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JACOBI'S FORMULA FOR HESSE CUBIC CURVES

K. MATSUMOTO, T. TERASOMA AND S. YAMAZAKI

Abstract. In this paper, we give analogs of Jacobi's formula for Hesse cubic curves $C(\mu)$ on theta constants and unnormalized period integrals. We also give several expressions of the modular invariant μ of $C(\mu)$ in terms of theta constants, which are reformulations of [Bi] and [Kr].

1. Introduction

An elliptic curve of the Jacobi canonical form with a parameter $\kappa \in \mathbb{C}-\{0,\pm 1\}$ is defined as

$$y^2 = (1-x^2)(1-\kappa^2x^2).$$

Its periods are given by

$$\psi_A(\kappa) = \int_1^{1/\kappa} \frac{2dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}}, \quad \psi_B(\kappa) = \int_0^1 \frac{4dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}}.$$

The ratio $\tau = 2\psi_A/\psi_B$ belongs to the upper half space $\mathbb{H}$ and ψ_B can be expressed by $2\pi F(\frac{1}{2},\frac{1}{2},1;\kappa^2)$, where $F(\alpha,\beta,\gamma;x)$ is the hypergeometric function defined as

$$F(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n, \quad (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1).$$

On the other hand, the parameter κ can be expressed by

$$\kappa = \frac{\vartheta^2_{\frac{1}{2},0}(\tau)}{\vartheta^2_{0,0}(\tau)},$$

where $\vartheta_{a,b}(\tau)$ is the theta constant defined by

$$\vartheta_{a,b}(\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+a)^2\tau + 2\pi i(n+a)b].$$

Moreover, Jacobi's formula states that

$$\vartheta^2_{0,0}(\tau) = \frac{1}{2\pi}\int_0^1 \frac{4dx}{\sqrt{(1-x^2)(1-\kappa^2x^2)}} = \frac{\psi_B}{2\pi} = F\left(\frac{1}{2},\frac{1}{2},1;\kappa^2\right).$$

In this paper, we study two periods ψ_A and ψ_B of a Hesse cubic curve

$$C(\mu) = \{[t_0,t_1,t_2] \in \mathbb{P}^2 \mid t_0^3+t_1^3+t_2^3-3\mu t_0t_1t_2 = 0\}$$

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with a parameter $\mu \in \mathbb{C} - \{1, \omega, \omega^2\}$, where ω denotes $\exp(\frac{2\pi i}{3})$. They are expressed by the hypergeometric function and the beta function in Theorem 1:

$$\psi_A(\mu) = (\omega - 1)B\left(\frac{1}{3}, \frac{1}{3}\right)F\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \mu^3\right) + (1 - \omega^2)\mu B\left(\frac{2}{3}, \frac{2}{3}\right)F(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; \mu^3\Big),$$

$$\psi_B(\mu) = B\left(\frac{1}{3}, \frac{1}{3}\right)F(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \mu^3\Big) - \mu B\left(\frac{2}{3}, \frac{2}{3}\right)F\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}; \mu^3\right).$$

On the other hand, the modular parameter μ is a modular function of $\tau = \psi_A(\mu)/\psi_B(\mu) \in \mathbb{H}$. The parameter μ is known to be expressed in terms of theta constants. For example, we have

$$\mu = \sqrt{3}i\frac{\vartheta^3_{\frac{1}{2},\frac{1}{6}}(\tau)}{\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)} + 1,$$

(see Proposition 3, or [Kr]). More strongly, we give relations between some theta constants and the period ψ_B when μ and τ are related as above. The main theorem (Theorem 2) states that

$$\begin{aligned}
3 \cdot (2\pi)^{12} \cdot \vartheta_{\frac{1}{6},\frac{1}{2}}(\tau)^{24} &= 81 \cdot \psi_B^{12}(\mu) \cdot (\mu^3 - 1),\\
3 \cdot (2\pi)^{12} \cdot \vartheta_{\frac{1}{2},\frac{1}{6}}(\tau)^{24} &= \psi_B^{12}(\mu) \cdot (\mu^3 - 1)(\mu - 1)^8,\\
3 \cdot (2\pi)^{12} \cdot \vartheta_{\frac{1}{6},\frac{1}{6}}(\tau)^{24} &= \psi_B^{12}(\mu) \cdot (\mu^3 - 1)(\mu - \omega^2)^8,\\
3 \cdot (2\pi)^{12} \cdot \vartheta_{\frac{-1}{6},\frac{1}{6}}(\tau)^{24} &= \psi_B^{12}(\mu) \cdot (\mu^3 - 1)(\mu - \omega)^8.
\end{aligned}$$

These formulas are analogs of Jacobi's formula for Hesse cubic curves.

Our proof of Theorem 2 is based on the elementary function theory and the modular property of $\vartheta_{ab}(\tau)$ with respect to the monodromy group of $\tau = \psi_A(\mu)/\psi_B(\mu)$. This idea is applied to the Picard curve version of Jacobi's formula studied in [MS]. Moreover, we have

$$\begin{aligned}
\vartheta_{\frac{1}{6},\frac{1}{6}}(\omega) &= \frac{3^{5/24}}{2\pi}\exp(-\frac{\pi i}{72})\Gamma(\frac{1}{3})^{3/2},\\
\vartheta_{\frac{-1}{6},\frac{1}{6}}(\omega) &= \frac{3^{3/8}}{2\pi}\exp(-\frac{5\pi i}{72})\Gamma(\frac{1}{3})^{3/2},
\end{aligned}$$

as corollary of these formulas. These formulas are obtained also from the Chowla-Selberg formula in [SC] of the η-function $\eta(\tau) = \exp(-\frac{\pi i}{6})\vartheta_{\frac{1}{6}\frac{1}{2}}(3\tau)$ (see also [FK]). These special values are used in [Ha], [MT] and [MS].

