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HODGE CLASSES ASSOCIATED TO 1-PARAMETER FAMILIES OF CALABI-YAU 3-FOLDS

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Abstract. We use L^2-Higgs cohomology to determine the Hodge numbers of the parabolic cohomology $H^1(\bar{S}, j_*\mathbb{V})$, where the local system $\mathbb{V}$ arises from the third primitive cohomology of family of Calabi-Yau threefolds over a curve $\bar{S}$. The method gives a way to predict the presence of algebraic 2-cycles in the total space of the family and is applied to some examples.

Introduction

We consider a smooth projective family of varieties $f : X \longrightarrow S$ over a smooth curve S. After a finite pull-back we may assume that the family has a semi-stable completion $\bar{f} : \bar{X} \longrightarrow \bar{S}$, giving rise to a diagram

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ f\downarrow & & \downarrow \bar{f} \\ S & \overset{j}{\hookrightarrow} & \bar{S} \end{array}$$

In such a situation the action of the local monodromy around each point of the boundary $D := \bar{S}\backslash S$ on the local systems of k-cohomologies $R^k f_*\mathbb{C}_X$ is unipotent. These local systems carry a polarised variation of Hodge structures (VHS), where the Hodge filtration arises from the truncation of the relative logarithmic deRham complex

$$\Omega^\bullet_{\bar{X}/\bar{S}}(\log \Delta)$$

where $\Delta := f^{-1}(D)$ is a reduced normal crossing divisor.

It is a very old idea that goes back to Poincaré to study cycles on $\bar{X}$ by looking at their slicing by the fibres of the map. If $D \neq \emptyset$, then the Leray spectral sequence shows that the space of cohomology classes of X that restrict to zero on fibres is the cohomology on S of the local system of k-cohomologies

$$\mathrm{Ker}(H^{k+1}(X) \longrightarrow H^0(S, R^{k+1}f_*\mathbb{C}_X)) = H^1(S, R^k f_*\mathbb{C}_X).$$

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It carries a canonical mixed Hodge structure and the subgroup

$$H^1(\bar{S}, j_* R^k f_* \mathbb{C}_X) = H^1(\bar{S}, R^k \bar{f}_* \mathbb{C}_{\bar{X}}) \subset H^1(S, R^k f_* \mathbb{C}_X)$$

can be identified with the part of lowest weight $k+1$. More generally, one can consider (irreducible) local systems $\mathbb{V}$ that underly a weight k VHS. The *intersection homology group* (also called the *parabolic cohomology group*)

$$H^1(\overline{S}, j_* \mathbb{V})$$

then carries a natural Hodge structure of weight $k+1$, a result due to S. Zucker, [Z]. In that paper, Zucker defined a local L^2-integrability condition for $\mathcal{C}^\infty$-sections of de Rham complex twisted by the local system $\mathbb{V}$, using the Hodge metric and the Poincaré metric near the boundary points. Using the fundamental estimate of W. Schmid on the growth of the Hodge metric, he showed that a local section of it is L^2-integrable if and only if it lies in the monodromy weight filtration in the certain given range. Using an L^2-Poincaré lemma, he showed that the subcomplex of L^2-integrable forms can be used to compute the intersection cohomology. In fact, using local information, one can define a certain complex $\Omega^\bullet_{(2)}(\mathbb{V})$, whose hypercohomology computes the intersection homology:

$$H^1_{(2)}(S, \mathbb{V}) := \mathbb{H}^1(\Omega^\bullet_{(2)}(\mathbb{V})) = H^1(\overline{S}, j_* \mathbb{V}).$$

The complex carries a natural Hodge filtration and the associated spectral sequence

$$\mathbb{H}^q(Gr^p_F \Omega^\bullet_{(2)}(\mathbb{V})) \Rightarrow H^{p+q}_{(2)}(S, \mathbb{V})$$

degenerates at E_1 and induces the pure Hodge structure on the limit.

In this paper we will focus attention to local systems that arise from non-trivial pencils of Calabi-Yau 3-folds. So we have $\bar{S} = \mathbb{P}^1$ and $f : X \to S = \mathbb{P}^1 \setminus D$ a smooth family of Calabi-Yau 3-folds whose Kodaira-Spencer map is not identically zero, hence

$$\dim_{\mathbb{C}} H^1(X_s, \Omega^2_{X_s}) = \dim_{\mathbb{C}} H^1(X_s, \Theta_{X_s}) = 1.$$

As a consequence, the local system $\mathbb{V} := R^3 f_* \mathbb{C}_X$ of (primitive) middle cohomology is irreducible and has Hodge numbers $(1,1,1,1)$. The cohomology group

$$H := H^1(\bar{S}, j_* \mathbb{V}) = H^1_{(2)}(S, \mathbb{V})$$

carries a pure Hodge structure of weight 4:

$$H = H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4}.$$

In this paper, we will identify the Hodge numbers of H in terms of geometric data of the family. We hope it will become clear to the reader how these ideas can be used to obtain formulae in far more general situations. Of particular interest is the Hodge space $H^{2,2}$, as any cycle $Z \in CH^2(\bar{X})$ (homologous to zero on all fibres of f) will give a class in it, as will be recalled in Section 4. On the other hand, a class in $H^1(\bar{S}, j_* \mathbb{V})$ also defines an extension of local systems, which is

the same as the one obtained from an inhomogeneous Picard-Fuchs differential equations of type

$$\mathcal{L}\Phi_Z = g$$

associated to the normal function Φ_Z of the cycle Z. Finally, we describe some examples that were recently considered in [W1] and [W2].

