm{II}^*)+3\nu(\mathrm{III})+9\nu(\mathrm{III}^*)+4\nu(\mathrm{IV})+8\nu(\mathrm{IV}^*), \tag{0.1}$$

where $\nu(\mathrm{T})$ denotes the number of singular fibers of type T, and j is the number of poles of the J-function of the family. Note that the J-function over C induces a holomorphic map of degree j from C to the moduli space $\overline{\mathcal{M}}_1$ of elliptic curves.

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If $C = \mathbb{P}^1$, then J is a rational function on C and $j = \deg J$. So j depends only on the generic fibers.

Kodaira proved that one can also compute j from the singular fibers F. Let $\widetilde{F}$ be the semistable model of F under an n-cyclic local base change totally ramified over $f(F)$. We define

$$\delta(F) = \frac{1}{n}\#\{\text{ double points of } \widetilde{F}\,\}.$$

$\delta(F)$ is independent of the choice of base changes. Then we have

$$j = \sum_F \delta(F). \tag{0.2}$$

When $g = 2$, Ogg [7] and Iitaka [3] gave a numerical classification of the singular fibers. Namikawa and Ueno [6] classified their numerical types completely and calculated their local monodromies. There are 126 numerical classes of singular fibers of genus 2. Uematsu [13] classified numerically singular fibers of genus 3.

On the other hand, if $g = 2$, then the surface X is generically a double cover over a ruled surface. Using the canonical resolution of the singularities of a double covering surface, Horikawa [4] gave a coarse classification of the singular fibers and got the local-global formula for $c_1^2(X)-2\chi(\mathcal{O}_X)$. Based on this classification, Xiao obtained finally the local-global formulas for $c_1^2(X)$ and $c_2(X)$. See §4 for the details.

In the general case $g \geq 2$, f induces also a holomorphic map from C to the moduli space of semistable curves of genus g:

$$J : C \longrightarrow \overline{\mathcal{M}}_g\,.$$

For each $\mathbb{Q}$-divisor class η in the moduli space (or stack) $\overline{\mathcal{M}}_g$, we can define an invariant $\eta(f) = \deg J^*\eta$ which satisfies the *base change property*, i.e., if $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ is the pullback fibration of f under a base change $\pi : \widetilde{C} \to C$ of degree d, then $\eta(\widetilde{f}) = d\eta(f)$.

There are three important $\mathbb{Q}$-divisor classes λ, δ and κ on the moduli space, where λ is the Hodge divisor class, $\delta = \delta_0 + \cdots + \delta_{[g/2]}$ is the boundary divisor class and $\kappa = 12\lambda - \delta$. The *modular invariants* of f are just $\kappa(f)$, $\lambda(f)$ and $\delta(f)$. Obviously, the modular invariants depend only on the generic fibers of f. In the case of elliptic fibrations, $\kappa = 0$ and $\delta = \infty$ consists of one point as a divisor, so $\kappa(f) = 0$ and $\delta(f) = j$.

In §1 of this paper, we will introduce the natural generalization of Kodaira's formula (0.1) given in [10] by using the stable reduction technique. Historically, Viehweg [14] and Xiao [15] are the first who tried to use the stable reduction technique to compute the contributions of a singular fiber to the invariants of the surface.

For a generalization of Kodaira's formula (0.1) to the higher genus case, one needs to understand the local contributions of a special fiber F. For a singular fiber F of genus g, we defined in [10] the Chern numbers $c_1^2(F)$, $c_2(F)$ and χ_F.

They are non-negative rational numbers. If $g = 1$, then $c_1^2(F) = 0$ and $c_2(F)$ is exactly the coefficient in (0.1) according to the type of the fiber F. If $g \geq 2$, then one of the three numbers vanishes if and only if F is semistable. Noether's formula holds true,

$$c_1^2(F) + c_2(F) = 12\chi_F.$$

The generalization of (0.1) is the following formulas.

$$\begin{cases} c_1^2(X) = \kappa(f) + 8(g-1)(g(C)-1) + \sum_{i=1}^s c_1^2(F_i), \\ c_2(X) = \delta(f) + 4(g-1)(g(C)-1) + \sum_{i=1}^s c_2(F_i), \\ \chi(\mathcal{O}_X) = \lambda(f) + (g-1)(g(C)-1) + \sum_{i=1}^s \chi_{F_i}, \end{cases} \tag{0.3}$$

where $F_1, \cdots, F_s$ are all singular fibers of f.

Therefore, the next main problem is to generalize the formula (0.2), namely, one needs to understand the local contributions to the modular invariants. In §4, we will explain the generalizations in the cases of $g = 2$ and 3.