The period integrals of $C(\mu)$ and expressions of $\mu(\tau)$ by modular forms are classically studied (c.f. [Bi] and [Kr]). Analogous formulas for certain type of $\mathbb{Z}/N\mathbb{Z}$-coverings of $\mathbb{P}^1$ ($\mathbb{Z}/N$-curves for short) is studied in [N], which is a generalization of Thomae's formula [Th] for hyper-elliptic curves. Another proofs of Thomae's formula for $\mathbb{Z}/N$-curves are discussed in the papers [EF] and [Ko]. Chan-Liu-Ng [CLN] show the equality

$$\frac{\eta(\tau)^3}{\eta(3\tau)} = F\left(\frac{1}{3}, \frac{1}{3}, 1; -27\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}\right),$$

which also implies Theorem 2.

2. Hesse cubic curves

A Hesse cubic curve $C(\mu)$ with a parameter $\mu \in \mathcal{M} = \mathbb{C} - \{1, \omega, \omega^2\}$ is defined as

$$C(\mu) = \{t = [t_0, t_1, t_2] \in \mathbb{P}^2 \mid F_\mu(t) = t_0^3 + t_1^3 + t_2^3 - 3\mu t_0 t_1 t_2 = 0\},$$

where ω denotes the third root $\exp(\frac{2\pi i}{3})$ of unity. Note that $C(\mu)$ is a smooth curve of genus 1 and its nine flex points are given by

$$(1)\qquad \begin{array}{lll} P_0 = [0, -1, 1], & P_1 = [1, 0, -1], & P_2 = [-1, 1, 0], \\ P_3 = [0, -\omega, \omega^2], & P_4 = [1, 0, -\omega^2], & P_5 = [-1, \omega, 0], \\ P_6 = [0, -\omega^2, \omega], & P_7 = [1, 0, -\omega], & P_8 = [-1, \omega^2, 0], \end{array}$$

for any $\mu \in \mathcal{M}$. Let $L_j = \{\ell_j = 0\}$ be the tangent line at the flex point P_j, where

$$(2)\qquad \begin{array}{lll} \ell_0 = \mu t_0 + t_1 + t_2, & \ell_1 = t_0 + \mu t_1 + t_2, & \ell_2 = t_0 + t_1 + \mu t_2, \\ \ell_3 = \mu t_0 + \omega^2 t_1 + \omega t_2, & \ell_4 = t_0 + \omega^2 \mu t_1 + \omega t_2, & \ell_5 = t_0 + \omega^2 t_1 + \omega \mu t_2, \\ \ell_6 = \mu t_0 + \omega t_1 + \omega^2 t_2, & \ell_7 = t_0 + \omega \mu t_1 + \omega^2 t_2, & \ell_8 = t_0 + \omega t_1 + \omega^2 \mu t_2; \end{array}$$

they depend on the parameter μ.

We consider the subgroup $C_3 \times C_3$ of translations on $C(\mu)$ generated by

$$(3)\qquad \rho_1 : [t_0, t_1, t_2] \mapsto [t_2, t_0, t_1], \quad \rho_2 : [t_0, t_1, t_2] \mapsto [t_0, \omega t_1, \omega^2 t_2].$$

Via the map $C_3 \times C_3 \ni \rho_1^j \rho_2^k \mapsto \rho_1^j \rho_2^k(P_0) \in \{P_0, \dots, P_8\}$, we identify the set $\{P_0, \dots, P_8\}$ with $C_3 \times C_3$. Then we have

$$(4)\qquad \rho_1^j \rho_2^k(P_0) = P_{3k+j} \quad (j, k = 0, 1, 2).$$

The linear transformations

$$\begin{aligned} (t_0', t_1', t_2') &\mapsto (t_0, t_1, t_2) = (\omega t_0', t_1', t_2'), \\ (t_0', t_1', t_2') &\mapsto (t_0, t_1, t_2) = \frac{1}{3}(t_0', t_1', t_2') \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \end{aligned}$$

induce the isomorphism from $C(\mu)$ to $C(\omega\mu)$ and that to $C(\frac{\mu+2}{\mu-1})$. The group $\mathcal{N}$ of linear fractional transformations generated by

$$(5)\qquad \nu_1 : \mu \mapsto \omega\mu, \quad \nu_2 : \mu \mapsto \frac{\mu+2}{\mu-1} = 1 + \frac{3}{\mu-1}$$

is isomorphic to the alternating group A_4. The product $\prod_{\nu \in \mathcal{N}} \nu(\mu)$ becomes

$$\frac{\mu^3(\mu^3 + 2^2)^3}{(\mu^3 - 1)^3},$$

which equals to 64 times the j-invariant of $C(\mu)$. Thus $C(\mu')$ is isomorphic to $C(\mu)$ if and only if there exists an element $\nu \in \mathcal{N}$ such that $\mu' = \nu(\mu)$.

3. Periods ψ_A and ψ_B

Let $f_\mu(x,y) = F_\mu(x,y,1)$ be the defining equation of the affine part of $C(\mu)$, where $x = \frac{t_0}{t_2}$, $y = \frac{t_1}{t_2}$. We fix a holomorphic 1-from φ on each $C(\mu)$ as

$$\varphi = \frac{3dy}{\frac{\partial f_\mu(x,y)}{\partial x}} = \frac{dy}{x^2 - \mu y} = \frac{-3dx}{\frac{\partial f_\mu(x,y)}{\partial y}} = \frac{-dx}{y^2 - \mu x}.$$

Note that

$$\rho_1^*(\varphi) = \rho_2^*(\varphi) = \varphi$$

for the isomorphisms ρ_1 and ρ_2 in (3).

In order to construct cycles A and B in $C(\mu)$, we consider a non-abelian covering $pr_y : C(\mu) \ni (x,y) \mapsto y \in \mathbb{P}^1$. Then the ramification points of pr_y are given by $\frac{\partial f_\mu}{\partial x}(x,y) = 0$. Their images $\eta_1, \dots, \eta_6$ in $\mathbb{P}^1$ are

$$\omega^k \sqrt[3]{2\mu^3 - 1 \pm 2\sqrt{\mu^3(\mu^3-1)}}, \quad (k = 0,1,2).$$

Note that the y is a local coordinate of $C(\mu)$ except η_j. We give cycles in $C(\mu)$ as liftings of those in $\mathbb{P}^1 - \{\eta_1, \dots, \eta_6\}$ by pr_y. To specify the cycles A and B in $C(\mu)$, we choose μ as $\mu_0 = \sqrt[-3]{2}$. For μ_0, we have

$$\eta_j = i(-\omega^2)^j, \quad (j = 1, \dots, 6).$$

For any $\mu \in \mathcal{M}$ and a path γ from μ_0 to μ in $\mathbb{P} - \{1, \omega, \omega^2, \infty\}$ we define the cycles A and B by the continuation according to γ. The homology classes of A and B depends only on the homotopy class of γ.

