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SOME REMARKS ON THE AR-PROBLEM

LE HOANG TRI AND NGUYEN HOANG THANH

ABSTRACT. The aim of this paper is to introduce a possible scheme for constructing counterexamples (if there are any) of the AR-problem for compact sets. In fact, this is an extension of Kalton-Peck-Roberts' result. We also have similar results on the AR-problem for the case of non-compact sets.

1. INTRODUCTION

Throughout most of this paper, by a linear metric space we mean a topological linear space X which is metrizable. By Kakutani's theorem (see, for instance [2]) there is an invariant metric ρ on X. We denote $\|x-y\| = \rho(x,y)$. Observe that $\|.\|$ is not a norm, in particular $\|\lambda x\| \neq |\lambda|\|x\|$.

However we assume that $\|.\|$ is monotonous, that is $\|\lambda x\| \leqslant \|x\|$ for every $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant 1$.

We recall that for $p \in (0,1)$ the linear metric space l_p is defined by $l_p = \{x = (x_n) | \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$ with metric $\rho(x,y) = \sum\limits_{n=1}^{\infty} |x_n - y_n|^p$ for each $x = (x_n), y = (y_n) \in l_p$.

For $p \in (0,1)$ the linear metric space L_p is defined by $L_p = \{f : [0,1] \to \mathbb{R} | \int\limits_0^1 |f(t)|^p dt < \infty\}$ with metric $\rho(f,g) = \int\limits_0^1 |f(t) - g(t)|^p dt$ for each $f, g \in L_p$.

The spaces l_p, L_p are non-locally convex linear metric spaces.

A topological space X is called to have the fixed point property if for every continuous map $f : X \to X$, there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.

Let X, Y be topological spaces. A continuous map $f : X \to Y$ is called compact if $f(X)$ is contained in a compact subset of Y.

A topological space X is called to have the fixed point property for compact maps if for every compact map $f : X \to X$, there exists a point $x_0 \in X$ such that $f(x_0) = x_0$.

A topological space Y is called an absolute retract whether

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(a) Y is metrizable and

(b) for any metrizable space X and any closed set $A \subset X$, each continuous map $f : A \to Y$ is extendable over X.

The class of absolute retracts is denoted by AR (see [2, 5]). We have

Theorem 1.1. (Borsuk, see [5]) *Every AR- space has the fixed point property for compact maps.*

In 1951 Dugundji proved the following theorem, see [2].

Theorem 1.2. (Dugundji) *Every convex subset of a locally convex linear metric space is an absolute retract.*

Let A be a convex subset of a linear metric space X. The set A is said to be admissible if, for every compact subset K of A, the id_K is the uniform limit of a sequence of continuous maps $f_n : K \to A$ such that each $\text{span} f_n(K)$ is finite-dimensional.

(Here $\text{span} f_n(K)$ denotes the linear subspace generated by $f_n(K)$).

Hence, a compact convex subset A of a linear metric space X is admissible if and only if id_A is the uniform limit of a sequence of continuous maps $f_n : A \to A$ such that each $\text{span} f_n(A)$ is finite-dimensional.

It can easily be proved that the admissibility is an invariant under homeomorphism (see [1]).

Every convex absolute retract in a linear metric space is admissible and every admissible convex subset of a linear metric space has the fixed point property for compact maps (see Lemma 2.1, Lemma 2.2).

The following is among the most outstanding problems in infinite dimensional topology.

Problem 1.1. (AR-problem) *Is every convex set in a linear metric space an AR?* (See [4, 7]).

Cauty has constructed a counterexample of the above problem for the case of non-compact sets (see [3]). For the case of compact sets, this problem is still open.

By Torunczyk's result in [10], Problem 1.1 for the case of compact sets becomes

Problem 1.2. *Is every infinite dimensional compact convex set in an F-space homeomorphic to the Hilbert cube.*

(Here an F-space is a complete linear metric space).

In [6], Kalton, Peck and Roberts have introduced a possible scheme for constructing counterexamples (if there are any) as follows: find a compact convex set K with finite dimensional compact convex subsets K_n and continuous maps $T_n : K_n \to K$ such that

(1) $D(K_n, K) \to 0$.

(2) If $x \in K_n$, then $\|T_n(x) - x\| \geqslant 1$.
(Here $D(K_n, K) = \sup\{d(x, K_n) | x \in K\}$ and $d(x, K_n) = \inf\{\|x - y\| | y \in K_n\}$).

Constructing of K is very difficult because before we have to construct a compact convex set K, then construct finite dimensional compact convex subsets K_n of K and finally construct continuous maps $T_n : K_n \to K$ satisfying (1), (2).