In particular, if f is an isotrivial fibration, then the modular invariants $\kappa(f)$, $\delta(f)$ and $\lambda(f)$ are all zero. So the numerical invariants of the surface are determined completely by the Chern numbers of the singular fibers. Recently, some authors are trying to find similar formulas for isotrivial fibration by viewing the surface X as the relative minimal model of the quotient surface $(F \times \widetilde{C})/G$ (see for example, [8]).

The Chern numbers of a singular fiber F inherit the properties of surfaces and curves. For example, we have the "local Miyaoka-Yau inequality" $c_1^2(F) \leq 2c_2(F)$ and the "local canonical class inequality" $c_1^2(F) \leq 4g - 4$ (see [10]). In §2, we will explain some improvements of these kinds of inequalities as well as their applications.

1. Chern numbers of a singular fiber

Denote by F a singular fiber of the fibration $f : X \to C$ over $p \in C$. We consider a semistable reduction $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ of F.

Let $\pi : \widetilde{C} \to C$ be a semistable reduction of F, i.e., π is ramified over $p = f(F)$ and some non-critical points of f, and the fibers of the pullback fibration $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ over $\pi^{-1}(p)$ are semistable. Here $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ is the unique relatively minimal birational model of $X \times_C \widetilde{C} \to \widetilde{C}$.

Denote by K_f the relative canonical divisor $K_{X/C} = K_X - f^*K_C$, it is well-known that K_f is compatible with the pullback over $C - p$ under the semistable reduction above.

Definition 1.1. The Chern numbers of F are defined as follows.

$$c_1^2(F) = K_f^2 - \frac{1}{d}K_{\widetilde{f}}^2, \quad c_2(F) = e_f - \frac{1}{d}e_{\widetilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d}\chi_{\widetilde{f}}, \tag{1.1}$$

where d is the degree of π and

$$\begin{cases} K_f^2 = c_1^2(X) - 8(g-1)(g(C)-1), \\ e_f = c_2(X) - 4(g-1)(g(C)-1), \\ \chi_f = \chi(\mathcal{O}_X) - (g-1)(g(C)-1). \end{cases}$$

From the definition, we have Noether's equality

$$\chi_F = \frac{1}{12}\left(c_1^2(F) + c_2(F)\right).$$

These invariants are independent of the choice of the semistable reductions π. Note that F is not necessarily minimal, i.e., F may contain (-1)-curves. Let $\sigma : X' \to X$ be the blowing up of X at a point p on F, $f' : X' \to C$ be the induced fibration, and $F' = \sigma^* F$ be the pullback of F. Then we know that $\widetilde{f}$ is also the semistable reduction of F'. By definition, we have the following "blow-up formulas"

$$c_1^2(F') = c_1^2(F) - 1, \quad c_2(F') = c_2(F) + 1, \quad \chi_{F'} = \chi_F. \tag{1.2}$$

In order to compute these invariants, we introduce some numerical invariants of the singularity of a plane curve.

A curve B on X is a nonzero effective divisor.

Definition 1.2. A *partial resolution* of the singularities of B is a sequence of blowing-ups $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r : \hat{X} \to X$

$$(\hat{X}, \sigma^* B) = (X_r, B_r) \xrightarrow{\sigma_r} X_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_2} (X_1, B_1) \xrightarrow{\sigma_1} (X_0, B_0) = (X, B),$$

satisfying the following conditions:

(i) $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities.

(ii) $B_i = \sigma_i^* B_{i-1}$ is the total transform of B_{i-1}.

(iii) σ_i is the blowing-up of X_{i-1} at a singular point $(B_{i-1,\mathrm{red}}, p_{i-1})$ which is not an ordinary double point.

Let $\sigma : \hat{X} \to X$ be the partial resolution of the singularities of F. $\sigma^*(F)$ is called the *normal crossing model* of F.

Note that the surface $\hat{X}$ is smooth, and $\sigma^* F$ may contain some redundant (-1)-curves (coming from the strict transform of F). A (-1)-curve in a fiber is called *redundant* if it meets the other components in at most two points. It is obvious that a redundant (-1)-curve can be contracted without introducing singularities worse than ordinary double points. After a finite number of such contractions, we will finally obtain a normal crossing fiber $\hat{F}$ containing no redundant (-1)-curves. We call $\hat{F}$ the *minimal normal crossing model* of F. $\hat{F}$ is determined uniquely by F.

