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# A NOTE ON LE-PHẠM'S PAPER - CONVERGENCE IN $\delta\mathcal{E}_p$ SPACES

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ABSTRACT. Let $\delta\mathcal{E}_p$, $p > 0$, be the real vector space containing functions of the form $u_1 - u_2$, where $u_1$ and $u_2$ are non-positive plurisubharmonic functions with finite pluricomplex $p$-energy. We prove a convergence theorem and give an example of interesting continuous mappings on this quasi-Banach space.

## 1. Introduction

Let the cones $\mathcal{E}_0$, $\mathcal{E}_p$ $(p > 0)$, $\mathcal{F}$, and $\mathcal{E}$ be defined as in [4, 5] (see also Section 2). If $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$, then we use the notation $\delta\mathcal{K} = \mathcal{K} - \mathcal{K}$. Let $p > 0$, and for $u \in \delta\mathcal{E}_p$ define:

$$\|u\|_p = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_p}} \left( \int_\Omega (-(u_1 + u_2))^p (dd^c(u_1 + u_2))^n \right)^{\frac{1}{n+p}}, \tag{1.1}$$

where $(dd^c\,\cdot)^n$ is the complex Monge-Ampère operator. If $p = 0$, then we shall use (1.1) with the convention that $(-(u_1 + u_2))^p = 1$. It was proved in [7] that $(\delta\mathcal{F}, \|\cdot\|_0)$ is a Banach space, and in [2] that $(\delta\mathcal{E}_p, \|\cdot\|_p)$ is a quasi-Banach space. In Section 2 we recall some definitions, and prove that $\mathcal{E}_0$ and $\delta\mathcal{E}_0$ are generally not closed neither in $(\delta\mathcal{F}, \|\cdot\|_0)$ nor in $(\delta\mathcal{E}_p, \|\cdot\|_p)$ (Proposition 2.1). We end Section 2 by proving that the inclusions $\overline{\mathcal{E}_0} \subseteq \mathcal{F}$, $\overline{\delta\mathcal{E}_0} \subseteq \delta\mathcal{F}$, are proper in $(\delta\mathcal{F}, \|\cdot\|_0)$ (Proposition 2.2). In Section 3, the following convergence theorem is proved.

**Theorem 3.2.** *Let $[u_j]$, $u_j \in \delta\mathcal{E}_p$, be a sequence that converges to a function $u$ in $\delta\mathcal{E}_p$ as $j$ tends to $\infty$, then $[u_j]$ converges to $u$ in capacity.*

Example 3.3 shows that convergence in capacity is weaker than convergence in $\delta\mathcal{E}_p$. It was proved in [8] that the convergence in $(\delta\mathcal{F}, \|\cdot\|_0)$ is stronger than the one in $C_n$-capacity.

Let now $\mathcal{M}(\Omega)$ denote the space of signed real Borel measures on $\Omega$ with the topology given by the usual system of semi-norms. Then $\mathcal{M}(\Omega)$ is a Fréchet

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space. Furthermore, let $\mathcal{M}_b(\Omega)$ consist of signed, real and finite Borel measures defined on $\Omega$ equipped with the norm given by the total variation on $\Omega$. Then $\mathcal{M}_b(\Omega)$ is a Banach space. In Theorem 3.6, we prove that the following maps are continuous:

$$T_1 : (\delta\mathcal{E}_p)^{n+1} \ni (v, u_1, \dots, u_n) \to T_1(v, u_1, \dots, u_n) = |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M}_b,$$
$$T_2 : (\delta\mathcal{E}_p)^n \ni (u_1, \dots, u_n) \to T_2(u_1, \dots, u_n) = dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M},$$
$$T_3 : \delta\mathcal{E}_p \ni u \to T_3(u) = u \in \delta\mathcal{E}\,.$$

In connection to these mappings it is worth to mention that the following two maps are continuous ([7, 8]):

$$T_4 : (\delta\mathcal{F})^n \ni (u_1, \dots, u_n) \to T_4(u_1, \dots, u_n) = dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M}_b,$$
$$T_5 : (\delta\mathcal{E})^n \ni (u_1, \dots, u_n) \to T_5(u_1, \dots, u_n) = dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M}\,,$$

## 2. Preliminaries

We start by recalling notations and definitions. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, connected, and open set. Recall that $\Omega$ is hyperconvex if there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in $\Omega$, for every $c \in (-\infty, 0)$. We say that a plurisubharmonic function $\varphi$ defined on $\Omega$ belongs to $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ if $\lim_{z\to\xi} \varphi(z) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < \infty$, where $(dd^c\,\cdot)^n$ is the complex Monge-Ampère operator.