1. The L^2-Higgs complex

Recall that a *logarithmic Higgs bundle* (E,θ) on $\bar{S}$ consists of a holomorphic vector bundle E, together with an $\mathcal{O}_{\bar{S}}$-linear map

$$E \xrightarrow{\theta} E \otimes \Omega^1_{\bar{S}}(\log D).$$

In the geometrical case, E is the sum of the Hodge bundles:

$$E = \oplus_{p+q=k} E^{p,q}, \quad E^{p,q} := R^q \bar{f}_*(\Omega^p_{\bar{X}/\bar{S}}(\log \Delta))$$

and the Higgs field is obtained from cup-product with the Kodaira-Spencer class. By Griffiths transversality, it has components

$$E^{i,j} \xrightarrow{\theta} E^{i-1,j+1} \otimes \Omega^1_{\bar{S}}(\log D).$$

In the paper of Jost, Yang and the last named author [JYZ], the local L^2-condition of Zucker was adapted to the more general setting of logarithmic Higgs bundles.

If (E,θ) is a logarithmic Higgs bundle on $\bar{S}$ and $p \in D := \bar{S} \setminus S$ is a boundary point, then the *residue* of the Higgs-field θ at p

$$N = N_p := Res_p(\theta) \in \mathrm{End}(E_p)$$

is a nilpotent endomorphism and so determines a weight filtration $W_\bullet$ on the fibre E_p.[1]

(Recall that any nilpotent endomorphism N of a vector space V determines in a canonical way an increasing filtration on V, called the *monodromy weight filtration*:

$$0 \subset W_{-m} \subset W_{-m+1} \subset \cdots \subset W_0 \subset W_1 \subset \cdots \subset W_m = V.$$

It is defined as follows: if $N^{m+1} = 0$ but $N^m \neq 0$ we put

$$W_{m-1} = \mathrm{Ker}(N^m), \quad W_{-m} = \mathrm{Im}(N^m).$$

Then inductively W_k is constructed in such a way that $N(W_k) = \mathrm{Im}(N) \cap W_{k-2} \subset W_{k-2}$ and

$$N^k : \mathrm{Gr}^W_k(V) \to \mathrm{Gr}^W_{-k}(V)$$

are isomorphisms.)

[1] In terms of the local system $\mathbb{V}$, the endomorphism N_p can be identified with $\log T_p$, the logarithm of the local monodromy operator around the puncture p.

These weight filtrations on $E_p,\ p \in D$ can be used to define a certain complex $(\Omega^\bullet_{(2)}(E), \theta)$, called the *$L^2$-Higgs complex.* It is determined by sub-sheaves

$$\Omega^0_{(2)}(E) \subset E, \quad \Omega^1_{(2)}(E) \subset E \otimes \Omega^1_{\bar{S}}(\log D)$$

which are defined near $p \in D$ by

$$\Omega^0_{(2)}(E) := W_0 + tE, \quad \Omega^1_{(2)}(E) := (W_{-2} + tE) \otimes \Omega^1_{\bar{S}}(\log D).$$

Here t is a local parameter near a boundary point $p = \{t = 0\} \in D$ and the weight filtration is defined by N_p.

It can be shown [JYZ] that the hypercohomology of the L^2- Higgs complex $(\Omega^\bullet_{(2)}(E), \theta)$ is isomorphic to the L^2-cohomology

$$H^k_{(2)}(S, E) = \mathbb{H}^k(\Omega^\bullet_{(2)}(E), \theta).$$

We now spell out the local structure of $(\Omega^\bullet_{(2)}, \theta)$ for the case of logarithmic Higgs bundles of type $(1, 1, 1, 1)$, so that

$$E = E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}$$

and each summand is a line bundle. At an ordinary point $p \in S$ the L^2-Higgs complex of course coincides with the Higgs-bundle itself, so that we have

$$\Omega^0_{(2)} = E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}, \quad \Omega^1_{(2)} = \left(E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}\right) \otimes \Omega^1_{\bar{S}}.$$

For the points $p \in D$ one has to distinguish cases corresponding to the possible Jordan-forms of the endomorphism $N = N_p$.

There are two subcases that appear when $m = 1$, that is $N^2 = 0$, but $N \neq 0$.

Point of type I. Here we have a single Jordan-block of the form

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The monodromy weight filtration has the following form:

$$0 = W_{-1} \subset W_0 = \mathrm{Ker}(N) \subset W_1 = V.$$

In this case in the sequence $E^{3,0} \xrightarrow{N} E^{2,1} \xrightarrow{N} E^{1,2} \xrightarrow{N} E^{0,3}$ at the point $p = \{t = 0\}$ the middle map must be an isomorphism, since $\dim \mathrm{Ker}(N) = 3$. Hence the left and right maps must both be zero. Therefore the L^2-complex near p looks like:

$$\begin{array}{ccccccccccc} \Omega^0_{(2)} & = & 0 & \oplus & E^{3,0} & \oplus & tE^{2,1} & \oplus & E^{1,2} & \oplus & E^{0,3} \\ \downarrow \theta & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^1_{(2)} & = & (\ tE^{3,0} & \oplus & tE^{2,1} & \oplus & tE^{1,2} & \oplus & tE^{0,3} & \oplus & 0\ ) \otimes \Omega^1_{\bar{S}}(\log D) \end{array}$$

Point of type II. Here we have two Jordan-blocks

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The monodromy weight filtration now has the following form

$$0 = W_{-1} \subset W_0 = \mathrm{Ker}(N) \subset W_1 = V.$$

In this case in the sequence $E^{3,0} \xrightarrow{N} E^{2,1} \xrightarrow{N} E^{1,2} \xrightarrow{N} E^{0,3}$ at the point $p = \{t = 0\}$, the middle map must be zero, since $\dim \mathrm{Ker}(N) = 2$ and the left and right maps are dual to each other. Therefore the L^2-subcomplex has the local form:

$$\begin{array}{ccccccccccc} \Omega^0_{(2)} & = & 0 & \oplus & tE^{3,0} & \oplus & E^{2,1} & \oplus & tE^{1,2} & \oplus & E^{0,3} \\ \downarrow \theta & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^1_{(2)} & = & (\, tE^{3,0} & \oplus & tE^{2,1} & \oplus & tE^{1,2} & \oplus & tE^{0,3} & \oplus & 0\,) \otimes \Omega^1_{\bar{S}}(\log D). \end{array}$$

Case m=2. Assume that $N^2 \neq 0$ and $N^3 = 0$, so we may assume

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

At the point p the first and the last maps in the sequence

$$E^{3,0} \xrightarrow{N} E^{2,1} \xrightarrow{N} E^{1,2} \xrightarrow{N} E^{0,3}$$

are dual to each other. If they are both isomorphisms, then $N^3 = 0$ implies that the middle map $E^{2,1} \xrightarrow{N} E^{1,2}$ is the zero map. Hence we already have $N^2 = 0$. Therefore this case cannot happen.