In Definition 1.2, we denote by m_i the multiplicity of $(B_{i,\mathrm{red}}, p_i)$ at p_i. If $q \in B_{r,\mathrm{red}}$ is a double point, and the two local components of (B_r, q) have multiplicities

a_q and b_q, then we define $[a_q, b_q] := \frac{\gcd(a_q,b_q)^2}{a_q b_q}$. In particular, if B_{red} has only one singular point $p = p_0$, then we define

$$\alpha_p = \sum_{i=1}^{r} (m_i - 2)^2, \qquad \beta_p = \sum_{q \in B_r} [a_q, b_q], \tag{1.3}$$

where q runs over all of the double points of $B_{r,\mathrm{red}}$. These two invariants are independent of the resolution. In fact, the δ-invariant of the singular point p can also be computed from these m_i's. Let

$$\alpha_B := \sum_{p \in B} \alpha_p, \qquad \beta_B := \sum_{p \in B} \beta_p.$$

In particular, we can define β_F. One can check easily that β_F is independent of the resolution, thus F, $\sigma^* F$ and $\hat{F}$ have the same β-invariants.

Definition 1.3. Let $\hat{F}$ be the minimal normal crossing model of F, and let $G(\hat{F})$ be the dual graph of $\hat{F}$. A H-J branch of rational curves in $G(\hat{F})$ is

$$\overset{n_1}{\underset{-e_1}{\circ}} \!\!-\!\!-\!\!-\!\!-\!\! \overset{n_2}{\underset{-e_2}{\circ}} \!\!-\!\!-\!\!-\!\!-\!\! \cdots \!\!-\!\!-\!\!-\!\!-\!\! \overset{n_r}{\underset{-e_r}{\circ}} \!\!-\!\!-\!\!-\!\!-\!\! \overset{n_{r+1}}{\bullet}$$

where $\overset{n_i}{\underset{-e_i}{\circ}}$ denotes a smooth rational curve Γ_i with $\Gamma_i^2 = -e_i$ whose multiplicity in $\hat{F}$ is n_i. $\bullet$ denotes either a curve $\Gamma \not\cong \mathbb{P}^1$, or a smooth rational curve meeting at 3 or more points with the other components.

Note that the r rational curves can be contracted to a Hirzebruch-Jung singularity of type (n, q) with defining equation $z^n = xy^{n-q}$ ([1], Ch. III, §5). Using the notations of ([1], Ch. III, §5), $n_i = \mu_i n_1$ for any i, $1 = \mu_1 < \mu_2 < \cdots < \mu_{r+1}$.

$$n = \mu_{r+1} = \frac{n_{r+1}}{n_1}, \quad q' = \mu_r = \frac{n_r}{n_1}$$

and q is the unique solution of the equation

$$qq' \equiv 1 \pmod{n}, \quad 1 \le q < n.$$

Since μ_i and μ_{i+1} are coprime, the contribution of the branch to $\beta_F = \beta_{\hat{F}}$ is

$$\beta' = \frac{1}{\mu_1 \mu_2} + \frac{1}{\mu_2 \mu_3} + \cdots + \frac{1}{\mu_r \mu_{r+1}}. \tag{1.4}$$

There is a relation ([1], p.81, eq(6))

$$\lambda_k \mu_{k+1} - \lambda_{k+1} \mu_k = n, \tag{1.5}$$

i.e.,

$$\frac{\lambda_k}{\mu_k} - \frac{\lambda_{k+1}}{\mu_{k+1}} = n \frac{1}{\mu_k \mu_{k+1}}. \tag{1.6}$$

Note that $\lambda_1 = q$ and $\lambda_{r+1} = 0$. Take the sum of (1.6) from $k = 1$ to r, we have

$$\beta' = \frac{1}{n} \left(\frac{\lambda_1}{\mu_1} - \frac{\lambda_{r+1}}{\mu_{r+1}} \right) = \frac{q}{n}. \tag{1.7}$$

Theorem 1.4. (Jun Lu) *The contribution of the H-J branch to β_F is $\frac{q}{n}$.*

Definition 1.5. $\beta_F^- = \sum \beta'$ is the total contribution of all H-J branches in $G(\hat{F})$.

The local semistable reduction $\pi : \widetilde{C} \to C$ of F can be constructed as follows. Let $\hat{f} : \hat{X} \to C$ be the fibration induced by $\sigma : \hat{X} \to X$. Denote by d an integer divided by all of the multiplicities of the components in the normal crossing model $\sigma^* F$. Then π can be chosen as a local d-cyclic covering totally ramified over $p = f(F) = \hat{f}(\sigma^* F) \in C$. Let X' be the normalization of $\hat{X} \times_C \widetilde{C}$, then X' admits at worst a finite number of Hirzebruch-Jung singular points. Let X'' be the minimal resolution of the singularities of X'. Then $\widetilde{X}$ is obtained by contracting all vertical (-1)-curves, $\tau : X'' \to \widetilde{X}$.

Theorem 1.6. (Gang Xiao [15]) *A curve in X'' is contracted by τ if and only if it comes from a H-J branch in $\hat{F}$.*

The theorem above is contained in the proof of Prop. 1 of [15].