Assume that $u$ is a plurisubharmonic function defined on $\Omega$ and $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, is a decreasing sequence that converges pointwise to $u$ on $\Omega$, as $j$ tends to $\infty$. If there can be no misinterpretation a sequence $[\,\cdot\,]_{j=1}^\infty$ will be denoted by $[\,\cdot\,]$. For $p > 0$ fixed, consider the following assertions:

(1) $\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty,$

(2) $\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\,.$

If the sequence $[\varphi_j]$ can be chosen such that (1) holds, then we say that $u$ belongs to $\mathcal{E}_p$ and if (2) holds, then $u$ belongs to $\mathcal{F}$. Let $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ be the class of plurisubharmonic functions $\varphi$ defined on $\Omega$, such that for each $z_0 \in \Omega$ there exist a neighborhood $\omega$ of $z_0$ in $\Omega$ and a decreasing sequence $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to $\varphi$ on $\omega$ and (2) holds. It was proved in [4, 5] that $(dd^c\,\cdot)^n$ is well defined on $\mathcal{E}$. Let $e_p(u)$ be defined by

$$e_p(u) = \int_\Omega (-u)^p (dd^c u)^n\,, \tag{2.1}$$

for $p > 0$. The integral $e_p(u)$ is the *pluricomplex $p$-energy* of the function $u$. Note that if $u \in \mathcal{E}_p$, then $0 \le e_p(u) < \infty$. It was proved in [2] that if $u \in \mathcal{E}_p$ then the quasi-norm of $u$ in the space $\delta\mathcal{E}_p$ is equal to $||u||_p = e_p(u)^{\frac{1}{n+p}}$.

**Proposition 2.1.** *Let $B = B(0,1) \subseteq \mathbb{C}^2$ be the unit ball in $\mathbb{C}^2$. Then*

(1) *the cone $\mathcal{E}_0$ and the space $\delta\mathcal{E}_0$ are not closed in $(\delta\mathcal{F}, \|\cdot\|_0)$.*
(2) *the cone $\mathcal{E}_0$ and the space $\delta\mathcal{E}_0$ are not closed in $(\delta\mathcal{E}_p, \|\cdot\|_p)$.*

*Proof.* For each $j \in \mathbb{N}$, let the function $\varphi_j : B \to \mathbb{R} \cup \{-\infty\}$ be defined by

$$\varphi_j(z) = \max\left(\frac{1}{2^j}\log|z|, -\frac{1}{j}\right).$$

Observe that $\varphi_j \in \mathcal{E}_0$ and therefore the function $u_k : B \to \mathbb{R}$ defined by $u_k = \sum_{j=1}^k \varphi_j$ belongs to $\mathcal{E}_0$. Note that for $k > l$ we have

$$\|u_k - u_l\|^2 = \|\sum_{j=l+1}^k \varphi_j\|^2 = \int_B \left(dd^c \textstyle\sum_{j=l+1}^k \varphi_j\right)^2 = (2\pi)^2 \left(\displaystyle\sum_{j=l+1}^k \frac{1}{2^j}\right)^2 \tag{2.2}$$