We define chains connecting flex points in (1). We define the chain λ_{01} from P_0 to P_1 in $\mathbb{R}^2$ as the lifting of the interval $[-1,0]$ in $C(\mu)$. We set

$$B = (1 + \rho_1 + \rho_1^2)(\lambda_{01}). \tag{6}$$

Since

$$\partial(\lambda_{01}) = P_1 - P_0, \quad \partial(\rho_1(\lambda_{01})) = P_2 - P_1, \quad \partial(\rho_1^2(\lambda_{01})) = P_0 - P_2,$$

B is a cycle. Note that it is the intersection of $C(\mu)$ and the real projective plane $\mathbb{P}^2(\mathbb{R})$.

We define the chain λ_{03} from P_0 to P_3 as the lifting of the segment from -1 to $-\omega^2$ in $C(\mu)$. (Figure 1.) We set

$$A = (1 + \rho_2 + \rho_2^2)(\lambda_{03}). \tag{7}$$

Since

$$\partial(\lambda_{03}) = P_3 - P_0, \quad \partial(\rho_2(\lambda_{01})) = P_6 - P_3, \quad \partial(\rho_2^2(\lambda_{01})) = P_0 - P_6,$$

the chain A is a cycle.

Lemma 1. *The intersection number $B \cdot A$ of cycles B and A is* 1.

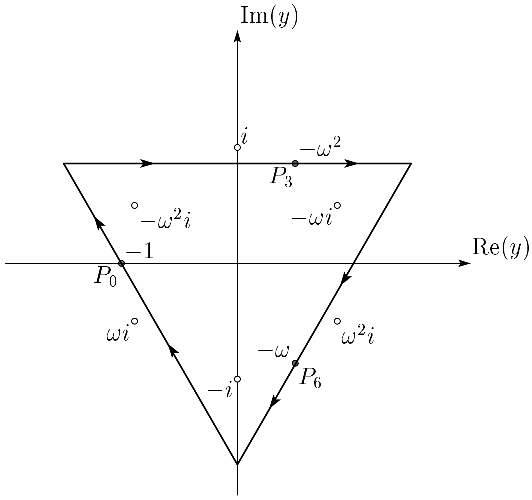

FIGURE 1. cycle A

Proof. Consider the projections $A|_y$ and $B|_y$ of cycles A and B under pr_y; $A|_y$ is in Figure 1 and $B|_y$ is the real axis. They have two intersection points. The one is $y = -1$, which corresponds to P_0. It is easy to see that the intersection number of B and A at this point is 1. The other $P|_y$ is the projection of $\rho_2(\lambda_{03})$ with real value. By following the continuation, the x-coordinate of $\rho_2(\lambda_{03})$ for real y does not belong to $\mathbb{R}$. Thus the preimage of $P|_y$ in A is different from that in B. □

We define periods ψ_A and ψ_B of $C(\mu)$ as

$$\psi_A = \psi_A(\mu) = \int_A \varphi, \quad \psi_B = \psi_B(\mu) = \int_B \varphi. \tag{8}$$

By Riemann's period relation, neither ψ_A nor ψ_B vanishes and

$$\tau = \tau(\mu) = \frac{\psi_A(\mu)}{\psi_B(\mu)} \tag{9}$$

belongs to the upper half space $\mathbb{H}$. For each $\mu \in \mathcal{M}$, the map

$$\Psi_\mu : C(\mu) \ni P \mapsto \frac{1}{\psi_B} \int_{P_0}^{P} \varphi \in E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \tag{10}$$

is an isomorphism. By (4), (6) and (7), we have

$$\Psi_\mu(P_{3k+j}) \equiv \frac{k}{3}\tau + \frac{j}{3} \mod (\mathbb{Z}\tau + \mathbb{Z}) \tag{11}$$

for $j, k \in \mathbb{F}_3 = \{0, 1, 2\}$.

The hypergeometric function $F(\alpha,\beta,\gamma;x)$ is defined as

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n,$$

where the variable z belongs to the unit disk $D=\{z\in\mathbb{C}\mid |z|<1\}$, $\gamma\neq 0,-1,-2,\dots$, and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)=\Gamma(a+n)/\Gamma(a)$. This function admits the integral representation:

$$F(\alpha,\beta,\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_1^{\infty}t^{\beta-\gamma}(t-1)^{\gamma-\alpha}(t-z)^{-\beta}\frac{dt}{t-1}.$$

Theorem 1. *For $\mu\in D=\{z\in\mathbb{C}\mid |z|<1\}$, we have*

$$\psi_A(\mu)=(\omega-1)B\left(\frac{1}{3},\frac{1}{3}\right)F\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};\mu^3\right)+(1-\omega^2)\mu B\left(\frac{2}{3},\frac{2}{3}\right)F\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};\mu^3\right),$$

$$\psi_B(\mu)=B\left(\frac{1}{3},\frac{1}{3}\right)F\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};\mu^3\right)-\mu B\left(\frac{2}{3},\frac{2}{3}\right)F\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};\mu^3\right).$$

Proof. We show the second equality. It is enough to show it for μ in the open interval $(0,1)$. Let $\tilde{C}(\mu)$ be the double cover of $C(\mu)$ defined by

$$\jmath:\tilde{C}(\mu)\ni(u,v)\mapsto(x,y)=(-u-v,\frac{uv}{\mu})\in C(\mu).$$

Then $\tilde{C}(\mu)$ is given by

$$v^3=\mu^3\frac{u^3-1}{u^3-\mu^3}.$$

Let $\tilde{\lambda}$ be the chain in $\tilde{C}(\mu)$ such that u varies from 1 to ∞ and v dependently varies from 0 to μ^3. It is easy to see that the image $\jmath(\tilde{\lambda})$ of $\tilde{\lambda}$ coincides with the chain $\rho_1(\lambda_{01})$ from P_1 to P_2. Since the pull-back of φ under the map $\jmath$ is

$$\jmath^*(\frac{dy}{x^2-\mu y})=\frac{\mu^2(u-v)du}{(u^3-\mu^3)v^2},$$

we have

$$\int_B\frac{dy}{x^2-\mu y}=3\int_{\rho_1(\lambda_{01})}\frac{dy}{x^2-\mu y}=3\int_{\tilde{\lambda}}\frac{\mu^2(u-v)du}{(u^3-\mu^3)v^2}.$$