β_F^- is just $c_{-1}(F)$ defined in [10] by the remark of ([10], p.666). Let $\beta_F^+ = \beta_F - \beta_F^-$.

$$\beta_F = \beta_F^+ + \beta_F^-.$$

Denote by μ_F the sum of the Milnor numbers of the singularities of F_{red}.

Let $N_F = g - p_a(F_{\mathrm{red}})$. One can prove that $0 \le N_F \le g$. $N_F = 0$ iff F is reduced, or $g = 1$ and F is of type ${}_m\mathrm{I}_b$. $N_F = g$ iff F is a tree of smooth rational curves.

The topological characteristic of F is equal to $2N_F + \mu_F + 2 - 2g$.

Then we have the following formulas for the computation of the Chern numbers of F.

Theorem 1.7.

$$\begin{cases} c_1^2(F) = 4N_F + F_{red}^2 + \alpha_F - \beta_F^-, \\ c_2(F) = 2N_F + \mu_F - \beta_F^+, \\ 12\chi_F = 6N_F + F_{red}^2 + \alpha_F + \mu_F - \beta_F. \end{cases} \tag{1.8}$$

From the blow-up formulas (1.2), we only need to compute the Chern numbers of the minimal normal crossing model $\hat{F}$.

Apply the formulas to elliptic fibrations, we have

Theorem 1.8. *(Kodaira) If $g = 1$ and F is minimal, then $c_1^2(F) = 0$ for any F, and $c_2(F)$ is exactly the coefficients as in Kodaira's formula* (0.1).

F	${}_m\mathrm{I}_k$	I_k^*	II	II*	III	III*	IV	IV*
$c_2(F)$	0	6	2	10	3	9	4	8
$\delta(F)$	k	k	0	0	0	0	0	0

Theorem 1.9. *If $g \geq 2$ and F is minimal, then the* 3 *invariants $c_1^2(F)$, $c_2(F)$ and χ_F are nonnegative, and one of them vanishes iff F is semistable.*

So in the case when $g \geq 2$, if one of the 3 invariants of F is zero, then the others are also zero.

Theorem 1.10. *Denote by $F_1, \cdots, F_s$ the singular fibers of $f : X \to C$. Then the Chern numbers of X can be computed by the formulas* (0.3).

The idea of the proof of (0.3) is simple. If f is semistable, then the relative invariants of f are exactly the modular invariants. Otherwise, we can apply it to a successive semistable reduction of one fiber after the other. Then (0.3) follows from the base change property of the modular invariants. See [10] and [11] for the details.

Note that f is isotrivial iff the smooth fibers of f are isomorphic to each other, iff $\kappa(f) = 0$, iff $\lambda(f) = 0$, which implies $\delta(f) = 0$. By (0.3), the Chern numbers depend only on the local Chern numbers of the singular fibers.

Corollary 1.11. *Assume that $f : X \to C$ is isotrivial. Then we have*

$$(1.9) \qquad \begin{cases} c_1^2(X) = 8(g-1)(g(C)-1) + \sum_{i=1}^s c_1^2(F_i), \\ c_2(X) = 4(g-1)(g(C)-1) + \sum_{i=1}^s c_2(F_i), \\ \chi(\mathcal{O}_X) = (g-1)(g(C)-1) + \sum_{i=1}^s \chi_{F_i}. \end{cases}$$

2. Inequalities between the Chern numbers

There are some inequalities between the Chern numbers $c_1^2(F)$ and $c_2(F)$, which are quite similar to those between the first and second Chern numbers of a surface X. For example, the Miyaoka-Yau type inequality and Vojta's canonical class inequality.

In what follows, we always assume that $g \geq 2$. Some inequalities have been obtained in [10].

Theorem 2.1. [10] *Assume that F contains no (-1)-curves. Then*

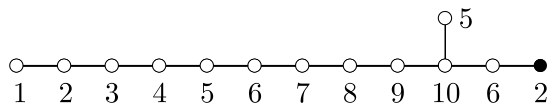
1) *$c_1^2(F) \geq 0$, $c_2(F) \geq 0$ and $\chi_F \geq 0$, and one of them vanishes iff F is semistable.*
2) *$c_1^2(F) \leq 2c_2(F)$, with equality iff F_{red} is a nodal curve and $F = nF_{\mathrm{red}}$ for some positive integer n.*
3) *$c_1^2(F) \leq 4g - 4$.*

Corollary 2.2. *Let $f : X \to C$ be an isotrivial fibration of genus $g \geq 1$. Then we have $c_1^2(X) \leq 2c_2(X)$, with equality iff all singular fibers are multiples of smooth curves.*

Proof. We assume that f is relatively minimal. Then the corollary follows from Corollary 1.10 and Theorem 2.1, 2). ∎

The third inequality can be viewed as the *local canonical class inequality.* In a forthcoming paper, we will prove that the local inequality is strict, i.e., $c_1^2(F) < 4g-4$.