and

$$\begin{aligned}
\|u_k - u_l\|_p^{n+p} &= \|\sum_{j=l+1}^k \varphi_j\|_p^{n+p} = e_p\left(\sum_{j=l+1}^k \varphi_j\right) \\
&= \int_B \left(-\left(\sum_{j=l+1}^k \varphi_j\right)\right)^p \left(dd^c \textstyle\sum_{j=l+1}^k \varphi_j\right)^2 \\
&= \sum_{j,r=l+1}^k \left(-\left(\sum_{m=l+1}^k \varphi_m\left(\max\left(e^{-\frac{2^j}{j}}, e^{-\frac{2^r}{r}}\right)\right)\right)\right)^p (2\pi)^2 \frac{1}{2^{j+r}} \\
&\le \sum_{r,j=l+1}^k \left(-u_k\left(e^{-\frac{2^r}{r}}\right)\right)^{\frac{p}{2}} \left(-u_k\left(e^{-\frac{2^j}{j}}\right)\right)^{\frac{p}{2}} (2\pi)^2 \frac{1}{2^{j+r}} \\
&= (2\pi)^2 \left(\sum_{j=l+1}^k \left(-u_k\left(e^{-\frac{2^j}{j}}\right)\right)^{\frac{p}{2}} \frac{1}{2^j}\right)^2.
\end{aligned}$$

Since

$$-u_k\left(e^{-\frac{2^j}{j}}\right) = \sum_{l=1}^j \frac{1}{2^l}\frac{2^l}{l} + \frac{2^j}{j}\sum_{l=j+1}^j \frac{1}{2^l} \le j+1,$$

we have

$$\|u_k - u_l\|_p^{n+p} \le (2\pi)^2 \left(\sum_{j=l+1}^k \frac{(j+1)^{\frac{p}{2}}}{2^j}\right)^2. \tag{2.3}$$

Let $u : B \to \mathbb{R} \cup \{-\infty\}$ be defined by $u = \lim_{k\to\infty} u_k$. Hence, $u$ is plurisubharmonic, since it is the limit of a decreasing sequence of plurisubharmonic functions and $u(\frac{1}{2}, 0) > -\infty$. Moreover $u \notin \mathcal{E}_0$ since $u(0) = -\infty$. Equality (2.2) implies that $[u_k]$ is a Cauchy sequence in $\delta\mathcal{F}$. The series $\sum_{j=1}^{\infty} \frac{(j+1)^{\frac{p}{2}}}{2^j}$ is convergent and therefore it follows by (2.3) that $[u_k]$ is a Cauchy sequence in $\delta\mathcal{E}_p$. □

**Proposition 2.2.** *We have*

$$\overline{\mathcal{E}_0} \subsetneq \mathcal{F}, \quad \textit{and } \overline{\delta\mathcal{E}_0} \subsetneq \delta\mathcal{F}$$

*in* $(\delta\mathcal{F}, \|\cdot\|_0)$.

*Proof.* The idea of this proof originates from [7]. We first recall the definition of the Lelong number:

$$\nu(u, z_0) = \lim_{r\to 0} \frac{1}{(2\pi)^n} \int_{B(z_0,r)} dd^c u \wedge (dd^c \log|z - z_0|)^{n-1} .$$

We have that the Lelong number $\nu(\cdot, z_0)$ at some point $z_0 \in \Omega$ is a continuous linear functional on $\delta\mathcal{F}$ since by [5] it holds that

$$\nu(u, z_0) \leq (dd^c u)^n(\{z_0\}) .$$

Assume that $\overline{\mathcal{E}_0} = \mathcal{F}$ and take $u(z) = g(z, z_0)$, where $g(z, z_0)$ is the pluricomplex Green function with pole at $z_0$. Then there exists a sequence $[u_j]$, $u_j \in \mathcal{E}_0$, that converges to $u$ in $\delta\mathcal{F}$, as $j \to \infty$ and therefore it follows that

$$0 = \nu(u_j, z_0) \to \nu(u, z_0) = 1.$$

Thus, a contradiction is obtained. □

## 3. On the convergence in $\delta\mathcal{E}_p$

Let us recall the definition of capacity and convergence in capacity.

**Definition 3.1.** The relative capacity of the Borel set $E \subset \Omega \subset \mathbb{C}^n$ with respect to $\Omega$ is defined by

$$cap(E, \Omega) = \sup\left\{ \int_E (dd^c u)^n : u \in \mathcal{PSH}(\Omega), -1 \leq u \leq 0 \right\} .$$

Let $u_j, u \in \mathcal{PSH}(\Omega)$. We say that a sequence $u_j$ converges to $u$ in capacity if for any $\epsilon > 0$ and $K \Subset \Omega$ we have

$$\lim_{j\to\infty} cap(K \cap \{|u_j - u| > \epsilon\}) = 0.$$

**Theorem 3.2.** *Let* $[u_j]$, $u_j \in \delta\mathcal{E}_p$, *be a sequence that converges to a function* $u$ *in* $\delta\mathcal{E}_p$, *as* $j$ *tends to* $\infty$, *then* $[u_j]$ *converges to* $u$ *in capacity.*

