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COUNTING CONICS IN COMPLETE INTERSECTIONS

LAURENT BONAVERO AND ANDREAS HÖRING

Abstract. We count the number of conics through two general points in complete intersections when this number is finite and give an application in terms of quasi-lines.

1. Introduction

Let X be a complex projective manifold of dimension n. A quasi-line l in X is a *smooth* rational curve $f\ :\mathbb{P}^1 \hookrightarrow X$ such that f^*T_X is the same as for a line in $\mathbb{P}^n$, *i.e.* is isomorphic to

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}.$$

Let X be a smooth projective variety containing a quasi-line l. Following Ionescu and Voica [IV03], we denote by $e(X,l)$ the number of quasi-lines which are deformations of l and pass through two given general points of X. We denote by $e_0(X,l)$ the number of quasi-lines which are deformations of l and pass through a general point x of X with a given general tangent direction at x. Note that one always has $e_0(X,l) \leq e(X,l)$, but in general the inequality may be strict [IN03, p.1066].

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+r}$ be a* general *smooth n-dimensional complete intersection of multi-degree $(d_1,\dots,d_r)$. Assume moreover that*

$$d_1+\cdots+d_r = \frac{n+1}{2}+r.$$

Then

(1) *the family of conics contained in X is a nonempty, smooth and irreducible component of the Chow scheme $\mathcal{C}(X)$,*
(2) *a general conic C contained in X is a quasi-line of X and*

$$e_0(X,C) = e(X,C) = \frac{1}{2}\prod_{i=1}^{r}(d_i-1)!d_i!.$$

The numerical assumption $d_1+\cdots+d_r = (n+1)/2+r$ assures that if C is a conic in X, then $-K_X\cdot C = n+1$. This numerical condition is of course necessary for a curve to be a quasi-line. Note that varieties appearing in our theorem are

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Fano varieties of dimension n and index $(n+1)/2$; they are well known to be the boundary Fano varieties with Picard number one being conic-connected (see [IR07], Theorem 2.2).

Using a degeneration argument, one can strengthen parts of the statement.

Corollary 1.2. *Let $X \subset \mathbb{P}^{n+r}$ be a smooth n-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$. If $d_1 + \cdots + d_r = (n+1)/2 + r$, the variety X contains a conic that is a quasi-line.*

By a theorem of Ionescu [Ion05], we obtain an immediate application of the theorem to formal geometry. Before stating it, let us recall that a subvariety Y of a variety X is G3 in X if the ring $K(X_{|Y})$ of formal-rational functions of X along Y is equal to $K(X)$.

Corollary 1.3. *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth n-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$ such that $d_1 + \cdots + d_r = (n+1)/2 + r$. Then any general conic C contained in X is* G3 *in X. In particular, if (X, C) and (X', C') are two such pairs such that the formal completions $X_{|C}$ and $X'_{|C'}$ are isomorphic as formal schemes, there exists an isomorphism from X to X' sending C to C'.*

When this note was almost finished, we learned from Laurent Manivel that Arnaud Beauville had obtained the formula $e(X, l) = \frac{1}{2} \prod_{i=1}^{r} (d_i - 1)! d_i!$ as a consequence of his computation of the quantum cohomology algebra $H^*(X, \mathbb{Q})$ of a complete intersection [Bea95]. We provide here a completely elementary proof. We end this note by mentioning a similar question where no elementary proof seems to be known.

2. Proofs

We start by explaining the enumerative argument in the simplest case.

2.1. **A well known example.** Suppose that $X = \{s = 0\}$ is a smooth cubic threefold in $\mathbb{P}^4$. A general conic C in X is a quasi-line [BBI00, Thm.3.2]. The basic idea of our proof is that counting conics in X through p and q can be reduced to counting 2-planes π through p and q such that the restriction $s|_\pi$ is a product of a polynomial of degree two and some residual polynomial. We will explain how to do this in general below, in the case of the cubic threefold we can use a geometric construction.

It is a classical fact that the lines in X form an irreducible smooth family of dimension two and that there are exactly six lines passing through a general point of X [AK77, Prop.1.7]. Fix now two general points p and q in X, then the line $[pq]$ intersects X in a third point u. For every line $l \subset X$ through u there exists a unique plane π_l containing l and $[pq]$. The intersection $X \cap \pi_l$ is the union of l and a residual conic C. Since l does not pass through p and q, the conic C passes through p and q. *Vice versa* the linear span of a conic $C \subset X$ passing through p and q is a 2-plane π_C containing the line $[pq]$. Since C does not pass through

u, the residual line passes through u. Thus the conics through p and q are in bijection with the lines through u, so $e(X, C) = 6$.