Theorem 2.3. *$c_1^2(F) \leq 4g-5$ except for $g=2$, and if the dual graph of F is as follows.*



Here $\circ$ is a (-2)-curve and $\bullet$ a (-3)-curve. The number is the multiplicity of the curve.

In the exceptional case, $c_2(F) = 16$ and $c_1^2(F) = \frac{16}{5} = 4g-5+\frac{1}{5}$.

The singular fibers of genus g with $c_1^2(F) = 4g-5$ are classified completely. The proof of this theorem will be given in a forthcoming paper. In general $c_1^2(F) \leq 4g-5+\frac{1}{5} < 4g-4$. The local canonical class inequality has some interesting applications. It has been used to establish the canonical class inequality for non-semistable fibrations. Now we give a new proof of the following well-known result.

Corollary 2.4. *Let $f : X \to \mathbb{P}^1$ be a nontrivial fibration of genus $g \geq 1$. Then f admits at least 2 singular fibers.*

Proof. If f is smooth, then it is trivial. Now we assume that f admits only one singular fiber F. In this case, f is isotrivial. So

$$c_1^2(X) = -8(g-1) + c_1^2(F), \quad c_2(X) = -4(g-1) + c_2(F). \tag{2.1}$$

If $g \geq 2$, we proved in [12] that $c_1^2(X) + 8(g-1) = K_f^2 \geq 4(g-1)$. By (2.1), we have $c_1^2(F) \geq 4g-4$, a contradiction.

If $g = 1$, then $12\chi(\mathcal{O}_X) = c_2(X) = c_2(F)$. So $c_2(F)$ is divided by 12. We know that $c_2(F) = 0$, and $F = nE$ for some smooth elliptic curve E and $n \geq 2$. Hence $\chi(\mathcal{O}_X) = 0$. By the formula for canonical class, we have $K_X \sim -(n+1)E$. Hence X is birationally ruled, $p_g(X) = 0$ and $q(X) = 1$. The Albanese map $\alpha : X \to B$ is the ruling. Let F' be a fiber of α. Then $2 = -K_X F' = (n+1)EF' \geq n+1 \geq 3$, a contradiction.

This proves that f admits at least 2 singular fibers. □

As for the local Miyaoka-Yau inequality $c_1^2(F) \leq 2c_2(F)$, Jun Lu recently classifies all singular fibers F of genus g with $c_1^2(F) > 2c_2(F) - 6$. As a consequence, all of these singular fibers have singular semistable models. In particular, if the semistable model of F is smooth, then $c_1^2(F) \leq 2c_2(F) - 6$. In particular, any singular fiber F in an isotrivial fibration satisfies this inequality unless $F = mC$ for some smooth curve C.

3. Elliptic fibrations over $\mathbb{P}^1$ with 2 singular fibers

In this section, we will try to explain that our formulas can be used to classify families of curves $f : X \to \mathbb{P}^1$ with small numbers of singular fibers $F_1, \cdots, F_s$. For example, $s = 2$, or $s = 3$ and f is not isotrivial. We consider only the case when $g = 1$ and $s = 2$ to explain the idea of the method. In fact, U Schmickler-Hirzebruch has classified all such elliptic fibrations with 2 or 3 singular fibers [9].

If $f : X \to \mathbb{P}^1$ admits two singular fibers over 0 and ∞, then the pullback fibration $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ under the cyclic base change $\pi_m : \mathbb{P}^1 \to \mathbb{P}^1$, $t \mapsto t^m$, has also two singular fibers over 0 and ∞ unless it is trivial.

Example 3.1. Let ξ be the primitive n-th root of unity, and let E be an elliptic curve with a torsion element δ of order n. The n-cyclic group $\mathbb{Z}_n = \{\sigma^k\}$ acts on $E \times \mathbb{P}^1$ by $\sigma^k(p, [x, y]) = (p + k\delta, [x, \xi^k y])$. Then $X = (E \times \mathbb{P}^1)/\mathbb{Z}_n \to \mathbb{P}^1 = \mathbb{P}^1/\mathbb{Z}_n$ admits two singular fibers of type ${}_n\mathrm{I}_0$.

Theorem 3.2. [9] *Let $f : X \to \mathbb{P}^1$ be an elliptic fibration with 2 singular fibers. Then f is isomorphic to one of the following families.*

I) $X = (E \times \mathbb{P}^1)/\mathbb{Z}_n$ *as in the above example;*
I*) $y^2 = \lambda(x^3 + x + c), \ \ 4 + 27c^2 \neq 0;$
II) $y^2 = x^3 + \lambda;$
III) $y^2 = x^3 + \lambda x;$
IV) $y^3 = x^3 + \lambda.$

The types of the singular fibers are respectively (${}_n\mathrm{I}_0$, ${}_n\mathrm{I}_0$), (I_0^*, I_0^*), (II, II*), (III, III*) *and* (IV, IV*).