*Proof.* Without lost of generality we can assume that $u = 0$. Let $[u_j]$, $u_j \in \delta\mathcal{E}_p$, be a sequence such that $\|u_j\|_p \to 0$, as $j \to \infty$. From the definition of $\delta\mathcal{E}_p$ there exist functions $v_j, w_j \in \mathcal{E}_p$ such that $u_j = v_j - w_j$ and $e_p(v_j + w_j) \to 0$, as $j \to \infty$. Since

$$\max(e_p(v_j), e_p(w_j)) \leq e_p(v_j + w_j),$$

we have that $e_p(v_j) \to 0$ and $e_p(w_j) \to 0$, as $j \to \infty$. Let $\epsilon > 0$ and $K \Subset \Omega$. For any $\psi \in \mathcal{PSH}(\Omega)$, $-1 \le \psi \le 0$, we have

$$\int_{\{|v_j|>\epsilon\}\cap K} (dd^c\psi)^n \le \frac{1}{\epsilon^{n+p}} \int_\Omega (-v_j)^{n+p} (dd^c\psi)^n \le \frac{C(n,p)}{\epsilon^{n+p}} e_p(v_j),$$

where $C(n,p)$ is a constant depending only on $n$ and $p$ (see [3]). Therefore we get

$$cap(\{|v_j| > \epsilon\} \cap K) \le \frac{C(n,p)}{\epsilon^{n+p}} e_p(v_j) \to 0,$$

as $j \to \infty$ and similarly

$$cap(\{|w_j| > \epsilon\} \cap K) \le \frac{C(n,p)}{\epsilon^{n+p}} e_p(w_j) \to 0,$$

as $j \to \infty$. Hence

$$cap(\{|u_j| > \epsilon\} \cap K) \le cap\left(\{|v_j| > \frac{\epsilon}{2}\} \cap K\right) + cap\left(\{|w_j| > \frac{\epsilon}{2}\} \cap K\right) \to 0,$$

as $j \to \infty$ and this proof is complete. □

The following example shows that convergence in capacity is weaker than convergence in $\delta\mathcal{E}_p$.

**Example 3.3.** Let $B(0,1)$ be the unit ball in $\mathbb{C}^n$. Let us define

$$u_j(z) = \max\left(j^{\frac{p}{n}} \log|z|, -\frac{1}{j}\right).$$

Then $u_j \in \mathcal{E}_0(B)$ and $\|u_j\|_p^{n+p} = e_p(u_j) = (2\pi)^n$. Thus, $[u_j]$ do not converge to 0 in $\delta\mathcal{E}_p$ as $j \to +\infty$. Observe also that for fixed $\epsilon > 0$ and for fixed $K \Subset B$ there exists $j_0$ such that for every $j \ge j_0$ we have $u_j = -\frac{1}{j} > -\epsilon$ on $K$. This implies that $K \cap \{u_j < -\epsilon\} = \emptyset$ and therefore $u_j \to 0$ in capacity. □

It was proved in [7, 8] that it is possible to extend the definition of the complex Monge-Ampère operator in a reasonable way to the spaces $\delta\mathcal{F}$ and $\delta\mathcal{E}$. Namely for $u \in \delta\mathcal{E}$ and $K \Subset \Omega$ there exist $u_1, u_2 \in \mathcal{F}$ such that $u = u_1 - u_2$. Define

$$(dd^cu)^n|_K = \sum_{k=0}^n (-1)^k \binom{n}{k} (dd^cu_1)^k \wedge (dd^cu_2)^{n-k}|_K\,.$$

It follows from [8] that the following operator is well defined. For $u_1, \dots, u_n \in \delta\mathcal{E}$ and $K \Subset \Omega$ there exist $w_j^1, w_j^2 \in \mathcal{F}$ for $1 \le j \le n$ such that $u_j = w_j^1 - w_j^2$ on $K$ for $1 \le j \le n$. Define

$$dd^cu_1 \wedge \cdots \wedge dd^cu_n|_K = dd^c(w_1^1 - w_1^2) \wedge \cdots \wedge dd^c(w_n^1 - w_n^2)|_K.$$