Suppose now that we are in the general situation of Theorem 1.1. We always assume that $X \subset \mathbb{P}^{n+r}$ is a general smooth n-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$ with $d_i \geq 2$ for all i and

$$d_1 + \ldots + d_r = \frac{n+1}{2} + r.$$

Let $l \subset X$ be a smooth rational curve contained in X. Then

$$-K_X \cdot l = (n + r + 1 - (d_1 + \ldots + d_r)) \deg(l) = \frac{n+1}{2} \deg(l)$$

therefore $-K_X \cdot l = n + 1$ if and only if l is a conic.

2.2. **The main step.** *For any general points p and q of X, there exists a conic contained in X passing through p and q.*

Fix two distinct points in $\mathbb{P}^{n+r}$, say $p = [1 : 0 : \ldots : 0]$ and $q = [0 : 0 : \ldots : 1]$. Suppose that X is a general complete intersection with equations

$$(s_1 = 0) \cap (s_2 = 0) \cap \ldots \cap (s_r = 0)$$

passing through p and q, where each $s_i \in H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d_i))$ is general among sections vanishing at p and q.

Suppose there is a conic C contained in X, passing through p and q and let π_C be the projective 2-plane generated by C. If s_C denotes the equation defining C in π_C, there exists for each $i = 1, \ldots, r$ a $\tilde{s}_i \in H^0(\pi_C, \mathcal{O}_{\pi_C}(d_i - 2))$ (defining the residual curve) such that

$$(s_i)_{|\pi_C} = s_C \cdot \tilde{s}_i.$$

Since X is general, it does not contain the 2-plane π_C [DM98, Thm. 2.1]. Therefore $(s_i)_{|\pi_C}$ and $\tilde{s}_i$ are not zero for at least one i.

Conversely, let π be a projective 2-plane containing p and q and assume there exists a non-zero $s_C \in H^0(\pi, \mathcal{O}_\pi(2))$ vanishing at p and q and, for each $i = 1, \ldots, r$, there exists a $\tilde{s}_i \in H^0(\pi, \mathcal{O}_\pi(d_i - 2))$ such that

$$(s_i)_{|\pi} = s_C \cdot \tilde{s}_i,$$

then the conic $(s_C = 0)$ is obviously contained in X.

Consider now the projective space of dimension $n + r - 2$ parametrizing the projective 2-planes in $\mathbb{P}^{n+r}$ containing p and q. Fixing homogeneous coordinates $[a_1 : \ldots : a_{n+r-1}]$ on this space, let

$$\pi_{[a_1:\ldots:a_{n+r-1}]} = \{[x : za_1 : \ldots : za_{n+r-1} : y] \mid [x : z : y] \in \mathbb{P}^2\}$$

be such a 2-plane. Then

$$(s_i)_{|\pi_{[a_1:\ldots:a_{n+r-1}]}}(x, z, y) = \sum_{k=0}^{d_i} \sum_{a=0}^{k} s^i_{a,k} x^a y^{k-a} z^{d_i - k}$$

where $s^i_{a,k}$ is a homogeneous polynomial of degree $d_i - k$ in the variables $a_1, \ldots, a_{n+r-1}$. The equation of an irreducible conic in this plane that passes through p and q is

$$s_C = s_2 z^2 + s_1 xz + s_1' yz + xy.$$

So for each $i = 1, \ldots, r$, the equation $(s_i)_{|\pi} = s_C \cdot \tilde{s}_i$ can be written explicitly

$$\sum_{k=0}^{d_i} \sum_{a=0}^{k} s^i_{a,k} x^a y^{k-a} z^{d_i-k} = (s_2 z^2 + s_1 xz + s_1' yz + xy) \times \sum_{k=0}^{d_i-2} \sum_{a=0}^{k} \tilde{s}^i_{a,k} x^a y^{k-a} z^{d_i-2-k}.$$

Thus we have to solve the equations

$$s^i_{a,k} = s_2 \tilde{s}^i_{a,k} + s_1 \tilde{s}^i_{a-1,k-1} + s_1' \tilde{s}^i_{a,k-1} + \tilde{s}^i_{a-1,k-2}$$

for any $0 \leq k \leq d_i$ and $0 \leq a \leq k$.