Proof. Let $F_0 = n_0 E_0$ and $F_\infty = n_\infty E_\infty$ be the two singular fibers over 0 and ∞, n_i is the multiplicity of F_i. Assume that f is relatively minimal. By Kodaira's formula,

$$12\chi(\mathcal{O}_X) = c_2(X) = c_2(F_0) + c_2(F_\infty) \leq 20. \tag{3.1}$$

So $\chi(\mathcal{O}_X) = 0$ or 1.

Now consider the n-cyclic base change $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified over $0 = f(F_0)$ and $\infty = f(F_\infty)$. Let $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ be the pullback fibration of f under π. It is well-known that $\widetilde{f}$ is semistable for some n. Because f is isotrivial, $\widetilde{f}$ must be a trivial fibration. Hence there is a generically finite n-cover $\Pi : \widetilde{X} = E \times \mathbb{P}^1 \dashrightarrow X$, which implies that $\kappa(X) = -\infty$. Thus either

A) $\chi(\mathcal{O}_X) = 0$, the Albanese map $\alpha : X \to B$ is a $\mathbb{P}^1$-fibration over an elliptic curve B, $p_g(X) = 0$, $q(X) = 1$; or
B) $\chi(\mathcal{O}_X) = 1$, X is rational with $p_g(X) = q(X) = 0$.

Case A: $\chi(\mathcal{O}_X) = 0$. We have $c_2(F_i) = 0$, and so $F_i = n_i F_i'$, F_i' is the reduced part of F_i. Since the semistable model of F_i is smooth, F_i' is a smooth elliptic curve and $n_i \geq 2$. By Kodaira's formula for the canonical class of the elliptic fibration $f : X \to \mathbb{P}^1$,

(3.2)
$$K_X \sim \left(-2 + \chi(\mathcal{O}_X) + 1 - \frac{1}{n_0} + 1 - \frac{1}{n_\infty}\right) F = -\left(\frac{1}{n_0} + \frac{1}{n_\infty}\right) F \sim -F_0' - F_\infty'.$$

Since $c_1^2(X) = 0$, X is minimal. Denote by $\bar{F} \cong \mathbb{P}^1$ a fiber of α. Then

$$2 = -K_X F' = F_0'\bar{F} + F_\infty'\bar{F},$$

hence $F_0'\bar{F} = F_\infty'\bar{F} = 1$ and $n_0 F_0'\bar{F} = F\bar{F} = n_\infty F_\infty'\bar{F}$, so $n_0 = n_\infty = n$. It is easy to see that $\Pi : \widetilde{X} \to X$ is an unramified n-cyclic cover. Because $\alpha : X \to B$ is a $\mathbb{P}^1$-fibration, the unramified n-cyclic cover $\Pi : \widetilde{X} \to X$ is induced by an unramified n-cyclic cover $E \to B$. Now we see easily that the family f is of type I.

Case B: $\chi(\mathcal{O}_X) = 1$. By (3.1), $c_2(F_0) + c_2(F_\infty) = 12$. We see that the two fibers $\{F_0, F_\infty\}$ are of four types: $\{\mathrm{I}_0^*, \mathrm{I}_0^*\}$, {II, II*}, {III, III*} and {IV, IV*}. In these cases,

$$K_X \sim (-2 + \chi(\mathcal{O}_X) + (1 - 1/1) + (1 - 1/1))F = -F.$$

Since $c_1^2(X) = 0$, X is not minimal. On the other hand, any (-1)-curve E must be a section of f since $FE = -K_X E = 1$. Therefore, X has a Weierstrass normal form as its defining equation

$$y^2 = x^3 + a(t)x + b(t), \quad t \in \mathbb{P}^1, \tag{3.3}$$

where $a(t)$ and $b(t)$ are polynomials in an affine variable t on the base $\mathbb{P}^1$, $\deg a(t) < 4$ or $\deg b(t) < 6$, and $4a(t)^3 + 27b(t)^2 \neq 0$ for $t \neq 0$ and ∞.

Because the family is of constant moduli, the J-invariant $J(t) = \frac{4a(t)^3}{4a(t)^3 + 27b(t)^2}$ is independent of t, we see that either $a(t) \equiv 0$, or $b(t) \equiv 0$, or $b(t)^2 \equiv c \cdot a(t)^3$ for some nonzero constant $c \in \mathbb{C}$.

If $a(t)$ is a zero polynomial, then $b(t)$ has only one zero $t = 0$, so $b(t) = b_0 t^m$, $m < 6$. The standard form is $y^2 = x^3 - t^m$.