The results of this paper are

Theorem 1.3. *Let $\{K_n\}$ be a sequence of finite - dimensional compact convex subsets of an F- space X satisfying the condition: for every $n \in \mathbb{N}$, there exists a continuous map $f_n : K_n \to X$ such that $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$, $D(K_n, f_n(K_n)) \to 0$ and $\cup_{n=1}^{\infty} K_n$ is totally bounded. Then there exists a compact convex subset which is not an AR set of X.*

Theorem 1.4. *Let $\{K_n\}$ be a sequence of finite - dimensional convex subsets of an F- space X satisfying the condition: for every $n \in \mathbb{N}$, there exists a continuous map $f_n : K_n \to X$ such that $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$, $D(K_n, f_n(K_n)) \to 0$ and $\cup_{n=1}^{\infty} f_n(K_n)$ is totally bounded. Then there exists a compact convex subset which is not an AR set of X.*

It can easily be proved that if there is not a sequence $\{K_n\}$ satisfying the hypothesis of Theorem 1.3 then every compact convex subset of a linear metric space has the fixed point property.

In fact, assume on the contrary that there exist a compact convex subset K of a linear metric space X and a continuous map $f : K \to K$ such that $f(x) \neq x$ for each $x \in K$.

Replacing X by its completion , we can assume that X is an F-space.

By the compactness of K, there exists an $\epsilon_0 > 0$ such that $\|f(x) - x\| \geqslant \epsilon_0$ for each $x \in K$.

Multiplying the metric of X by a constant, we derive that there exist a compact convex subset L of a linear metric space Y and a continuous map $g : L \to L$ such that $\|g(x) - x\| \geqslant 1$. For each $n \in \mathbb{N}$, by the compactness of L, we choose a finite- dimensional compact convex subset K_n of L such that $D(K_n, L) < \frac{1}{n}$. Let $f_n : K_n \to Y$ be defined by $f_n(x) = g(x)$ for each $x \in K_n$. Then $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$, $D(K_n, f_n(K_n)) \to 0$ and $\cup_{n=1}^{\infty} K_n \subset L$. This implies that $\cup_{n=1}^{\infty} K_n$ is totally bounded.

Similarly, if there is not a sequence $\{K_n\}$ satisfying the hypothesis of Theorem 1.4 then every convex subset of a linear metric space has the fixed point property for compact maps. We can prove that in the space l_p ($p \in (0, 1)$), for every $\epsilon > 0$ there exist a finite - dimensional compact convex subset K and a continuous map $f : K \to l_p$ such that $\|f(x) - x\| \geqslant 1$ for each $x \in K$ and $D(K, f(K)) < \epsilon$ (Remark 2.1).

2. Some remarks on the AR-problem for compact sets

The main result of this section is

Theorem 2.1. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of finite - dimensional compact convex subsets of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$ there exists a continuous map $f_n : K_n \to X$ such that $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$; for each $y \in f_n(K)$ there exists an element $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} K_n$ is totally bounded. Then there exists a compact convex subset which is not an AR set of $\hat{X}$ (Here $\hat{X}$ is the completion of the linear metric space X).*

We note that Theorem 2.1 is an equivalent form of Therem 1.3.

For every $n \in \mathbb{N}$ we let $A_n = f_n(K_n)$. By using the Brouwer fixed point theorem (see [5]), for each continuous map $g_n : A_n \to K_n$ there exists an element $x \in A_n$ such that $\|g_n(x) - x\| \geqslant 1$.

Therefore this theorem is a corollary of the following theorem.

Theorem 2.2. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of finite - dimensional compact convex subsets and $\{A_n\}$ be a sequence of subsets of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$, for each continuous map $g_n : A_n \to K_n$, there exists an element $x \in A_n$ with $\|g_n(x) - x\| \geq 1$; for each $y \in A_n$ there exists $x \in K_n$ with $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} K_n$ is totally bounded. Then there exists a compact convex subset which is not an AR set of $\hat{X}$.*

Before proving this theorem, we have some remarks.

Remark 2.1. For every $p \in (0, 1)$, for every $\epsilon > 0$ there exist a finite - dimensional compact convex subset K of the space l_p and a continuous map $f : K \to l_p$ such that, for each $x \in K$ $\|f(x) - x\| \geqslant 1$ and for each $y \in f(K)$ there exists an element $x \in K$ such that $\|y - x\| < \epsilon$.

Proof. For each $n \in \mathbb{N}$, let $a = n^{2-\frac{1}{p}}$ and

$$\begin{aligned}
e_1 &= (a, 0, 0, \ldots, 0, 0, \ldots) \in l_p, \\
e_2 &= (0, a, 0, \ldots, 0, 0, \ldots) \in l_p, ..., \\
e_n &= (0, 0, 0, \ldots, a, 0, \ldots) \in l_p, \\
p_n &= (\underbrace{\frac{a}{n^2}, \frac{a}{n^2}, \frac{a}{n^2}, \ldots, \frac{a}{n^2}}_{n \text{ times}}, 0, \ldots) \in l_p, \\
K_n &= \operatorname{conv}\{e_1, e_2, \ldots, e_n\},
\end{aligned}$$

we have $\|p_n\| = 1$.