Let us first solve this system (whose unknown variables are s_2, s_1, s_1' defining the conic and the $\tilde{s}^i_{a,k}$'s defining the residual curve) for each i separately. Note that X passes through p and q if and only if $s^i_{0,d_i} = s^i_{d_i,d_i} = 0$. Therefore writing the $d_i - 1$ equations $s^i_{a,d_i} = \tilde{s}^i_{a-1,d_i-2}$ for $1 \leq a \leq d_i - 1$ provides the $\tilde{s}^i_{a-1,d_i-2}$'s.

Considering the equations corresponding to $(a,k) = (0, d_i - 1)$ and $(d_i - 1, d_i - 1)$ allows to find s_2 and s_1. Considering then the equations corresponding to $(a,k) = (1, d_i - 1)$ and $(0, d_i - 2)$ gives $\tilde{s}^i_{0,d_i-3}$ and s_1' (in particular this determines the conic, if it exists !). Write down successively the equations for (a,k), $a = 1, \ldots, k-1$, $k = d_i - 1, \ldots, 2$ to find all the $\tilde{s}^i_{a,k}$'s (this determines the residual curve $(\tilde{s}^i = 0)$!).

Therefore, the r systems have a common solution if and only if the remaining equations for each system are satisfied and the corresponding conic is the same for each i. For each i, the remaining equations are "universal formulas" (meaning the coefficients just depend on the equations defining X) corresponding to $(a,k) = (0, d_i - 3), \ldots, (0,0)$ and $(a,k) = (d_i - 2, d_i - 2), \ldots, (1,1)$. This gives $2d_i - 4$ equations of respective degrees $3, \ldots, d_i$ and $2, 3, \ldots, d_i - 1$ in the variables $a_1, \ldots, a_{n+r-1}$. The $3r - 3$ equations saying that the conic is the same for each $i = 1, \ldots, r$ are $2r - 2$ equations of degree 1 and $r - 1$ equations of degree 2 in the variables $a_1, \ldots, a_{n+r-1}$.

Altogether, using the relation $d_1 + \cdots + d_r = (n+1)/2 + r$, this gives exactly $n + r - 2$ homogeneous equations in the variables $a_1, \cdots, a_{n+r-1}$. We therefore get at least one solution. Moreover since X is general, the coefficients $s^i_{a,k}$ appearing in the initial equations are general. Since they completely determine the remaining $n+r-2$ homogeneous equations, these equations are general. Thus the space of solutions is smooth and of the expected dimension, so there are exactly $\frac{1}{2} \prod_{i=1}^{r} (d_i - 1)! d_i!$ solutions by Bezout's theorem.

Let us briefly indicate how the same method gives *the number of conics contained in X, passing through p and tangent to the line (pq)*. With the above notations, we have $s^i_{d_i-1,d_i} = s^i_{d_i,d_i} = 0$ and we have to solve the r systems

$$\sum_{k=0}^{d_i}\sum_{a=0}^{k} s^i_{a,k} x^a y^{k-a} z^{d_i-k} = (s_2z^2 + s_1xz + s'_1yz + y^2) \times \sum_{k=0}^{d_i-2}\sum_{a=0}^{k} \tilde{s}^i_{a,k} x^a y^{k-a} z^{d_i-2-k}$$

which means

$$s^i_{a,k} = s_2\tilde{s}^i_{a,k} + s_1\tilde{s}^i_{a-1,k-1} + s'_1\tilde{s}^i_{a,k-1} + \tilde{s}^i_{a,k-2}$$

for any $0 \leq k \leq d_i$ and $0 \leq a \leq k$. The remaining details are left to the reader.

2.3. **The space of conics is irreducible.** Let $\mathbb{G}(2, n+r)$ be the Grassmannian of projective 2-planes contained in $\mathbb{P}^{n+r}$ and E be the tautological rank 3-bundle on $\mathbb{G}(2, n+r)$. The Hilbert scheme of conics in $\mathbb{P}^{n+r}$ is the projectivisation[1] of S^2E^*. Denote by $\varphi : \mathbb{P}(S^2E^*) \to \mathbb{G}(2, n+r)$ the natural map. We have an exact sequence on $\mathbb{P}(S^2E^*)$:

$$(*) \qquad 0 \to \bigoplus_{i=1}^{r} \varphi^* S^{d_i-2}E^* \otimes \mathcal{O}_{\mathbb{P}(S^2E^*)}(-1) \to \bigoplus_{i=1}^{r} \varphi^* S^{d_i}E^* \to \mathcal{Q} \to 0$$