Point of type III. Assume now that $m = 3$, so $N^3 \neq 0$, so this case corresponds to points of *maximal unipotent monodromy*,

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The monodromy weight filtration has the following form:

$0 \subset W_{-3} = W_{-2} = \mathrm{Ker}(N) \subset W_{-1} = W_0 = \mathrm{Ker}(N^2) \subset W_1 = W_2 = \mathrm{Ker}(N^3) \subset V.$

Since $N^3 \neq 0$ and the $E^{p,q}$ have rank one, the composition

$$E^{3,0} \xrightarrow{N} E^{2,1} \xrightarrow{N} E^{1,2} \xrightarrow{N} E^{0,3}$$

at the point $p = \{t = 0\}$ is an isomorphism. The L^2-subcomplex takes the following form

$$\begin{array}{ccccccccccc}
\Omega^0_{(2)} & = & 0 & \oplus & tE^{3,0} & \oplus & tE^{2,1} & \oplus & E^{1,2} & \oplus & E^{0,3} \\
\downarrow \theta & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^1_{(2)} & = & (\, tE^{3,0} & \oplus & tE^{2,1} & \oplus & tE^{1,2} & \oplus & E^{0,3} & \oplus & 0\ ) \otimes \Omega^1_{\bar{S}}(\log D).
\end{array}$$

2. Hodge Filtration

One of the main points of the constructions of the L^2-Higgs complex is that one obtains a more or less *explicit* description of the Hodge-decomposition on $H^1(\overline{S}, j_*\mathbb{V})$. The obvious Hodge filtration

$$F^k \subset F^{k-1} \subset \ldots \subset F^0, \quad F^p := \bigoplus_{i \geq p} E^{i,k-i}$$

on E defines by intersection with the L^2-subcomplex $\Omega^\bullet_{(2)}(E)$ a filtration and the corresponding hypercohomology spectral sequence degenerates at E_1.

It appears from description of the previous section, that the complex $\Omega^\bullet_{(2)}(E)$ is in fact the direct sum of graded pieces

$$\Omega^\bullet_{(2)}(E) = \oplus_{p+q=k} \Omega^\bullet_{(2)}(E)^{p,q},$$

where

$$\Omega^0_{(2)}(E)^{p,q} := \Omega^0_{(2)}(E) \cap E^{p,q}$$

$$\Omega^1_{(2)}(E)^{p,q} := \Omega^1_{(2)}(E) \cap E^{p-1,q+1} \otimes \Omega^1_{\bar{S}}(\log D)$$

In the case of a logarithmic Higgs bundle of type $(1,1,1,1)$ considered before, the F-graded pieces have the following form:

	0		1
$Gr^4_F \Omega^\bullet_{(2)}(E)$	0	$\longrightarrow$	$\Omega^1_{(2)}(E)^{3,0}$
$Gr^3_F \Omega^\bullet_{(2)}(E)$	$\Omega^0_{(2)}(E)^{3,0}$	$\stackrel{\theta}{\longrightarrow}$	$\Omega^1_{(2)}(E)^{2,1}$
$Gr^2_F \Omega^\bullet_{(2)}(E)$	$\Omega^0_{(2)}(E)^{2,1}$	$\stackrel{\theta}{\longrightarrow}$	$\Omega^1_{(2)}(E)^{1,2}$
$Gr^1_F \Omega^\bullet_{(2)}(E)$	$\Omega^0_{(2)}(E)^{1,2}$	$\stackrel{\theta}{\longrightarrow}$	$\Omega^1_{(2)}(E)^{0,3}$
$Gr^0_F \Omega^\bullet_{(2)}(E)$	$\Omega^0_{(2)}(E)^{0,3}$	$\stackrel{\theta}{\longrightarrow}$	0

From this we obtain the following expression for the Hodge spaces $H^{p,q}$ of $H^1_{(2)}(S, E)$:

Theorem 2.1.

$$\begin{array}{rcl}
H^{4,0} & = & H^0(\bar{S}, E^{30} \otimes \Omega^1_{\bar{S}}) \\
H^{3,1} & = & H^0(\bar{S}, \Omega^1_{(2)}(E)^{2,1} / \theta(\Omega^0_{(2)}(E)^{3,0})) \\
H^{2,2} & = & H^0(\bar{S}, \Omega^1_{(2)}(E)^{1,2} / \theta(\Omega^0_{(2)}(E)^{2,1})) \\
H^{1,3} & = & H^0(\bar{S}, \Omega^1_{(2)}(E)^{3,0} / \theta(\Omega^0_{(2)}(E)^{1,2})) \\
H^{0,4} & = & H^1(\bar{S}, E^{03})
\end{array}$$

Proof. From the local description of the L^2- Higgs complex one sees

$$\Omega^1_{(2)}(E)^{3,0} = E^{3,0} \otimes \Omega^1_{\bar{S}}, \quad \Omega^0_{(2)}(E)^{0,3} = E^{0,3}$$

which gives the $H^{4,0}$ and $H^{0,4}$-spaces. For the other three we remark that the map θ is injective, so the two-term complex is quasi-isomorphic to the corresponding cokernel (at spot one). □

3. Numerical consequences

We can use the above description of the L^2-Higgs complex and its Hodge-graded pieces to obtain explicit formulae for the Hodge numbers of

$$H := H^1_{(2)}(S, E)$$

in terms of elementary geometric data. As before, E is a logarithmic Higgs-bundle on a curve $\bar{S}$ (of genus $g(\bar{S})$) of type $(1,1,1,1)$.