Let $f : K_n \to l_p$ be defined by $f(x) = x + p_n$ for each $x \in K_n$.

Then f is a continuous map and $\|f(x) - x\| = \|p_n\| = 1$ for each $x \in K_n$.

For each $x + p_n \in f(K_n)$ there exist the numbers $\alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \ldots, \alpha_n \geqslant 0$, $\sum_{i=1}^{n} \alpha_i = 1$ such that $x = \sum_{i=1}^{n} \alpha_i e_i = (a\alpha_1, a\alpha_2, \ldots, a\alpha_n, 0, \ldots)$. Thus $x + p_n = (a\alpha_1 + \frac{a}{n^2}, a\alpha_2 + \frac{a}{n^2}, \ldots, a\alpha_n + \frac{a}{n^2}, 0, \ldots)$.

Without loss of generality we can assume $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_n$; therefore $\alpha_1 \geqslant \frac{1}{n}$. We have

$$a\alpha_2 + \frac{a}{n^2} + \ldots + a\alpha_n + \frac{a}{n^2} = a(1-\alpha_1) + \frac{a(n-1)}{n^2} = a(\frac{n^2+n-1}{n^2} - \alpha_1)$$
$$\leqslant a(\frac{n^2+n-1}{n^2} - \frac{1}{n}) = a(\frac{n^2-1}{n^2}) \leqslant a.$$

Hence, if we put

$$y = \big(a - (a\alpha_2 + \ldots + a\alpha_n + \frac{a(n-1)}{n^2}), a\alpha_2 + \frac{a}{n^2}, \ldots, a\alpha_n + \frac{a}{n^2}, 0, \ldots\big),$$

then $y \in \operatorname{conv}\{e_1, e_2, \ldots, e_n\}$ and

$$\|x + p_n - y\| = \|\big(a\alpha_1 + \frac{a}{n^2} - a + (a\alpha_2 + \ldots + a\alpha_n + \frac{a(n-1)}{n^2}), 0, 0, \ldots\big)\|$$
$$= \|\big(\frac{a}{n^2} + \frac{a(n-1)}{n^2}, 0, 0, \ldots\big)\| = \|(\frac{a}{n}, 0, 0, \ldots)\|$$
$$= \frac{a^p}{n^p} = \frac{n^{2p-1}}{n^p} = \frac{1}{n^{1-p}}.$$

It follows that there exists a number $n_0 \in \mathbb{N}$ such that $\frac{1}{n^{1-p}} < \epsilon$ for every $n \geqslant n_0$.

Thus, for every $\epsilon > 0$, there exist a finite - dimensional compact convex subset K (namely, K_{n_0}) of the space l_p and a continuous map $f : K \to l_p$ such that $\|f(x) - x\| \geqslant 1$ for every $x \in K$ and for all $y \in f(K)$ there exists an element $x \in K$ such that $\|y - x\| < \epsilon$. □

By using the Brouwer fixed point theorem, we have

Remark 2.2. For every $p \in (0,1)$, for each $\epsilon > 0$, there exist a finite - dimensional compact convex subset K of the space l_p and a subset A of the space l_p such that for each $x \in A$ there exists an $y \in K$ with $\|x - y\| < \epsilon$ and for each continuous map $f : A \to K$ there exists an element $x \in A$ with $\|f(x) - x\| \geqslant 1$.

Applying similar arguments to needle point spaces (see [7, 8, 9]), we have

Remark 2.3. Let X be an arbitrary needle point space with an element $x_0 \in X$ such that $\|x_0\| \geqslant 1$. Then for each $\epsilon > 0$, there exist a finite - dimensional compact convex subset K of X and a continuous map $f : K \to X$ such that for every $x \in K$ $\|f(x) - x\| \geqslant 1$ and for each $y \in f(K)$ there exists an element $x \in K$ such that $\|y - x\| < \epsilon$.

Remark 2.4. Let X be an arbitrary needle point space with an element $x_0 \in X$ such that $\|x_0\| \geqslant 1$. Then for each $\epsilon > 0$, there exist a finite - dimensional compact convex subset K of X and a subset A of X such that for every $x \in A$ there exists an element $y \in K$ with $\|y - x\| < \epsilon$ and for every continuous map $f : A \to K$ there exists an element $x \in A$ with $\|f(x) - x\| \geqslant 1$.