Now we can extend the definition of the mutual $p$-energy to the space $\delta\mathcal{E}_p$. For $v, u_1, \dots, u_n \in \delta\mathcal{E}_p$ there exist functions $v^1, v^2, w_j^1, w_j^2 \in \mathcal{E}_p$ for $1 \le j \le n$ such that $v = v^1 - v^2$ and $u_j = w_j^1 - w_j^2$ for $1 \le j \le n$. Define

$$e_p(v, u_1, \dots, u_n) = \int_\Omega |v|^p dd^cu_1 \wedge \cdots \wedge dd^cu_n$$

$$= \int_\Omega |v^1 - v^2|^p dd^c(w_1^1 - w_1^2) \wedge \cdots \wedge dd^c(w_n^1 - w_n^2).$$

We write $e_p(u)$ for the case when $v = u_1 = \ldots = u_n = u$. If $u \in \mathcal{E}_p$ then $e_p(u) < \infty$, so $(dd^c u)^n = 0$ on the set $\{z \in \Omega : u(z) = -\infty\}$. For $u \in \delta\mathcal{E}_p$, $u = u_1 - u_2$, $u_1, u_2 \in \mathcal{E}_p$ we have that $(dd^c u)^n = 0$ on the set $\{z \in \Omega : u_1(z) = -\infty\} \cup \{z \in \Omega : u_2(z) = -\infty\}$. By previous observation, and by [8] the mutual $p$-energy is well defined.

In the rest of this section we shall need the following theorem. Theorem 3.4 was proved in [9] (see also [4, 6]), and for $0 < p < 1$ in [1]. If $p = 0$, then (3.1) can be interpreted as Corollary 5.6 in [5].

**Theorem 3.4.** *Let $p > 0$ and $u_0, u_1, \ldots, u_n \in \mathcal{E}_p$. Then*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n$$
$$\le D(n,p)\, e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)} \cdots e_p(u_n)^{1/(p+n)}, \tag{3.1}$$

*where $D(n,p) \ge 1$ is a constant depending only on $n$ and $p$.*

**Lemma 3.5.** *For $v, u_1, \ldots, u_n \in \delta\mathcal{E}_p$ we have*

$$|e_p(v, u_1, \ldots, u_n)| \le D(n,p)\|v\|_p^p \|u_1\|_p \ldots \|u_n\|_p$$

*and*

$$|e_p(v)| \le \|v\|_p^{n+p}.$$

*Proof.* Let $v, u_1, \ldots, u_n \in \delta\mathcal{E}_p$ then there exist $v^1, v^2, w_j^1, w_j^2 \in \mathcal{E}_p$ for $1 \le j \le n$ such that $v = v^1 - v^2$ and $u_j = w_j^1 - w_j^2$ for $1 \le j \le n$. Note that

$$\begin{aligned} |e_p(v, u_1, \ldots, u_n)| &= \left| \int_\Omega |v^1 - v^2|^p dd^c(w_1^1 - w_1^2) \wedge \cdots \wedge dd^c(w_n^1 - w_n^2) \right| \\ &\le \int_\Omega (-v^1 - v^2)^p dd^c(w_1^1 + w_1^2) \wedge \cdots \wedge dd^c(w_n^1 + w_n^2) \\ &\le D(n,p) e_p(v^1 + v^2)^{\frac{p}{n+p}} e_p(u_1^1 + u_1^2)^{\frac{1}{n+p}} \cdots e_p(u_n^1 + u_n^2)^{\frac{1}{n+p}}. \end{aligned}$$

By taking infimum over all decomposition of the functions $v, u_1, \ldots, u_n$ we get

$$|e_p(v, u_1, \ldots, u_n)| \le D(n,p)\|v\|_p^p \|u_1\|_p \ldots \|u_n\|_p\,.$$

For the second part we observe that

$$\begin{aligned} |e_p(v)| &= \left| \int_\Omega |v^1 - v^2|^p \sum_{k=0}^n (-1)^k \binom{n}{k} (dd^c v^1)^k \wedge (dd^c v^2)^{n-k} \right| \\ &\le \int_\Omega (-v^1 - v^2)^p \sum_{k=0}^n \binom{n}{k} (dd^c v^1)^k \wedge (dd^c v^2)^{n-k} = e_p(v^1 + v^2). \end{aligned}$$