By the inequalities $u-1>0$, $u-\mu>0$, $0<v<\mu$ on the chain $\tilde{\lambda}$, v can be expressed as

$$v=\mu(u^3-1)^{\frac{1}{3}}(u^3-\mu^3)^{-\frac{1}{3}},$$

where the cubic roots take positive real values. Thus we have

$$\begin{aligned}&3\int_{\tilde{\lambda}}\frac{\mu^2(u-v)du}{(u^3-\mu^3)v^2}\\ =\ &3\int_1^{\infty}u(u^3-1)^{-\frac{2}{3}}(u^3-\mu^3)^{-\frac{1}{3}}du-3\mu\int_1^{\infty}(u^3-1)^{-\frac{1}{3}}(u^3-\mu^3)^{-\frac{2}{3}}du.\end{aligned}$$

Put $u^3=t$, then we see that the integrals can be expressed by the hypergeometric function. We can similarly show the first equality. □

4. Monodromy

We give relations between $(\psi_A(\nu_i(\mu)), \psi_B(\nu_i(\mu)))$ and $(\psi_A(\mu), \psi_B(\mu))$ for $i = 1, 2$, where ν_1 and ν_2 are given in (5).

Lemma 2. *The values of ψ_A and ψ_B at $\nu_1(\mu) = \omega\mu$ are given by*

$$\begin{pmatrix}\psi_A(\omega\mu)\\ \psi_B(\omega\mu)\end{pmatrix} = \omega^2 \begin{pmatrix}-2 & -3\\ 1 & 1\end{pmatrix}\begin{pmatrix}\psi_A(\mu)\\ \psi_B(\mu)\end{pmatrix},$$

where μ is in the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

Proof. We have

$$\begin{pmatrix}\psi_A(\omega\mu)\\ \\ \psi_B(\omega\mu)\end{pmatrix} = \begin{pmatrix}\omega-1 & 1-\omega^2\\ \\ 1 & -1\end{pmatrix}\begin{pmatrix}B\left(\frac{1}{3},\frac{1}{3}\right)F\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};\mu^3\right)\\ \\ \omega\mu B\left(\frac{2}{3},\frac{2}{3}\right)F\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};\mu^3\right)\end{pmatrix}$$

$$= \begin{pmatrix}\omega-1 & 1-\omega^2\\ \\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ \\ 0 & \omega\end{pmatrix}\begin{pmatrix}B\left(\frac{1}{3},\frac{1}{3}\right)F\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};\mu^3\right)\\ \\ \mu B\left(\frac{2}{3},\frac{2}{3}\right)F\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};\mu^3\right)\end{pmatrix}$$

$$= \omega^2\begin{pmatrix}-2 & -3\\ \\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega-1 & 1-\omega^2\\ \\ 1 & -1\end{pmatrix}\begin{pmatrix}B\left(\frac{1}{3},\frac{1}{3}\right)F\left(\frac{1}{3},\frac{1}{3},\frac{2}{3};\mu^3\right)\\ \\ \mu B\left(\frac{2}{3},\frac{2}{3}\right)F\left(\frac{2}{3},\frac{2}{3},\frac{4}{3};\mu^3\right)\end{pmatrix},$$

which implies this lemma. □

Put $\mu' = \nu_2(\mu) = 1 + \frac{3}{\mu-1}$. By considering the map

$$C(\mu) \ni (x, y) \mapsto (x', y') = \left(\frac{x+y+1}{x+\omega^2 y+\omega}, \frac{x+\omega y+\omega^2}{x+\omega^2 y+\omega}\right) \in C(\mu'),$$

we can obtain the following lemma.

Lemma 3. *The values of ψ_A and ψ_B at $\nu_2(\mu) = 1 + \frac{3}{\mu-1}$ are given by*

$$\frac{\begin{pmatrix}\psi_A(\nu_2(\mu))\\ \psi_B(\nu_2(\mu))\end{pmatrix} = 1-\mu}{\sqrt{-3}\begin{pmatrix}-3 & -5\\ 2 & 3\end{pmatrix}\begin{pmatrix}\psi_A(\mu)\\ \psi_B(\mu)\end{pmatrix}},$$

where μ is in a small neighborhood U of $1-\sqrt{3}$ which is the fixed point of ν_2 in the unit disk D.

We put

$$N_1 = \begin{pmatrix}-2 & -3\\ 1 & 1\end{pmatrix}, \quad N_2 = \begin{pmatrix}-3 & -5\\ 2 & 3\end{pmatrix}, \tag{12}$$

They belong to $SL_2(\mathbb{Z})$ and satisfy $N_1^3 = -N_2^2 = I_2$.

Proposition 1. *If μ follows an anti-clockwise path along a small circle around ω^j, then $\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$ is analytically continued to $M_{\omega^j}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$, where*

$$M_1=\begin{pmatrix}1&3\\0&1\end{pmatrix},\quad M_\omega=\begin{pmatrix}7&12\\-3&-5\end{pmatrix},\quad M_{\omega^2}=\begin{pmatrix}4&3\\-3&-2\end{pmatrix}.$$

These matrices satisfy

$$M_\infty^{-1}=M_1M_\omega M_{\omega^2}=\begin{pmatrix}1&0\\3&1\end{pmatrix}$$

and generate the principal congruence subgroup

$$\mathbf{\Gamma}(3)=\{g\in SL_2(\mathbb{Z})\mid g\equiv I_2 \bmod 3\}.$$

Proof. Proposition 12.2 in [Mu] together with our construction of cycles by the actions of ρ_1 and ρ_2 implies that M_{ω^j} belongs to $\mathbf{\Gamma}(3)$.