Using the local descriptions of the previous section, we can give a global description of the L^2-Higgs complex.

Proposition 3.1. *Let*

$$\mathcal{O}_{\bar{S}}(-I), \quad \mathcal{O}_{\bar{S}}(-II), \quad \mathcal{O}_{\bar{S}}(-III),$$

denote the ideal sheaves of the points of type I, II and III respectively. Then the graded pieces of the terms of the L^2-Higgs complex $\Omega^\bullet_{(2)}(E)$ are given by

			$\Omega^1_{(2)}(E)^{3,0}$	$=$	$E^{3,0} \otimes \Omega^1_{\bar{S}}$
$\Omega^0_{(2)}(E)^{3,0}$	$=$	$E^{3,0}(-II-III)$	$\Omega^1_{(2)}(E)^{2,1}$	$=$	$E^{2,1} \otimes \Omega^1_{\bar{S}}$
$\Omega^0_{(2)}(E)^{2,1}$	$=$	$E^{2,1}(-I-III)$	$\Omega^1_{(2)}(E)^{1,2}$	$=$	$E^{1,2} \otimes \Omega^1_{\bar{S}}$
$\Omega^0_{(2)}(E)^{1,2}$	$=$	$E^{1,2}(-II)$	$\Omega^1_{(2)}(E)^{0,3}$	$=$	$E^{0,3}(III) \otimes \Omega^1_{\bar{S}}$
$\Omega^0_{(2)}(E)^{0,3}$	$=$	$E^{0,3}$			

Proof. This can be read off immediately by looking at the positions of the t's in the local description of the L^2-Higgs complex. □

We put

$$a := \deg(E^{3,0}), \quad b := \deg(E^{2,1}),$$

so that by self-duality of the Higgs-bundle we have

$$\deg(E^{0,3}) = -a, \quad \deg(E^{1,2}) = -b.$$

Let A, B, C be the numbers of singular points of type I, II, III described above.

Theorem 3.2. *The Hodge numbers of $H := H^1_{(2)}(S, \mathbb{V})$ are:*

$$\begin{aligned} h^{4,0} &= h^{0,4} = a - 1 + g(\bar{S}), \\ h^{3,1} &= h^{1,3} = B + C + b - a - 2 + 2g(\bar{S}), \\ h^{2,2} &= A + C - 2b - 2 + 2g(\bar{S}). \end{aligned}$$

In particular we have

$$\dim_{\mathbb{C}} H = A + 2B + 3C + 8(g(\bar{S}) - 1).$$

Proof. We use the description of the Hodge spaces from the previous section. We have $H^{4,0} = H^0(\bar{S}, E^{3,0} \otimes \Omega^1_{\bar{S}})$ and so by Riemann-Roch $h^{4,0} = a+2g-2-g+1 = a - 1 + g$ as $H^1(\bar{S}, E^{3,0} \otimes \Omega^1_{\bar{S}}) = 0$ by positivity of $E^{3,0}$ ($g = g(\bar{S})$).

Recall that the space $H^{3,1}$ is represented as global sections of the cokernel of the map $\Omega^0_{(2)}(E)^{3,0} \stackrel{\theta}{\longrightarrow} \Omega^1_{(2)}(E)^{2,1}$, so we have to count the zeros of a bundle map

$$E^{3,0}(-II - III) \longrightarrow E^{2,1} \otimes \Omega^1_{\bar{S}}.$$

Their number is given by the difference in degree of the bundles involved, so

$$h^{3,1} = \deg E^{2,1} \otimes \Omega^1_{\bar{S}} - \deg E^{3,0}(-II - III) = b + 2g - 2 - a + B + C$$

Similarly, the space $H^{2,2}$ is represented as global sections of the cokernel of the map $\Omega^0_{(2)}(E)^{2,1} \stackrel{\theta}{\longrightarrow} \Omega^1_{(2)}(E)^{1,2}$, so we have to count the zeros of a bundle map

$$E^{2,1}(-I - III) \longrightarrow E^{1,2} \otimes \Omega^1_{\bar{S}}.$$

and one obtains the dimension in a similar way. By duality, the other Hodge numbers are determined. □

Corollary 3.3. *In the special case* $\dim_{\mathbb{C}} H \leq 1$ *we have* $H = H^{2,2}$ *and it follows that*

$$g(\bar{S}) = 0, \quad a = 1, \quad b = \frac{1}{2}(A + C - 2 - \dim_{\mathbb{C}} H).$$

Remark 3.4. The 3-fold iterated Kodaira-Spencer map θ gives rise to a homomorphism

$$\theta^{(3)} : (E^{3,0})^{\otimes 2} \longrightarrow S^3\Omega^1_{\bar{S}}(\log D)$$

which is called the Griffiths-Yukawa coupling. The number of zeros of $\theta^{(3)}$ counted with multiplicity is therefore given by the formula

$$\sharp\{p \in \mathbb{P}^1 \mid \theta^{(3)}(p) = 0\} = 3(N - 2 + 2g(\bar{S})) - 2a.$$

Here

$$N := A + B + C$$

denotes the total number of singularities.

Remark 3.5. For a non-trivial, irreducible local system $\mathbb{V}$ there is a simple classical formula for the dimension

$$h^1(\mathbb{V}) := \dim_{\mathbb{C}} H^1(\mathbb{P}^1, j_*\mathbb{V})$$

in terms of the *ramification indices*

$$R(p) := \mathrm{Rank}(\mathbb{V}) - \dim(\mathbb{V}^{I(p)}),$$

where $I(p)$ is the local monodromy around the point p.