defining a vector bundle $\mathcal{Q}$ of rank $n+1+3r$. Since X is a complete intersection $(s_1 = 0) \cap (s_2 = 0) \cap \cdots \cap (s_r = 0)$, the s_i's induce by restriction to 2-planes, pull-back and projection onto $\mathcal{Q}$ a section of $\mathcal{Q}$ whose zero locus Z is precisely the set of conics contained in X. Since E^* is globally generated, the images of sections $(s_1, \ldots, s_r)$ give a vector space $V \subseteq H^0(\mathbb{P}(S^2E^*), \mathcal{Q})$ that globally generates $\mathcal{Q}$. Applying Bertini's theorem to this subspace we see that the zero locus of a general section in V is smooth. Since X is supposed to be a *general* complete intersection, Z is smooth and proving its irreducibility reduces to showing that $h^0(Z, \mathcal{O}_Z) = 1$. By the Koszul resolution of $\mathcal{O}_Z$, it is enough to show that for any $1 \leq j \leq \operatorname{rk}\mathcal{Q}$

$$h^j(\mathbb{P}(S^2E^*), \wedge^j\mathcal{Q}^*) = 0.$$

Using the exact sequence $(*)$, this easily reduces to showing that for any $1 \leq j \leq \operatorname{rk}\mathcal{Q}$ and any $0 \leq k \leq j$,

$$H^k(\mathbb{P}(S^2E^*), \wedge^k(\oplus_{i=1}^r \varphi^* S^{d_i}E) \otimes S^{j-k}((\oplus_{i=1}^r \varphi^* S^{d_i-2}E) \otimes \mathcal{O}_{\mathbb{P}(S^2E^*)}(1))) = 0.$$

Since the higher direct images with respect to φ vanish, it is sufficient to show that for any $1 \leq j \leq \operatorname{rk}\mathcal{Q}$ and for any $0 \leq k \leq j$, we have

$$H^k(\mathbb{G}(2, n+r), \wedge^k(\oplus_{i=1}^r S^{d_i}E) \otimes S^{j-k}((\oplus_{i=1}^r S^{d_i-2}E) \otimes S^2E))) = 0.$$

This will follow from Bott's theorem applied on $\mathbb{G}(2, n+r)$. Indeed, using Schur functor notation, let S_bE be an irreducible factor appearing in the decomposition of $\wedge^k(\oplus_{i=1}^r S^{d_i}E) \otimes S^{j-k}((\oplus_{i=1}^r S^{d_i-2}E) \otimes S^2E)$ where $b = (b_1, b_2, b_3)$ is a triple of integers $b_1 \geq b_2 \geq b_3 \geq 0$. By the Littlewood-Richardson rule, we get

$$(**) \qquad b_2 + b_3 \geq k - r \text{ and } b_3 \geq k - (d_1 + \ldots + d_r) - r = k - (n+1)/2 - 2r.$$

On the other hand by Bott's theorem, the whole cohomology of S_bE vanishes except maybe in the following cases :

[1] In this article we follow the convention that the projectivisation of a vector bundle E is the variety of lines of E.

(1) $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 0, 0)$ with $b_1 \geq n + r - 1$,
(2) $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 1, 0)$ with $b_1 \geq n + r - 1$,
(3) $k = n + r - 2$ and $(b_1, b_2, b_3) = (b_1, 1, 1)$ with $b_1 \geq n + r - 1$,
(4) $k = 2(n + r - 2)$ with $b_2 \geq n + r$ and $b_3 = 0, 1, 2$,
(5) $k = 3(n + r - 2)$ with $b_3 \geq n + r + 1$.

The case $n = 3$ has been dealt with by Bădescu, Beltrametti and Ionescu [BBI00], so we may assume $n \geq 5$ since n is odd.

In the first three cases, we get $k - r = n - 2 \geq 3 > b_2 + b_3 = 0, 1, 2$, which is excluded by $(**)$. In case (4), since $n \geq 5$, we get $k - (n+1)/2 - 2r = 3(n-3)/2 > b_3 = 0, 1, 2$, which is again excluded by $(**)$. Case (5) is also excluded since we are only interested in the situation where $k \leq \operatorname{rk} \mathcal{Q} = n+1+3r$, but $3(n + r - 2) > n + 1 + 3r$ when $n \geq 5$.

We obtain the following corollary of the proof.

Corollary 2.1. *Let $X \subset \mathbb{P}^{n+r}$ be a general smooth n-dimensional complete intersection of multi-degree $(d_1, \ldots, d_r)$. Assume moreover that*

$$d_1 + \cdots + d_r \leq \frac{n+1}{2} + r$$

and $n \geq 5$. Then the family of conics contained in X is a nonempty, smooth and irreducible component of the Chow scheme $\mathcal{C}(X)$.