We show the equality $M_\omega=(N_1^2N_2)^3$. Let $\mu_1=1-\sqrt{3}$, $\mu_2=(\nu_1^2\circ\nu_2)(\mu_1)$ and $\mu_3=(\nu_1^2\circ\nu_2)^2(\mu_1)$, and let $\overline{\mu_1\mu_2}$ be the segment from μ_1 to μ_2. The composite γ_ω of the paths $\overline{\mu_1\mu_2}$, $(\nu_1^2\circ\nu_2)(\overline{\mu_1\mu_2})$ and $(\nu_1^2\circ\nu_2)^2(\overline{\mu_1\mu_2})$ turns around the point ω. By Lemmas 2 and 3, we have

$$\begin{pmatrix}\psi'_A(\mu)\\ \psi'_B(\mu)\end{pmatrix}=c'(\mu)(N_1^2N_2)^3\begin{pmatrix}\psi_A(\mu)\\ \psi_B(\mu)\end{pmatrix},$$

where ${}^t\left(\psi'_A(\mu),\psi'_B(\mu)\right)$ denotes the analytic continuation of ${}^t\left(\psi_A(\mu),\psi_B(\mu)\right)$ along the path γ_ω and $c'(\mu)$ is a rational function of μ. Since $(N_1^2N_2)^3\in\mathbf{\Gamma}(3)$, we have $c'(\mu)=1$ and $M_\omega=(N_1^2N_2)^3$. It is easy to see that

$$M_\omega=N_1M_1N_1^{-1},\quad M_{\omega^2}=N_1^2M_1N_1^{-2}.$$

Thus we get the first part of the proposition.

A fundamental domain of the quotient space $\mathbb{H}/\mathbf{\Gamma}(3)$ and a system of generators of $\mathbf{\Gamma}(3)$ are given in [FK] and [Kl]. These results imply that $\mathbf{\Gamma}(3)$ is freely generated by M_1, M_∞ and ${}^tM_{\omega^2}^{-1}$. □

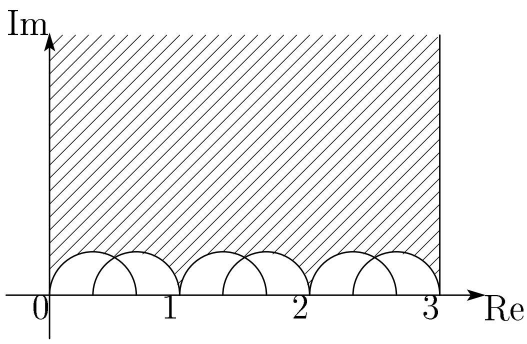

FIGURE 2. Fundamental domain for $\mathbb{H}/\mathbf{\Gamma}(3)$

5. Modular function μ in terms of theta constants.

The theta function with characteristic (a, b) is defined as

$$\vartheta_{a,b}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)],$$

where (z, τ) is a variable in $\mathbb{C}\times\mathbb{H}$, a, b are real parameters. This function satisfies

$$\begin{aligned}
\vartheta_{a,b}(z,\tau) &= \exp(\pi i a^2\tau + 2\pi i a(z+b))\vartheta_{0,0}(z+a\tau+b,\tau),\\
\vartheta_{a,b}(z+1,\tau) &= \exp(2\pi i a)\vartheta_{a,b}(z,\tau),\\
\vartheta_{a,b}(z+\tau,\tau) &= \exp(-2\pi i b)\exp(-\pi i\tau - 2\pi i z)\vartheta_{a,b}(z,\tau),\\
\vartheta_{a,b}(-z,\tau) &= \vartheta_{-a,-b}(z,\tau),\\
\vartheta_{a+p,b+q}(z,\tau) &= \exp(2\pi i a q)\vartheta_{a,b}(z,\tau),\\
\vartheta_{a,b}(z+c\tau+d,\tau) &= \exp(-\pi i c^2\tau - 2\pi i c(z+b+d))\vartheta_{a+c,b+d}(z,\tau),
\end{aligned}$$

where $c, d \in \mathbb{R}$, $p, q \in \mathbb{Z}$. It is known that $\theta_{a,b}(z,\tau) = 0$ if and only if $z = \frac{1+\tau}{2} - (a+p)\tau - (b+q)\tau$, where $p, q \in \mathbb{Z}$. Thus for

$$(a,b) = \frac{1}{2}(1,1) - \frac{1}{3}(k,j),\ (a',b') = \frac{1}{2}(1,1) - \frac{1}{3}(k',j'),\quad j,k,j',k' \in \mathbb{F}_3,$$

$\vartheta^3_{a,b}(z,\tau)/\vartheta^3_{a',b'}(z,\tau)$ is a meromorphic function on E_τ and it has a zero of order 3 at $\frac{k\tau+j}{3}$ and a pole of order 3 at $\frac{k'\tau+j'}{3}$. Since the pull-back of this function under the map Ψ_μ has the divisor $3P_{3k+j} - 3P_{3k'+j'}$, it is a constant multiple of the ratio $\ell_{3k+j}/\ell_{3k'+j'}$ of the linear forms in (2). More precisely, we have the following proposition.

Proposition 2. *We have the equality*

$$(\ell_0 : \ell_1 : \cdots : \ell_8) = ((-1)^j\omega^{-jk}\vartheta_0^3 : (-1)^j\omega^{-jk}\vartheta_1^3 : \cdots : (-1)^j\omega^{-jk}\vartheta_8^3)$$

in $\mathbb{P}^8$, *where* $j, k \in \{0, 1, 2\}$ *and*

$$\vartheta_{3k+j} = \Psi_\mu^*(\vartheta_{\frac{1}{2}-\frac{k}{3},\frac{1}{2}-\frac{j}{3}}(z,\tau)).$$

Proof. The value of the meromorphic function ℓ_1/ℓ_0 at $P_2 = [-1, 1, 0]$ is

$$\ell_1/\ell_0(P_2) = \frac{t_0+\lambda t_1+t_2}{\lambda t_0+t_1+t_2}(P_2) = \frac{-1+\lambda}{-\lambda+1} = -1.$$