Proposition 3.6.

$$h^1(\mathbb{V}) = \sum_{p\in D} R(p) + (2g(\bar{S}) - 2)\mathrm{Rank}(\mathbb{V}).$$

Proof. From topology one obtains

$$\chi(j_*\mathbb{V}) = \chi(S)\cdot \mathrm{Rank}(\mathbb{V}) + \sum_{p\in D} \dim \mathbb{V}^{I(p)} = (2-2g)\mathrm{Rank}(\mathbb{V}) - \sum_{p\in D} R(p).$$

From the exact sequence

$$0 \longrightarrow j_!\mathbb{V} \longrightarrow j_*\mathbb{V} \longrightarrow i^*j_*\mathbb{V} \longrightarrow 0$$

(where $i : D \hookrightarrow \bar{S}$) one gets $H^2(j_!\mathbb{V}) = H^2(j_*\mathbb{V})$ and from Poincaré duality one gets

$$H^2(j_!\mathbb{V}) = H^2_c(S,\mathbb{V}) = H^0(S,\mathbb{V}^*)^*$$

As $H^0(S,\mathbb{V}^*) = 0$ the result follows. □

4. Cycle classes

Recall that the space $H^1(S,\mathbb{V})$ is naturally identified with the space of *extension classes*

$$0 \longrightarrow \mathbb{V} \longrightarrow \widehat{\mathbb{V}} \longrightarrow \mathbb{C}_S \longrightarrow 0$$

of local systems on S. The parabolic subspace $H^1(\bar{S}, j_*\mathbb{V}) \subset H^1(S,\mathbb{V})$ consists precisely of those extensions which split locally near each point $p \in D$, see e.g. [S2]. If the local system is described as horizontal sections or solutions to a homogeneous differential equation on S

$$\mathcal{L}\Phi = 0,$$

then extensions are described in terms of inhomogeneous equations of the form

$$\mathcal{L}\Psi = g,$$

where $g \in \mathcal{M}_S$ is a meromorphic function on S. The local system of solutions to the extended equation

$$\widehat{\mathcal{L}}\Phi = 0, \quad \widehat{\mathcal{L}} = (g\frac{d}{dt} - g')\mathcal{L}$$

makes up the local system $\widehat{\mathbb{V}}$. This leads to a description of the parabolic cohomology by an exact sequence of the form

$$0 \longrightarrow \mathcal{L}(\mathcal{M}_S) \longrightarrow \mathcal{M}_S^{para} \longrightarrow H^1(\bar{S}, j_*\mathbb{V}) \longrightarrow 0,$$

where $\mathcal{M}^{para}$ is the set of meromorphic functions on $\bar{S}$ that satisfy a *parabolic condition* in the points $p \in D$. We refer to [S1] and [S2] for details of this construction.

Consider as in the introduction a semi-stable morphism $\bar{f} : \bar{X} \to \bar{S}$ which extends a smooth map $f : X \longrightarrow S$. We set $\Delta = \bar{f}^{-1}(D)$, the inverse image of $D := \bar{S} \setminus S$, a reduced simple normal crossing divisor. We consider algebraic cycles $Z \in CH^2(\bar{X})$, whose classes restrict to zero on the fibres of f, i.e. are in

the kernel of the restriction map $H^4(\bar{X}) \longrightarrow H^4(fibre)$. Such a cycle determines a locally split extension

$$0 \longrightarrow \mathbb{V} \longrightarrow \widehat{\mathbb{V}} \longrightarrow \mathbb{C}_S \longrightarrow 0$$

of local systems on S that we will describe now. Its fibre over $t \in S$ is

$$0 \longrightarrow H^3(X_t) \longrightarrow \widehat{\mathbb{V}}_t \longrightarrow \mathbb{C} \longrightarrow 0$$

$$\widehat{\mathbb{V}}_t = \rho^{-1}(\mathbb{C} \cdot [Z_t]), \quad \rho : H^3(X_t \setminus |Z_t|) \longrightarrow H^4_{|Z_t|}(X_t)$$

where $X_t := f^{-1}(t)$ and $Z_t := Z \cap X_t$. The choice of a 3-form $\omega \in H^0(S, E^{3,0})$ determines the Picard-Fuchs differential operator $\mathcal{L}$ which describes the local system $\mathbb{V} = R^3 f_* \mathbb{C}_X$ and has all periods

$$\Phi := \int_{\gamma_t} \omega, \quad \gamma_t \in H_3(X_t).$$

as solutions. The cycle Z and the form ω determine a *normal function* in the sense of Poincaré-Griffiths:

$$\Phi_Z(t) = \int_{\Gamma_t} \omega,$$

where the 3-chain Γ_t has the property that $\partial \Gamma_t = Z_t$. It satisfies an inhomogeneous Picard-Fuchs ODE of the form

$$\mathcal{L}\Phi_Z = g,$$

where the source function g is a meromorphic function on $\bar{S}$ [dAMS].

There is a Higgs part to this story: each cycle $Z \in CH^2(\bar{X})$ will also determine an element in the group $H := H^1_{(2)}(S, E)$, which in fact will lie in $(2,2)$ component $H^{2,2}$ of H. Let us indicate how to obtain directly a class in

$$H^0(\bar{S}, E^{1,2} \otimes \Omega^1_{\bar{S}}(\log D)/\theta(E^{2,1}))$$

or rather its L^2-version

$$H^0(\bar{S}, \Omega^1_{(2)}(E)^{1,2}/\theta(\Omega^0_{(2)}(E)^{2,1})).$$

From the class of Z in $H^4(\bar{X})$ we first obtain by restriction to X a class in $H^2(\bar{X}, \Omega^2_{\bar{X}}(\log \Delta))$ which is mapped to a class in