Let us also mention that Harris, Roth and Starr have shown the irreducibility of the space of smooth rational curves of arbitrary degree e for general hypersurfaces of low degree d [HRS04].

2.4. **Conics are quasi-lines.** By the first step, there exists a conic C passing through two general points. Such a conic is necessarily smooth: a line d contained in X and passing through a general point satisfies

$$T_X|_d \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \frac{n-3}{2}} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \frac{n+1}{2}},$$

so an easy dimension count shows that two general points are not connected by a chain of two lines. Thus C is smooth and its deformations with a fixed point cover a dense open subset in X. This implies that the normal bundle $N_{C/X}$ is ample [Deb01, Prop.4.10] and since $-K_X \cdot C = n + 1$, the curve C is a quasi-line.

2.5. **Proof of the Corollary 1.3.** The irreducibility of the variety of conics gives

$$e_0(X, l) = e(X, l) = \frac{1}{2} \prod_{i=1}^{r} (d_i - 1)! d_i!.$$

The equality $e_0(X, l) = e(X, l)$ implies that general conics are G3 in X [Ion05, Cor. 4.6], in particular [Ion05, Cor. 4.7, Cor.1.9] apply.

3. A similar question

Using exactly the same method as developed in §2.3 , one can prove the following result, left to the reader.

Proposition 3.1. *Let $X_d \subset \mathbb{P}^{n+1}$ be a general smooth n-dimensional hypersurface of degree d. Then, for $n \geq 7$ and $d \leq n+1$, the family of conics contained in X_d is a nonempty, smooth and irreducible component of dimension $3n-2d+1$ of the Chow scheme $\mathcal{C}(X_d)$.*

In the case of $d = n+1$, there is a finite number of conics passing through a general point of X_{n+1}. Let us denote by N_{n+1} this number. It seems that there are no known elementary method to compute this number. A general formula comes from the calculation of some Gromov-Witten invariants using mirror symmetry and an ordinary differential equation introduced by Givental. The following lines were written while reading [JNS04] and [Jin05].

Proposition 3.2. (Coates, Givental - Jinzenji, Nakamura, Suzuki) *Let $X_n \subset \mathbb{P}^n$ be a general smooth hypersurface of degree n in $\mathbb{P}^n$. Let N_n be the number of conics passing through a general point of X_n. Then*

$$N_n = \frac{(2n)!}{2^{n+1}} - \frac{(n!)^2}{2}.$$

Let us briefly explain where this result comes from. If a, b, c and d are four integers, let $\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle_d$ be the Gromov-Witten invariant counting the number (possibly infinite) of rational curves of degree d contained in X_n and meeting 3 general subspaces of $\mathbb{P}^n$, of respective codimension a, b and c. When a, b or c are equal to 1, each such rational curve has to be counted d times since the intersection of a degree d curve intersects a general hyperplane in d points. Since a general line meets X_n in n points, we get that $N_n = \langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_{n-1} \rangle_2 / 4n$. In [Jin05] are introduced some constants $\tilde{L}_m^{n+1,n,d}$, called "structure constants of the quantum cohomology ring of X_n". They satisfy the following formula:

$$\sum_{m=0}^{n-1} \tilde{L}_m^{n+1,n,1} w^m = n \prod_{j=1}^{n-1} (jw + (n-j))$$

and

$$\sum_{m=0}^{n-2} \tilde{L}_m^{n+1,n,2} w^m = \sum_{j_2=0}^{n-2} \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \tilde{L}_{j_1}^{n+1,n,1} \tilde{L}_{j_2+1}^{n+1,n,1} w^{j_1-j_0} \left(\frac{1+w}{2} \right)^{j_2-j_1}.$$

It is also shown in [Jin05] that for every integer m, $0 \leq m \leq n-2$, we have $\tilde{L}_m^{n+1,n,2} = \langle \mathcal{O}_1 \mathcal{O}_{n-1-m} \mathcal{O}_{m+1} \rangle_2 / n$.

Then the proposition follows by evaluating the w^{n-2} coefficient in the second formula above, the w^{n-1} coefficient in the first and putting $w = 2$.

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Institut Fourier, UMR 5582, Université de Grenoble 1, BP 74
38402 Saint Martin d'Hères, France
E-mail address: laurent.bonavero@ujf-grenoble.fr

Université Paris 6, Institut de Mathématiques de Jussieu
Equipe de Topologie et Géométrie Algébrique
175, rue du Chevaleret, 75013 Paris, France
E-mail address: hoering@math.jussieu.fr