If $m = 1$, let $\lambda = -t$. This proves that the family is of type II, the two singular fibers are of types II and II*.

If $m = 2$, replace Z by $t^{-1}Z$ in the equation $Y^2 Z = X^3 - t^2 Z^3$, we get $tX^3 = Z(Z - Y)(Z + Y)$. The polynomial on the right hand side has three roots 0, 1 and -1. By a linear transformation of the homogeneous coordinates, we assume that the three roots are $1, \omega, \omega^2$ with $\omega^3 = 1$. Then the equation is transformed into the one as in IV.

Similarly, the cases $m = 3$, 4 and 5 are transformed respectively to the cases I*, IV, II.

If $b(t)$ is a zero polynomial, then $a(t) = t^m$, $m \leq 3$. The equation is $Y^2 Z = X^3 + t^m XZ^2$. Similarly, we can prove that the types correspond to the cases $m = 1$, 2 and 3 are respectively III, I* and III.

Now we assume that $a(t)$ and $b(t)$ are nonzero polynomials and $b(t)^2 \equiv c\cdot a(t)^3$. Because the discriminant $4a(t)^3 + 27b(t)^2$ has only one zero, we see that the W-equation is $y^2 = x^3 + t^{2m}x + c \cdot t^{3m}$, where $2m < 4$ or $3m < 6$, so we have $m = 1$. This is birationally equivalent to the case IV. The two singular fibers are of types IV and IV*. □

4. Families of curves of genus two

Let F be a singular fiber in a family $f : X \to C$ of curves of genus two. Inspired by Kodaira's work on elliptic surfaces, Ogg Iitaka, Namikawa-Ueno have classified completely all singular fibers of genus two. There are about 126 configurations.

In order to compute the global invariants of the surfaces, Horikawa expresses the surface X as a generically double cover over a ruled surface. By using the canonical resolution of the singularities of a double covering surface, Horikawa classified the singular fibers of genus two into 5 types, I, II, III, IV and V. He got the following global formula.

$$\begin{aligned} &c_1^2(X) - 2\chi(\mathcal{O}_X) = 6(g(C) - 1)+ \\ &+\sum_k \{(2k-1)(\nu(\mathrm{I_k}) + \nu(\mathrm{III_k})) + 2\mathrm{k}(\nu(\mathrm{II_k}) + \nu(\mathrm{IV_k}))\} + \nu(\mathrm{V}) \end{aligned} \tag{4.1}$$

The coefficient is called the Horikawa index of the fiber.

Furthermore, Xiao introduced the singularity indices $s_2(F)$ and $s_3(F)$ for each singular fiber F. In fact, $s_3(F)$ is the Horikawa index of F. Then he got the following formulas.

$$\begin{cases} c_1^2(X) = 8(g(C) - 1) + \sum_{i=1}^s \left(\frac{1}{5}s_2(F_i) + \frac{7}{5}s_3(F_i)\right), \\ \chi(\mathcal{O}_X) = g(C) - 1 + \sum_{i=1}^s \left(\frac{1}{10}s_2(F_i) + \frac{1}{5}s_3(F_i)\right), \\ c_2(X) = 4(g(C) - 1)) + \sum_{i=1}^s (s_2(F_i) + s_3(F_i)). \end{cases} \tag{4.2}$$

When F is semistable, $s_2(F)$ (resp. $s_3(F)$) is the number of inseparable (resp. separable) double points p of F. A double point p is called *separable* if F becomes disconnected when normalize F locally at p, otherwise, p is called *inseparable.* This provides us formulas for the computation of the three modular invariants $\kappa(f)$, $\lambda(f)$ and $\delta(f)$.

Definition 4.1. Let $\widetilde{F}$ be the semistable model of F under an n-cyclic base change totally ramified over $f(F)$. We define

$$\delta_0(F) = \frac{1}{n}\#\{\text{ inseparable double points of } \widetilde{F}\,\},$$
$$\delta_1(F) = \frac{1}{n}\#\{\text{ separable double points of } \widetilde{F}\,\}.$$

$\delta_0(F)$ and $\delta_1(F)$ are independent of the choice of the base changes. If fact, as $\mathbb{Q}$-divisors, $\delta = \delta_0 + \delta_1$ has two components δ_0 and δ_1. $\delta_i(f)$ is the total intersection number of δ_i with the curve C in the moduli space. $\delta_i(F)$ is just the local intersection number.

$$\delta_0(f) = \deg J^*\delta_0 = \sum_{i=1}^{s} \delta_0(F_i), \quad \delta_1(f) = \deg J^*\delta_1 = \sum_{i=1}^{s} \delta_1(F_i). \tag{4.3}$$