$$H^0(\bar{S}, R^2 \bar{f}_* \Omega^2_{\bar{X}}(\log \Delta))$$

by the edge-homomorphism $H^2(\bar{X}, \Omega^2_{\bar{X}}(\log \Delta)) \to H^0(\bar{S}, R^2 \overline{f}_* \Omega^2_{\bar{X}}(\log \Delta))$ of the Leray spectral sequence. The exact sequence

$$0 \to \overline{f}^* \Omega^1_{\bar{S}}(\log D) \otimes \Omega^1_{\bar{X}/\bar{S}}(\log \Delta) \to \Omega^2_{\bar{X}}(\log \Delta) \to \Omega^2_{\bar{X}/\bar{S}}(\log \Delta) \to 0$$

gives by taking the direct image under $\bar{f}_*$ a long exact sequence of sheaves on $\bar{S}$:

$$\begin{aligned} \ldots \longrightarrow E^{2,1} \xrightarrow{\theta} \Omega^1_{\bar{S}}(\log D) \otimes E^{1,2} &\longrightarrow R^2 \bar{f}_* \Omega^2_{\bar{X}}(\log \Delta) \\ &\longrightarrow R^2 \bar{f}_* \Omega^2_{\bar{X}/\bar{S}}(\log \Delta) \longrightarrow \ldots \end{aligned}$$

From this sequence we get
$H^0(E^{1,2}\otimes\Omega^1_{\bar S}(\log D)/E^{2,1}) = \mathrm{Ker}(H^0(\bar S, R^2\bar f_*\Omega^2_{\bar X}(\log\Delta)) \to H^0(\bar S, R^2\bar f_*\Omega^2_{\bar X/\bar S}(\log\Delta)))$.
As Z is assumed to be homologous to zero on the fibres, the class of Z is indeed in the kernel and hence we obtain an element

$$\delta(Z)\in H^0(E^{1,2}\otimes\Omega^1_{\bar S}(\log D)/\theta(E^{2,1}))$$

which is called *Griffiths' infinitesimal invariant* of Z. However, we know a priori that the class of Z is of type $(2,2)$, so that the element lies in the $H^{2,2}$-subspace of $H^1_{(2)}(S, R^3f_*\mathbb{C}_X)$. Hence we must have that

$$\delta(Z)\in H^0(\bar S,\Omega^1_{(2)}(E)^{1,2}/\theta(\Omega^0_{(2)}(E)^{2,1})).$$

One can also understand $H^1_{(2)}(E,\theta)$ in terms of *extensions of Higgs-bundles*. If (E,θ) is the Higgs bundle associated to $\mathbb{V}$, then there is a Higgs bundle $(\widehat E,\widehat\theta)$ associated to $\widehat{\mathbb{V}}$. It is obtained as sum of graded pieces for the Hodge filtration of the Deligne extension of $\widehat{\mathbb{V}}$. It sits in an exact sequence

$$0\to(E,\theta)\to(\widehat E,\widehat\theta)\to(\mathcal{O}_{\bar S},0)\to 0,$$

where $(\mathcal{O}_{\bar S},0)$ is the trivial Higgs bundle. This extension produces an extension of L^2-Higgs complexes

$$0\longrightarrow\Omega^\bullet_{(2)}(E)\longrightarrow\Omega^\bullet_{(2)}(\widehat E)\longrightarrow\Omega^\bullet_{(2)}(\mathcal{O})\longrightarrow 0$$

and the connecting homomorphism of hypercohomology gives an exact sequence

$$\ldots\longrightarrow H^0(\bar S,\mathcal{O})\xrightarrow{\partial}H^1_{(2)}(S,E)\longrightarrow H^1_{(2)}(S,\widehat E))\longrightarrow\ldots$$

Clearly one has

$$\partial(1)=\delta(Z).$$

5. Examples

5.1. **Mirror Quintic.** The Dwork pencil is the following pencil of quintics in $\mathbb{P}^4$

$$x_1^5+x_2^5+x_3^5+x_4^5+x_5^5-5\psi x_1x_2x_3x_4x_5=0$$

which plays a prominent role in mirror symmetry [C].

Apart from the degeneration into the union of 5 hyperplanes $x_1x_2x_3x_4x_5=0$ occurring for $\psi=\infty$, the fibres aquire singularities over the fifth roots of unity $\psi^5=1$. In fact, these fibres have 125 nodes, which form a single orbit under a group $G\approx(\mathbb{Z}/5)^3$ that acts on the whole pencil by scaling the coordinates that preserve each monomial of the equation.

The quotient by the pencil by G has a birational model as pencil of quintics [M]:

$$(x_1+x_2+x_3+x_4+x_5)^5-(5\psi)^5x_1x_2x_3x_4x_5=0.$$

A crepant resolution of the general fibre is a Calabi-Yau 3-fold with Hodge numbers $h^{1,2}=1$, $h^{1,1}=101$, called the *mirror quintic*.

The local system $\mathbb{V} = R^3 f_* \mathbb{C}$ of this family is unipotent and has 6 singular fibres; there is one point of type III over $\psi = \infty$ and the 5 points of type II over the fifth unit roots. So we have $N = 6$, $A = 5$, $B = 0$, $C = 1$ and from the dimension formula one finds $h^1(\mathbb{V}) = 0$. From our formulas we deduce $a = 1$ and $b = 2$. It follows that Griffiths-Yukawa Coupling $\theta^{(3)}$ has 10 zeros. It vanishes twice at the fifth roots of unity.

After taking a further base change $z \mapsto z^2$, we obtain a family with $A = 10$, $B = 0$, $C = 1$ and $N = 11$. By the dimension formula we have $h^1(\mathbb{V}) = 5$. As we pull back a unipotent system, the Hodge bundles are obtained by base-change. So we have $a = 2, b = 4$. Our formula gives $h^{4,0} = h^{3,1} = h^{2,2} = 1$.