Now by taking infimum over all decomposition of the function $v$ it follows that

$$|e_p(v)| \le \|v\|_p^{p+n}\,.$$

□

Recall from the introduction that $\mathcal{M}(\Omega)$ denotes the space of signed real Borel measures on $\Omega$. For every $\mu \in \mathcal{M}(\Omega)$, and every $K \Subset \Omega$, we denote by $\|\mu\|_K$ the variation of $\mu$ on $K$. The space $\mathcal{M}(\Omega)$ with the topology given by the system of semi-norms $\|\cdot\|_K$ is a Fréchet space. Furthermore, the space $\mathcal{M}_b(\Omega)$ consisting of signed, real and finite Borel measure on $\Omega$ equipped with the norm given by the total variation on $\Omega$ is a Banach space.

**Theorem 3.6.** *The following mappings are continuous.*

$$T_1 : (\delta\mathcal{E}_p)^{n+1} \ni (v, u_1, \ldots, u_n) \to T_1(v, u_1, \ldots, u_n) = |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M}_b,$$
$$T_2 : (\delta\mathcal{E}_p)^{n} \ni (u_1, \ldots, u_n) \to T_2(u_1, \ldots, u_n) = dd^c u_1 \wedge \cdots \wedge dd^c u_n \in \mathcal{M},$$
$$T_3 : \delta\mathcal{E}_p \ni u \to T_3(u) = u \in \delta\mathcal{E}\,.$$

*Proof.* $T_1$: Let $v^j, u_1^j, \ldots, u_n^j, v, u_1, \ldots, u_n \in \delta\mathcal{E}_p$ be such that $v^j \to v$, $u_k^j \to u_k$, for $1 \le k \le n$ in $\delta\mathcal{E}_p$. Then there exist $v_1, v_2, v_1^j, v_2^j, w_j^1, w_j^2, \varphi_k^j, \psi_k^j, \alpha^j, \beta^j, x_k^j, y_k^j \in \mathcal{E}_p$ for $1 \le k \le n$ such that $v = v_1 - v_2$, $v^j = v_1^j - v_2^j$, $v^j - v = \alpha^j - \beta^j$, $u_k = w_k^1 - w_k^2$, $u_k^j = x_k^j - y_k^j$, $u_k^j - u_k = \varphi_k^j - \psi_k^j$ and

$$e_p(\alpha^j + \beta^j) \to 0, \quad e_p(\varphi_k^j + \psi_k^j) \to 0,$$

as $j \to \infty$. Moreover, observe that we can choose the functions above so that there exists a constant $C > 0$ not depending on $j$ satisfying

$$\sup_{j \ge 1, k=1,\ldots,n} \left\{ e_p(v_1^j + v_2^j), e_p(w_j^1 + w_j^2), e_p(\varphi_k^j + \psi_k^j), e_p(\alpha^j + \beta^j), e_p(x_k^j + y_k^j) \right\} \le C\,.$$

We prove that $T_1$ is continuous. Note that

$$\begin{aligned}
&T_1(v^j, u_1^j, \ldots, u_n^j) - T_1(v, u_1, \ldots, u_n)\\
&= |v_j|^p dd^c u_1^j \wedge \cdots \wedge dd^c u_n^j - |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n\\
&= \sum_{k=1}^{n} |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c (u_k^j - u_k) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j\\
&\quad + |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n - |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n = \sum_{k=1}^{n} \mu_k^j + \nu^j.
\end{aligned}$$