On the other hand, the value of

$$\Psi_\mu^*(\vartheta^3_{\frac{1}{2},\frac{1}{2}-\frac{1}{3}}(z,\tau)/\vartheta^3_{\frac{1}{2},\frac{1}{2}}(z,\tau))$$

at P_2 is

$$\frac{\vartheta^3_{\frac{1}{2},\frac{1}{2}-\frac{1}{3}}(\frac{2}{3},\tau)}{\vartheta^3_{\frac{1}{2},\frac{1}{2}}(\frac{2}{3},\tau)} = \frac{\vartheta^3_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(0,\tau)}{\vartheta^3_{\frac{1}{2},\frac{1}{2}+\frac{2}{3}}(0,\tau)} = \frac{\vartheta^3_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(0,\tau)}{\vartheta^3_{-\frac{1}{2},-\frac{1}{2}-\frac{2}{3}}(0,\tau)} = \frac{\vartheta^3_{\frac{1}{2},\frac{1}{2}+\frac{1}{3}}(0,\tau)}{\vartheta^3_{\frac{1}{2}-1,\frac{1}{2}+\frac{1}{3}-2}(0,\tau)} = 1.$$

Thus we have

$$\frac{\ell_1}{\ell_0} = -\frac{\vartheta_1^3}{\vartheta_0^3}.$$

Similarly we can calculate the rests. □

The function $\vartheta_{ab}(\tau) = \theta_{ab}(0,\tau)$ of $\tau \in \mathbb{H}$ is called the theta constant. Statements in the rest of this section are classically known. We give their proves for the convenience of readers.

Proposition 3 (c.f.[Kr])**.** *The parameter μ can be expressed as*

$$\begin{aligned}
\mu - 1 &= \sqrt{3}i\frac{\vartheta^3_{\frac{1}{2},\frac{1}{6}}(\tau)}{\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)},\\
\mu - \omega &= \sqrt{3}i\frac{\vartheta^3_{\frac{-1}{6},\frac{1}{6}}(\tau)}{\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)},\\
\mu - \omega^2 &= \sqrt{3}i\frac{\vartheta^3_{\frac{1}{6},\frac{1}{6}}(\tau)}{\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)},\\
\mu^3 - 1 &= -3\sqrt{3}i\frac{\vartheta^3_{\frac{1}{2},\frac{1}{6}}(\tau)\vartheta^3_{\frac{-1}{6},\frac{1}{6}}(\tau)\vartheta^3_{\frac{1}{6},\frac{1}{6}}(\tau)}{\vartheta^9_{\frac{1}{6},\frac{1}{2}}(\tau)}.
\end{aligned}$$

Proof. The value of ℓ_1/ℓ_3 at P_0 is $-(\mu-1)/(\omega-\omega^2)$. On the other hand, $\vartheta_1^3/\vartheta_3^3$ at P_0 is $-\vartheta^3_{\frac{1}{2},\frac{1}{6}}(\tau)/\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)$. Thus we have

$$\mu - 1 = \sqrt{3}i\frac{\vartheta^3_{\frac{1}{2},\frac{1}{6}}(\tau)}{\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau)}.$$

In order to get the second equality, consider the values of ℓ_3/ℓ_5 and $\vartheta_3^3/\vartheta_5^3$ at P_0. In order to get the third equality, consider the values of ℓ_3/ℓ_4 and $\vartheta_3^3/\vartheta_4^3$ at P_0. By multiplying these three equalities, we have the last equality. □

Corollary 1 (c.f.[FK], p.193, Thm. 3.12)**.** *We have*

$$\begin{aligned}
\vartheta^3_{\frac{1}{6}\frac{1}{6}}(\tau) &= \vartheta^3_{\frac{1}{2}\frac{1}{6}}(\tau) - \omega\vartheta^3_{\frac{1}{6}\frac{1}{2}}(\tau),\\
\vartheta^3_{\frac{1}{6}\frac{-1}{6}}(\tau) &= \vartheta^3_{\frac{1}{2}\frac{1}{6}}(\tau) + \omega^2\vartheta^3_{\frac{1}{6}\frac{1}{2}}(\tau).
\end{aligned}$$

Proof. Eliminate μ from the equalities in Proposition 3. □

Proposition 4 (c.f.[Kr])**.** *The map $E_\tau \ni z \mapsto [h_0(z), h_1(z), h_2(z)] \in C(\mu)$ is the inverse of Ψ_μ, where*

$$\begin{pmatrix} h_0(z)\\ h_1(z)\\ h_2(z)\end{pmatrix} = \begin{pmatrix} \vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)\vartheta_{\frac{1}{6},\frac{1}{2}}(z,\tau)\vartheta_{\frac{-1}{6},\frac{1}{2}}(z,\tau)\\ -\vartheta_{\frac{1}{2},\frac{1}{6}}(z,\tau)\vartheta_{\frac{1}{6},\frac{1}{6}}(z,\tau)\vartheta_{\frac{-1}{6},\frac{1}{6}}(z,\tau)\\ \vartheta_{\frac{1}{2},\frac{-1}{6}}(z,\tau)\vartheta_{\frac{1}{6},\frac{-1}{6}}(z,\tau)\vartheta_{\frac{-1}{6},\frac{-1}{6}}(z,\tau)\end{pmatrix}.$$

Proof. By following the proof of Proposition 2, we can show that

$$\left(\frac{t_0}{t_2}, \frac{t_1}{t_2}\right) = \left(\frac{\vartheta_0\vartheta_3\vartheta_6}{\vartheta_2\vartheta_5\vartheta_8}, -\frac{\vartheta_1\vartheta_4\vartheta_7}{\vartheta_2\vartheta_5\vartheta_8}\right)$$

as meromorphic functions on $C(\mu)$. □

6. Analogs of Jacobi's formula for Hesse cubic curves

In this section, we prove the following formulas, which are analogs of Jacobi's formula for Hesse cubic curves.