The Yukawa-coupling $\theta^{(3)}$ vanishes twice at each of the 10 points of type I. Furthermore, $\theta^{(3)}$ now has a triple zero at 0 due to the ramification, whereas at ∞ the ramification does not produce a zero, as the pull-back of a point of type III remains of type III. So $\theta^{(3)}$ has 23 zeros in total, in accordance with our formula.

Hence, $H^1(\bar{S}, j_* \mathbb{V})$ is a remarkable Hodge structure of dimension 5, all whose Hodge numbers are one. In particular there should be one interesting 2-cycle on the total space. It was determined explicitly by Walcher, using matrix factorizations, [W1].

5.2. **Hypergeometric pencils.** Note that the pencil of the mirror quintic of the previous paragraph is the pull-back under the map $z = (5\psi)^{-5}$ of the pencil

$$z(x_1 + x_2 + x_3 + x_4 + x_5)^5 - x_1 x_2 x_3 x_4 x_5 = 0$$

The local system $\mathbb{L} = R^3 f_* \mathbb{C}$ of this family can be identified with the local system of solutions to the hypergeometric operator

$$\theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad (\theta = z\partial/\partial z)$$

which has three singular points $z = 0, z = 5^{-5}, \infty$. The monodromy around $z = 0$ of type III, around $z = 5^{-5}$ of type I and around $z = \infty$ it is of order five. The dimension formula gives

$$\dim_{\mathbb{C}} H^1(\bar{S}, j_* \mathbb{L}) = 1 + 3 + 4 - 8 = 0.$$

When we pull-back over the map $z \longrightarrow z^2$, we get a local system $\widetilde{\mathbb{L}}$ with

$$\dim_{\mathbb{C}} H^1(\bar{S}, j_* \widetilde{\mathbb{L}}) = 2 + 3 + 4 - 8 = 1$$

so there is a cycle. We will need an extension of our theory to the quasi-unipotent case in order to see via L^2-Higgs theory that this is indeed a $(2,2)$-class. This gives an explanation for the appearance of $\sqrt{z}$ in the inhomogeneous term

$$\mathcal{L}\Phi = \frac{c}{(2\pi i)^2} \sqrt{z}, \quad c \in \mathbb{Q}$$

for the inhomogeneous Picard-Fuchs equation of the cycle in $H^{2,2}$.

In total there are 14 such hypergeometric local systems that carry a variation of Hodge structures [DM]: they all arise as mirror families of Calabi-Yau complete intersections in weighted projective spaces. We consider pull-backs $z \mapsto z^e$ of

order e, which have 1 point of type III, e points of type I and possibly a further singularity at ∞.

Seven cases with semi-simple monodromy at infinity.

Operator	Model	e	h^1	$h^{4,0}$	$h^{3,1}$	$h^{2,2}$
$\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$	$\mathbb{P}^4[5]$	1	0	0	0	0
		2	1	0	0	1
		5	0	0	0	0
		10	1	1	1	1
$\theta^4 - 36z(6\theta+1)(3\theta+1)(3\theta+2)(6\theta+5)$	$\mathbb{P}(1,1,1,1,2)[6]$	1	0	0	0	0
		2	1	0	0	1
		6	1	0	0	1
$\theta^4 - 16z(8\theta+1)(8\theta+3)(8\theta+5)(8\theta+7)$	$\mathbb{P}(1,1,1,1,4)[8]$	1	0	0	0	0
		2	1	0	0	1
		8	3			
$\theta^4 - 80z(10\theta+1)(10\theta+3)(10\theta+7)(10\theta+9)$	$\mathbb{P}(1,1,1,2,5)[10]$	1	0	0	0	0
		2	1	0	0	1
		10	5			
$\theta^4 - 48z(6\theta+1)(4\theta+1)(4\theta+3)(6\theta+5)$	$\mathbb{P}(1,1,1,1,1,2)[3,4]$	1	0	0	0	0
		2	1	0	0	1
		4	1	0	0	1
		6	3			
		12	7			
$\theta^4 - 144z(12\theta+1)(12\theta+5)(12\theta+7)(12\theta+11)$	$\mathbb{P}(1,1,1,1,2,12)[2,12]$	1	0	0	0	0
		2	1	0	0	1
		12	7			
$\theta^4 - 12z(4\theta+1)(3\theta+1)(3\theta+2)(4\theta+3)$	$\mathbb{P}(1,1,1,2,2,3)[4,6]$	1	0	0	0	0
		2	1	0	0	1
		3	0	0	0	0
		4	1	0	0	1
		12	7			

Three cases with one Jordan-block at ∞:

Operator	Model	e	h^1	$h^{4,0}$	$h^{3,1}$	$h^{2,2}$
$\theta^4 - 16z(4\theta+1)(2\theta+1)^2(4\theta+3)$	$\mathbb{P}^5[2,4]$	1	0	0	0	0
		2	0	0	0	0
		4	0	0	0	0
		8	4	1	1	0
$\theta^4 - 12z(3\theta+1)(2\theta+1)^2(3\theta+2)$	$\mathbb{P}^6[2,2,3]$	1	0	0	0	0
		2	0	0	0	0
		3	0	0	0	0
		6	2			
$\theta^4 - 48z(6\theta+1)(2\theta+1)^2(6\theta+5)$	$\mathbb{P}^5[2,6]$	1	0	0	0	0
		2	0	0	0	0
		6	2			

For example, the pull-back of the $\mathbb{P}^5[2,4]$-operator by $z \mapsto z^4$ has $A = 5$, $C = 1$, so still $h^1 = 3 + 5 - 8 = 0$, and thus it follows that $a = 1$, $b = 2$. Making an additional quadratic pull-back $z \mapsto z^2$ gives a local system with $a = 2, b = 4$, $A = 9$, $B = 0$, $C = 1$, so we get $h^{4,0} = h^{3,1} = 1, h^{2,2} = 0$!