So

$$\begin{cases} \kappa(f) = \sum_{i=1}^{s}\left(\frac{1}{5}\delta_0(F_i) + \frac{7}{5}\delta_1(F_i)\right), \\ \lambda(f) = \sum_{i=1}^{s}\left(\frac{1}{10}\delta_0(F_i) + \frac{1}{5}\delta_1(F_i)\right), \\ \delta(f) = \sum_{i=1}^{s}(\delta_0(F_i) + \delta_1(F_i)). \end{cases} \tag{4.4}$$

Thus

$$c_1^2(F) = \frac{1}{5}\left(s_2(F) - \delta_0(F)\right) + \frac{7}{5}\left(s_3(F) - \delta_1(F)\right),$$
$$\chi_F = \frac{1}{10}(s_2(F) - \delta_0(F)) + \frac{1}{5}(s_3(F) - \delta_1(F)).$$

Namely

$$\begin{cases} s_2(F) = \delta_0(F) + 14\chi_F - 2c_1^2(F), \\ s_3(F) = \delta_1(F) + c_1^2(F) - 2\chi_F. \end{cases} \tag{4.5}$$

Therefore, the singularity indices of F can be computed directly from the singular fiber itself.

$$\begin{cases} c_1^2(X) = 8(g(C)-1) + \sum_{i=1}^{s}\left(\dfrac{1}{5}\delta_0(F_i) + \dfrac{7}{5}\delta_1(F_i) + c_1^2(F_i)\right), \\ \chi(\mathcal{O}_X) = g(C) - 1 + \sum_{i=1}^{s}\left(\dfrac{1}{10}\delta_0(F_i) + \dfrac{1}{5}\delta_1(F_i) + \chi_{F_i}\right), \\ c_2(X) = 4(g(C)-1) + \sum_{i=1}^{s}(\delta_0(F_i) + \delta_1(F_i) + c_2(F_i)). \end{cases} \tag{4.6}$$

Based on the complete list of the singular fibers of genus two, Cheng Gong computed the four invariants c_1^2, c_2, δ_0 and δ_1 for each singular fiber of genus two. As a consequence, we see that for any singular fiber F of genus two,

$$\frac{1}{11}c_2(F) \le c_1^2(F) \le \frac{1}{2}c_2(F)$$

i.e.,

$$\chi_F \le c_1^2(F) \le 4\chi_F$$

From (4), it is obvious that

$$2\lambda(f) \le \kappa(f) \le 7\lambda(f).$$

Because $c_1^2(F) \le 4\chi_F$, the inequality $\kappa(f) \le 7\lambda(f)$ between modular invariants implies $K_f^2 \le 7\chi_f$, with equality if and only if f is semistable and any double point in a singular fiber is separable. In this case, the stable model of the singular fiber consists of two nonsingular elliptic curves meeting normally at a point (the singular fiber itself is obtained by replacing the intersection point by a chain of (-2)-curves of type A_k).

On the other hand, the inequality $2\chi_f \le K_f^2$ holds true [4] as a consequence of the nonnegativity of $s_3(F)$, however it is not an immediate consequence of the inequality $2\lambda(f) \le \kappa(f)$ because we have only a weak inequality $c_1^2(F) \ge \chi_F$. The nonnegativity of the Horikawa index $s_3(F)$ is an inequality of Noether's type:

$$2\chi_F - \delta_1(F) \le c_1^2(F). \tag{4.7}$$

We see that there are exactly 10 types of singular fibers of genus two whose semistable models are smooth. Therefore, there are at most 10 types of singular fibers in an isotrivial fibration of genus two. By using Corollary 1.11, we can classify completely genus two fibrations over $\mathbb{P}^1$ with two singular fibers.

By [14] the modular invariants are determined by the singular fibers and by the singularities of the closure of the Weierstrass points of the general fiber. It is a very interesting problem to give explicitly the computation formulas.

In the case of $g \ge 3$, G. Xiao [16] has obtained such formulas when the generic fibers are hyperelliptic. In the non-hyperelliptic case, Z. Chen, J. Lu and the author [2] have just found such formulas when the genus $g = 3$:

$$\begin{cases} \kappa(f) = \frac{1}{3}\delta_0(f) + 3\delta_1(f) + \frac{4}{3}h(f), \\ \lambda(f) = \frac{1}{9}\delta_0(f) + \frac{1}{3}\delta_1(f) + \frac{1}{9}h(f), \\ \delta(f) = \delta_0(f) + \delta_1(f), \end{cases}$$

where $\delta_k(f) = \sum_{i=1}^s \delta_k(F_i)$ and $\delta_k(F)$ is defined as in Definition 4.1. $h(f) = \sum_F h(F)$ and $h(F)$ is the Horikawa index of F, i.e., the local intersection number $I_F(C, H)$ of C with the hyperelliptic locus H in $\overline{\mathcal{M}}_3$.

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