Theorem 2. *We have*

$$\begin{aligned}
3\cdot(2\pi)^{12}\cdot\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau) &= 81\cdot\psi_B(\mu)^{12}\cdot(\mu^3-1),\\
3\cdot(2\pi)^{12}\cdot\vartheta^{24}_{\frac{1}{2},\frac{1}{6}}(\tau) &= \psi_B(\mu)^{12}\cdot(\mu^3-1)(\mu-1)^8,\\
3\cdot(2\pi)^{12}\cdot\vartheta^{24}_{\frac{1}{6},\frac{1}{6}}(\tau) &= \psi_B(\mu)^{12}\cdot(\mu^3-1)(\mu-\omega^2)^8,\\
3\cdot(2\pi)^{12}\cdot\vartheta^{24}_{\frac{-1}{6},\frac{1}{6}}(\tau) &= \psi_B(\mu)^{12}\cdot(\mu^3-1)(\mu-\omega)^8,
\end{aligned}$$

where we regard $\tau = \psi_A(\mu)/\psi_B(\mu)$ as a function of $\mu \in \mathcal{M} = \mathbb{P}^1 - \{1, \omega, \omega^2, \infty\}$.

In order to prove this theorem, we give some facts and lemmas.

Fact 1 ([Ig], p.176)**.** For any $M = (m_{jk}) \in SL_2(\mathbb{Z})$, we have

$$\vartheta_{M\cdot(a,b)}(M\cdot\tau)^8 = \varepsilon_{a,b}(M)^8(m_{21}\tau+m_{22})^4\vartheta_{a,b}(\tau)^8,$$

where

$$\begin{aligned}
M\cdot\tau &= \frac{m_{11}\tau+m_{12}}{m_{21}\tau+m_{22}}, \quad M\cdot(a,b) = (a,b)M^{-1} + \frac{1}{2}(m_{21}m_{22}, m_{11}m_{12}),\\
\varepsilon_{a,b}(M) &= \exp[-\pi i(a^2m_{12}m_{22} - 2abm_{12}m_{21} + b^2m_{21}m_{11})]\\
&\quad \times\exp[\pi i(am_{22}-bm_{21})m_{11}m_{12}].
\end{aligned}$$

For generators $T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $S = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ of $SL_2(\mathbb{Z})$, we have

$$\begin{aligned}
\vartheta_{a,b-a+\frac{1}{2}}(\tau+1)^8 &= \exp[-\pi i a(a-1)]^8\vartheta_{a,b}(\tau)^8,\\
\vartheta_{-b,a}\left(\frac{-1}{\tau}\right)^8 &= \exp[-2\pi i ab]^8\tau^4\vartheta_{a,b}(\tau)^8.
\end{aligned}$$

Lemma 4. *We have*

$$\lim_{\mu\to 1}\psi_B = \frac{2\pi}{\sqrt{3}} = B(1/3, 2/3),$$

where μ goes to 1 in the unit disk D.

Proof. The period ψ_B is given as

$$\int_{-1}^{0}\frac{3dy}{x^2-\mu y}.$$

If μ belongs to the open interval $(0,1)$ then x varies from 0 to -1 in the closed interval $[-1,0]$ as y varies from -1 to 0. When $\mu=1$, the curve $C_\mu^\circ : x^3+y^3+1-3\mu xy=0$ reduces to the union of three lines

$$x+y+1=0, \quad \omega x+\omega^2 y+1=0, \quad \omega^2 x+\omega y+1=0.$$

Since the cycle B is in $C_\mu \cap \mathbb{P}^2(\mathbb{R})$, its limit chain as $\mu \to 1$ is in the line $x+y+1=0$. Thus we have

$$\lim_{\mu\to 1} 3\int_{-1}^{0} \frac{dy}{x^2-\mu y} = \int_{-1}^{0} \frac{3dy}{(1+y)^2-y}$$
$$= \int_{-1}^{0} \frac{3dy}{1+y+y^2} = \Big[2\sqrt{3}\arctan\big(\frac{2y+1}{\sqrt{3}}\big)\Big]_{-1}^{0} = \frac{2\pi}{\sqrt{3}}.$$

On the other hand, we have

$$B(1/3,2/3) = \frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1)} = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}}$$

by the formula $\Gamma(\alpha)\Gamma(1-\alpha)=\pi/\sin(\pi\alpha)$. □

Similarly we can show the following.

Lemma 5. *We have*

$$\lim_{\mu\to 1} \frac{\psi_A(\mu)}{\psi_B(\mu)} = i\infty.$$

Lemma 6.

$$\vartheta^4_{\frac{1}{6},\frac{1}{2}}(3\tau) = \frac{i}{\sqrt{3}}\vartheta_{\frac{1}{6},\frac{1}{2}}(\tau)\vartheta_{\frac{1}{6},\frac{1}{6}}(\tau)\vartheta_{\frac{1}{2},\frac{1}{6}}(\tau)\vartheta_{\frac{-1}{6},\frac{1}{6}}(\tau).$$

Proof. Use infinite product expressions of theta constants in §13 of [Mu]. □

Lemma 7.

$$\lim_{\mathrm{Im}(\tau)\to\infty} \frac{\vartheta^{36}_{\frac{1}{6},\frac{1}{2}}(\tau)}{\vartheta^{12}_{\frac{1}{6},\frac{1}{2}}(3\tau)} = 1.$$

Proof. Express $\vartheta_{\frac{1}{6},\frac{1}{2}}(\tau)$ and $\vartheta_{\frac{1}{6},\frac{1}{2}}(3\tau)$ by $q=\exp(\pi i\tau)$. We have only to note that $\exp(\pi i(3\tau))=q^3$ and that $q\to 0$ as $\mathrm{Im}(\tau)\to\infty$. □

Remark 1. The function $\vartheta_{\frac{1}{6},\frac{1}{2}}(3\tau)^{24}$ is known to be Jacobi's Δ-function, which is a cusp form with respect to $SL_2(\mathbb{Z})$ and admits the infinite product expression:

$$\Delta(\tau) = \exp(2\pi i\tau)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau))^{24}.$$

Proof of Theorem 2. We show the equality

$$3\cdot(2\pi)^{12}\cdot\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau(\mu)) = 81\cdot\psi_B(\mu)^{12}\cdot(\mu^3-1).$$