Three cases with two Jordan-blocks at infinity:

Operator	Model	e	h^1	$h^{4,0}$	$h^{3,1}$	$h^{2,2}$
$\theta^4 - 9z(3\theta+1)^2(3\theta+2)^2$	$\mathbb{P}^5[3,3]$	1	0	0	0	0
		2	1	0	0	1
		3	0	0	0	0
		6	3	1	0	1
$\theta^4 - 16z(4\theta+1)^2(4\theta+3)^2$	$\mathbb{P}^5(1,1,1,1,2,2)[4,4]$	1	0	0	0	0
		2	1	0	0	1
		4	1	0	0	1
		8	5	1	0	3
$\theta^4 - 144z(6\theta+1)^2(6\theta+5)^2$	$\mathbb{P}^5[2,6]$	1	0	0	0	0
		2	1	0	0	1
		6	3			

For example, the pull-back of the $\mathbb{P}^5[3,3]$-operator by $z \mapsto z^3$ has $A = 3$, $B = 1$, $C = 1$, so still $h^1 = 3 + 2 + 3 - 8 = 0$, and thus it follows that $a = 1$, $b = 1$. Making an additional quadratic pull-back $z \mapsto z^2$ gives a local system with $a = 2, b = 2$, $A = 6$, $B = 1$, $C = 1$, so we get $h^{4,0} = 1, h^{3,1} = 0, h^{2,2} = 1$.

One case with a maximal Jordan block at infinity:

Operator	Model	e	h^1	$h^{4,0}$	$h^{3,1}$	$h^{2,2}$
$\theta^4 - 16z(2\theta+1)^4$	$\mathbb{P}^7[2,2,2,2]$	1	0	0	0	0
		2	0	0	0	0
		$2k$	$2k-2$	$k-1$	0	0

Pulling-back by $z \mapsto z^2$ produces a local system with $A = 2, B = 0, C = 2$, so $h^1 = 0$, hence $a = 1, b = 1$. A further pull-back by $z \mapsto z^k$, gives $A = 2k, B = 0, C = 2$, $a = k, b = k$ and hence $h^{4,0} = k - 1$,$h^{3,1} = h^{2,2} = 0$; we never get any $(2,2)$-classes.

This gives some apriori explanation for which cases one expects the appearance of $\sqrt{z}$ in the inhomogeneous term

$$\mathcal{L}\Phi = \frac{c}{(2\pi i)^2}\sqrt{z}, \quad c \in \mathbb{Q}$$

for Picard-Fuchs equation of the cycle in $H^{2,2}$, [W1],[W2].

5.3. **A Grassmannian Example.** In [AESZ] over 300 examples of differential equations of Calabi-Yau type are collected. Conjecturally all of these have a geometrical origin and underlie a variation of Hodge structures of type $(1,1,1,1)$. After the 14 hypergeometric examples there are cases that arise from mirror symmetry for complete intersections in a Grassmannian.

The canonical class of the six dimensional Grassmannian $Z := Gr(2,5)$ is $5H$, where H is the ample generator of $H^2(Z)$ which defines the Plücker-embedding of Z in $\mathbb{P}^{10}$. Consequently, the complete intersection of Z with three hypersurfaces of degree $1,2,2$ is a smooth Calabi-Yau threefold with $h^{1,1} = 1$. In [BCKS]

a mirror family was described. It is a pencil with 4 singular points $0, \alpha, \beta, \infty$, where

$$\alpha := \frac{1}{32}(5\sqrt{5} - 11), \quad \beta := \frac{1}{32}(-5\sqrt{5} - 11).$$

The Picard-Fuchs operator for this family was computed in [BCKS]:

$$\theta^4 - 4z(11\theta^2 + 11\theta + 3)(2\theta + 1)^2 - 16x^2(2\theta + 1)^2(2\theta + 3)^2, \quad (\theta = z\partial/\partial z).$$

which has a holomorphic solution

$$\Phi(z) = \sum_{n=0}^{\infty} A_n z^n, \quad A_n := \binom{2n}{n}^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

(it is Nr.25 in the list [AESZ]). Around 0 we have maximal unipotent monodromy, around the points α and β we have a single Jordan-block of size two. After a base change $x \mapsto x^2$ we are in a unipotent situation also at ∞. The points α and β each have two preimages of type A. The points ∞ is of type B (two Jordan blocks). This implies immediately that $1 = h^1(\mathbb{V}) = h^{2,2}(\mathbb{V})$ and the other Hodge numbers are zero. The constants are $A = 4$, $B = 1$, $C = 1$, $a = 1$ and $b = 1$.

5.4. **A remarkable Hadamard Product.** The second order operator

$$\theta^2 - z(32\theta^2 + 32\theta + 12) - 256(\theta + 1)^2$$

is Picard-Fuchs operator of a rational elliptic surface and defines a local system $\mathbb{L}$ of rank two. It has a unique period that is holomorphic near 0 which has an expansion $\phi(z) = 1 + 12z + 164z^2 + \ldots +$

$$\phi(z) = \sum_{n=0}^{\infty} A_n z^n, \quad A_n := \sum_{k=0}^{n} 4^{n-k} \binom{2k}{k}^2 \binom{2n-2k}{n-k}.$$

Its *Hadamard square* is the function

$$\Phi(z) := \sum_{n=0} A_n^2 z^n$$

which satisfied the fourth order equation

$$\mathcal{L}\Phi = 0$$

where $\mathcal{L}$ is a rather complicated operator (Nr. 115 in the list [AESZ]).

The local system $\mathbb{V}$ of solutions of $\mathcal{L}$ is the multiplicative convolution of the local system $\mathbb{L}$ with itself and defines a VHS of type $(1, 1, 1, 1)$. It has only three points of maximal unipotent monodromy, $A = 0, B = 0, C = 3$. One has $h^1(\mathbb{V}) = 3.3 - 8 = 1$, and hence $a = 1, b = 0$(!), and $h^{2,2} = 1$. Furthermore, the Griffiths-Yukawa coupling vanishes at a single point $(3(3 - 2) - 2 = 1)$.

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