For $1 \le k \le n$ it holds that

$$\begin{aligned}
\|\mu_k^j\| &= \Big\| |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c (u_k^j - u_k) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j \Big\|\\
&= \Big\| |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c (\varphi_k^j - \psi_k^j) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j \Big\|\\
&\le \int_\Omega (-v_1^j - v_2^j)^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_{k-1}^1 + w_{k-1}^2) \wedge dd^c (\varphi_k^j + \psi_k^j)\\
&\qquad \wedge dd^c (x_{k+1}^j + y_{k+1}^j) \wedge \cdots \wedge dd^c (x_n^j + y_n^j)\\
&\le D(n,p) e_p(v_1^j + v_2^j)^{\frac{p}{n+p}} e_p(\varphi_k^j + \psi_k^j)^{\frac{1}{n+p}} \times
\end{aligned}$$

$$\times \prod_{l=1}^{k-1} e_p(w_l^1 + w_l^2)^{\frac{1}{n+p}} \prod_{l=k+1}^{n} e_p(x_l^j + y_l^j)^{\frac{1}{n+p}}$$
$$\leq D(n,p) C^{\frac{n+p-1}{n+p}} e_p(\varphi_k^j + \psi_k^j)^{\frac{1}{n+p}} \to 0, \quad \text{as} \quad j \to \infty.$$

We shall now prove that $\|\nu^j\| \to 0$, as $j \to \infty$. First assume that $0 < p < 1$, and observe that for $x, y \geq 0$ we have $|x^p - y^p| \leq |x - y|^p$. Using this inequality we get

$$\big|\, |v^j|^p - |v|^p \,\big| \leq \big|\, |v^j| - |v| \,\big|^p \leq |v^j - v|^p \leq (-\alpha^j - \beta^j)^p \,,$$

hence

$$\begin{aligned} \|\nu^j\| &= \|\, |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n - |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \| \\ &\leq \int_\Omega (-\alpha^j - \beta^j)^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \\ &\leq D(n,p) e_p(\alpha^j + \beta^j)^{\frac{p}{n+p}} \prod_{l=1}^{n} e_p(w_l^1 + w_l^2)^{\frac{1}{n+p}} \\ &\leq D(n,p) C^{\frac{n}{n+p}} e_p(\alpha^j + \beta^j)^{\frac{p}{n+p}} \to 0, \quad \text{as} \quad j \to \infty. \end{aligned}$$

We have proved that $T_1$ is continuous for $0 < p < 1$. Now assume that $p \geq 1$. For $x, y \geq 0$ we have that $|x^p - y^p| \leq p(\max(x,y))^{p-1}|x - y|$, and therefore it holds that

$$\begin{aligned} \big|\, |v^j|^p - |v|^p \,\big| &\leq p\,(\max(|v_j|, |v|))^{p-1} \,\big|\, |v^j| - |v| \,\big| \\ &\leq p\,(\max(-v_1^j - v_2^j, -v_1 - v_2))^{p-1}(-\alpha^j - \beta^j)\,. \end{aligned}$$

Hölder's inequality yields that

$$\begin{aligned} \|\nu^j\| &= \|\, |v_j|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n - |v|^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \| \\ &\leq \int_\Omega p(\max(-v_1^j - v_2^j, -v_1 - v_2))^{p-1}(-\alpha^j - \beta^j) dd^c (w_1^1 + w_1^2) \\ &\quad \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \\ &\leq p \left( \int_\Omega (-\alpha^j - \beta^j)^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \right)^{\frac{1}{p}} \times \\ &\quad \times \left( \int_\Omega (\max(-v_1^j - v_2^j, -v_1 - v_2))^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \right)^{\frac{p-1}{p}} \\ &\leq p D(n,p)^{\frac{1}{p}} e_p(\alpha^j + \beta^j)^{\frac{1}{n+p}} \prod_{l=1}^{n} e_p(w_l^1 + w_l^2)^{\frac{1}{p(n+p)}} \\ &\quad \times \Big[ \int_\Omega (-v_1^j - v_2^j)^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \\ &\quad + \int_\Omega (-v_1 - v_2)^p dd^c (w_1^1 + w_1^2) \wedge \cdots \wedge dd^c (w_n^1 + w_n^2) \Big]^{\frac{p-1}{p}} \\ &\leq p D(n,p)^{\frac{1}{p}} C^{\frac{n}{p(n+p)}} e_p(\alpha^j + \beta^j)^{\frac{1}{n+p}} D(n,p)^{\frac{p-1}{p}} \times \end{aligned}$$

$$\times \Big[e_p(v_1^j+v_2^j)^{\frac{p}{n+p}}\prod_{l=1}^n e_p(w_l^1+w_l^2)^{\frac{1}{n+p}}$$
$$+ e_p(v_1+v_2)^{\frac{p}{n+p}}\prod_{l=1}^n e_p(w_l^1+w_l^2)^{\frac{1}{n+p}}\Big]^{\frac{p-1}{p}}$$
$$\le 2^{\frac{p-1}{p}} pD(n,p)C^{\frac{p+n-1}{n+p}} e_p(\alpha^j+\beta^j)^{\frac{1}{n+p}} \to 0 \text{ as } j\to\infty.$$