The rests can be easily obtained from this equality and Proposition 3. Consider the function $f(\mu) = \vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau(\mu))/\psi_B(\mu)^{12}$ on $\mathcal{M}$. We claim that $f(\mu)$ is single-valued and holomorphic on $\mathcal{M}$. By the continuation of $\begin{pmatrix}\psi_A(\mu)\\ \psi_B(\mu)\end{pmatrix}$ along a closed path, it is multiplied by an element $M = (m_{jk}) \in \mathbf{\Gamma}(3)$ from the left. Then $\psi_B(\mu)^{12}$ is transformed into

$$(m_{21}\psi_A(\mu) + m_{22}\psi_B(\mu))^{12}$$

and $\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)$ is transformed into

$$\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(M\cdot\tau) = (m_{21}\tau + m_{22})^{12}\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)$$

by Fact 1. Note that $(\kappa(M)\varepsilon_{a,b}(M))^{24} = 1$ and the characteristic $(\frac{1}{6},\frac{1}{2})$ is invariant by the action of M by $M \in \mathbf{\Gamma}(3)$. Since $\tau = \psi_A/\psi_B$, it turns out that $\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau(\mu))/\psi_B(\mu)^{12}$ is invariant under this continuation. Thus $f(\mu)$ is single-valued. We have pointed out that ψ_B never vanishes for any $\mu \in \mathcal{M}$. Hence this claim is shown.

Next we show that the meromorphic function $f(\mu)/(\mu^3-1)$ is a constant. Since $\vartheta_{\frac{1}{6},\frac{1}{2}}(\tau)$ never vanishes on $\mathbb{H}$, $f(\mu)/(\mu^3-1)$ is a non-zero holomorphic function on $\mathbb{P}^1 - \{1,\omega,\omega^2,\infty\}$. We study the behavior of $f(\mu)/(\mu^3-1)$ around the isolated singular points $1,\omega,\omega^2$ and ∞.

If μ goes to 1, then ψ_B converges to $2\pi/\sqrt{3}$ and τ goes to $i\infty$ by Lemmas 4 and 5. The function $\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)/(\mu^3-1)$ is a constant multiple of $\vartheta^{36}_{\frac{1}{6},\frac{1}{2}}(\tau)/\vartheta^{12}_{\frac{1}{6},\frac{1}{2}}(3\tau)$ by the last equality of Proposition 3 and Lemma 6. Its limit as $\tau \to i\infty$ is 1 by Lemma 7. Thus we have

$$\lim_{\mu\to 1}\frac{\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau(\mu))}{(\mu^3-1)\psi_B(\mu)^{12}} = \frac{27}{(2\pi)^{12}}. \tag{13}$$

In order to study the behavior of $f(\mu)/(\mu^3-1)$ as $\mu\to\omega$, we use Lemma 2. We have

$$\begin{aligned}\lim_{\mu\to\omega}\frac{\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)}{\psi_B(\mu)^{12}(\mu^3-1)} &= \lim_{\mu\to 1}\frac{\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(N_1\cdot\tau)}{(\psi_A(\mu)+\psi_B(\mu))^{12}(\mu^3-1)}\\ = \lim_{\mathrm{Im}(\tau)\to\infty}&\frac{(\psi_A(\mu)/\psi_B(\mu)+1)^{12}\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)}{(\psi_A(\mu)+\psi_B(\mu))^{12}(\mu^3-1)} = \lim_{\mathrm{Im}(\tau)\to\infty}\frac{\vartheta^{24}_{\frac{1}{6},\frac{1}{2}}(\tau)}{\psi_B(\mu)^{12}(\mu^3-1)},\end{aligned}$$

where $\tau = \psi_A(\mu)/\psi_B(\mu)$ and $N_1 = \begin{pmatrix}-2 & -3\\ 1 & 1\end{pmatrix} \in SL_2(\mathbb{Z})$. Thus the behavior of $f(\mu)/(\mu^3-1)$ as $\mu\to\omega$ reduces to that as $\mu\to 1$, and we have $\lim_{\mu\to\omega} f(\mu)/(\mu^3-1) \neq 0$. Similarly, we have $\lim_{\mu\to\omega^2} f(\mu)/(\mu^3-1)\neq 0$.

Since the meromorphic function $f(\mu)/(\mu^3-1)$ is extended to a non-vanishing holomorphic function on $\mathbb{P}^1 - \{\infty\}$, it should be a constant. This constant has been evaluated in (13). $\square$

The following is a direct corollary to Theorem 2. These statements can be obtained also by the Chowla-Selberg formula.

Corollary 2. *The values of the above theta constants at ω are given by*

$$\begin{aligned}
\vartheta_{\frac{1}{6},\frac{1}{2}}(\omega) &= \frac{3^{5/24}}{2\pi}\exp(\frac{5\pi\sqrt{-1}}{24})\Gamma(\frac{1}{3})^{3/2},\\
\vartheta_{\frac{1}{2},\frac{1}{6}}(\omega) &= \frac{3^{5/24}}{2\pi}\exp(-\frac{\pi\sqrt{-1}}{8})\Gamma(\frac{1}{3})^{3/2},\\
\vartheta_{\frac{1}{6},\frac{1}{6}}(\omega) &= \frac{3^{5/24}}{2\pi}\exp(-\frac{\pi\sqrt{-1}}{72})\Gamma(\frac{1}{3})^{3/2},\\
\vartheta_{\frac{-1}{6},\frac{1}{6}}(\omega) &= \frac{3^{3/8}}{2\pi}\exp(-\frac{5\pi\sqrt{-1}}{72})\Gamma(\frac{1}{3})^{3/2}.
\end{aligned}$$

Proof. Substitute $\mu = 0$ into Theorem 1, then $\tau = \psi_A/\psi_B = \omega - 1$. Use Theorem 2 and Fact 1 for T, and select a suitable 144-th root of unity. □

Proposition 5.

$$\vartheta^3_{\frac{1}{6},\frac{1}{2}}(\tau) = \frac{(1-\omega)i}{2\pi}\cdot\psi_B\cdot\vartheta_{\frac{1}{6},\frac{1}{2}}(3\tau).$$

Proof. Use Proposition 3, Theorem 2 and Lemma 6. □

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Department of Mathematics
Hokkaido University
Sapporo, 060-0810, Japan

E-mail address: matsu@math.sci.hokudai.ac.jp

Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguro, Tokyo, 153-8914, Japan

E-mail address: terasoma@ms.u-tokyo.ac.jp

Kitami Computer Business Inc
Nishi-3 Kita-3, Kitami, 090-0043, Japan