Thus, $T_1$ is continuous for $p \ge 1$.

$T_2$: Now we continue by proving that $T_2$ is a continuous mapping. We have

$$\begin{aligned}
&T_2(u_1^j,\dots,u_n^j) - T_2(u_1,\dots,u_n)\\
&= dd^c u_1^j \wedge \cdots \wedge dd^c u_n^j - dd^c u_1 \wedge \cdots \wedge dd^c u_n\\
&= \sum_{k=1}^n dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c(u_k^j - u_k) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j\\
&= \sum_{k=1}^n \tilde{\mu}_k^j.
\end{aligned}$$

Fix $K \Subset \Omega$. The relative extremal function for $K$ is defined by

$$h_K(z) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega), u \le 0, u \le -1 \text{ on } K\}.$$

It is well known that $h_K^* \in \mathcal{E}_0$, $h_K^*(z) = -1$ on $K$ and $-1 \le h_K^* \le 0$. For $1 \le k \le n$ it holds that

$$\begin{aligned}
\|\tilde{\mu}_k^j\|_K &= \Big\| dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c(u_k^j - u_k) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j \Big\|_K\\
&= \Big\| dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c(\varphi_k^j - \psi_k^j) \wedge dd^c u_{k+1}^j \wedge \cdots \wedge dd^c u_n^j \Big\|_K\\
&\le \int_\Omega (-h_K^*)^p dd^c(w_1^1+w_1^2) \wedge \cdots \wedge dd^c(w_{k-1}^1+w_{k-1}^2) \wedge dd^c(\varphi_k^j+\psi_k^j)\\
&\quad \wedge dd^c(x_{k+1}^j+y_{k+1}^j) \wedge \cdots \wedge dd^c(x_n^j+y_n^j)\\
&\le D(n,p)e_p(h_K^*)^{\frac{p}{n+p}} e_p(\varphi_k^j+\psi_k^j)^{\frac{1}{n+p}} \prod_{l=1}^{k-1} e_p(w_l^1+w_l^2)^{\frac{1}{n+p}} \prod_{l=k+1}^{n} e_p(x_l^j+y_l^j)^{\frac{1}{n+p}}\\
&\le D(n,p)C^{\frac{n-1}{n+p}} e_p(h_K^*)^{\frac{p}{n+p}} e_p(\varphi_k^j+\psi_k^j)^{\frac{1}{n+p}} \to 0 \text{ as } j\to\infty.
\end{aligned}$$

Thus, $T_2$ is continuous.

$T_3$: Fix $u \in \delta\mathcal{E}_p$, $u_1, u_2 \in \mathcal{E}_p \cap \mathcal{F}$ such that $u = u_1 - u_2$ and fix $K \Subset \Omega$. Let $h_K$ be the relative extremal function for $K$ in $\Omega$. Then we have

$$\int_K (dd^c(u_1+u_2))^n \le \int_\Omega (-h_K^*)^p (dd^c(u_1+u_2))^n \le D(n,p)e_p(h_K^*)^{\frac{p}{n+p}} e_p(u_1+u_2)^{\frac{n}{n+p}},$$

so

$$\left(\int_K (dd^c(u_1+u_2))^n\right)^{\frac{1}{n}} \le C(K,n,p)e_p(u_1+u_2)^{\frac{1}{n+p}},$$

where the constant $C(K, n, p)$ depends only on $K$, $n$ and $p$. Taking infimum over all decomposition of the function $u$ we obtain

$$\|u\|_K \leq C(K, n, p)\|u\|_p,$$

which proves that $T_3$ is continuous. □

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