Let $\{L_n\}$ be a sequence of nonempty subsets in a complete linear metric space X, we denote by

$\lim^{-} L_n$ the set of all elements x of X such that for each $n \in \mathbb{N}$ there exists an element $x_n \in L_n$ satisfying $x = \lim_{n\to\infty} x_n$.

$\lim^{+} L_n$ the set of all elements x of X such that there exists a subsequence $\{m_n\}$ of the sequence $\{n\}$ such that for each $n \in \mathbb{N}$ there exists $x_{m_n} \in L_{m_n}$ satisfying $x = \lim_{n\to\infty} x_{m_n}$.

Note that $\lim^{-} L_n \subset \lim^{+} L_n$ and $\lim^{+} L_n$ is a closed subset of X.

Lemma 2.1. *Every admissible convex set has the fixed point property for compact maps.*

Proof. Let A be an admissible convex subset of an arbitrary linear metric space (X, d). Suppose A has not the fixed point property for compact maps, then there exists a compact map $f : A \to A$ and $f(x) \neq x$, for each $x \in A$. Since f is a compact map, then there exists $\epsilon_0 > 0$ such that $d(x, f(x)) \geqslant \epsilon_0$ for each $x \in A$. Let K be a compact subset of A such that $f(A) \subset K$. Since A is admissible, there exists a continuous map $g : K \to A$ such that $d(x, g(x)) < \epsilon_0$ for each $x \in K$ and $g(K)$ is contained in a finite - dimensional linear subspace L of X. Therefore $g(K) \subset L \cap A$ and $L \cap A$ is a convex set in the finite dimensional linear metric space L.

Consider $g \circ f|_{L\cap A} : L \cap A \to L \cap A$. We know that every finite - dimensional linear metric space is a locally convex linear metric space, hence $L \cap A$ is an AR (Dugundji Theorem). Thus $L \cap A$ has the fixed point property for compact maps and $g \circ f|_{L\cap A}(L \cap A) \subset g(K)$, $g(K)$ is a compact set. Therefore there exists $x_0 \in L \cap A$ such that $g \circ f(x_0) = x_0$. Thus $d(g(f(x_0)), f(x_0)) < \epsilon_0$, hence $d(x_0, f(x_0)) < \epsilon_0$. This contradicts our assumption. □

Lemma 2.2. *Every convex absolute retract in a linear metric space is admissible.*

Proof. Let A be an AR convex subset of an arbitrary linear metric space (X, d). By Arens - Eells Theorem (see [2]), there exists $\varphi : A \to A'$ which is an isometric embedding of A in a closed set A' of a normed linear space E.

Let K be compact in A. We denote $K' = \varphi(K)$, then K' is compact in A', let $r : E \to A'$ be a retraction.

Let $\epsilon, \delta > 0$. Because E is a normed linear space, then E is admissible (using Schauder projection, see [5]). Therefore there exists a continuous map $f : K' \to E$ such that $f(K')$ is contained in a finite dimensional linear subspace of E and $\|f(x) - x\| < \delta$ for each $x \in K'$.

Consider $\varphi^{-1} \circ r|_{f(K')} : f(K') \to A$, since $f(K')$ is a finite dimensional compact set and A is convex then there exists a continuous map $g : f(K') \to A$ such that $\|g(x) - \varphi^{-1} \circ r(x)\| < \delta$, for each $x \in f(K')$ and $g \circ f(K')$ is contained in a finite - dimensional linear subspace of X.

Note that $g \circ f \circ \varphi|_K : K \to A$ is a map with image contained in a finite dimensional linear subspace of X and for every $x \in K$,

$$\begin{aligned}
&\|g \circ f \circ \varphi|_K(x) - x\| \\
&\leqslant \|g \circ f \circ \varphi|_K(x) - \varphi^{-1} \circ r \circ f \circ \varphi(x)\| + \|\varphi^{-1} \circ r \circ f \circ \varphi(x) - x\| \\
&\leqslant \delta + \|\varphi^{-1} \circ r \circ f \circ \varphi(x) - \varphi^{-1} \circ \varphi(x)\| \\
&= \delta + \|r \circ f \circ \varphi(x) - \varphi(x)\| \\
&= \delta + \|r \circ f \circ \varphi(x) - r \circ \varphi(x)\|.
\end{aligned}$$

We will show that there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, $\|r \circ f \circ \varphi(x) - r \circ \varphi(x)\| < \frac{\epsilon}{2}$ (*).

Suppose this claim is false, then for every $n \in \mathbb{N}$ there exist a continuous map $f_n : K' \to E$ and a sequence $\{y_n\} \in K'$ such that $\|r \circ f_n(y_n) - r(y_n)\| \geqslant \frac{\epsilon}{2}$ and $\|f_n(y_n) - y_n\| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Since K' is a compact set there is no loss generality in assuming that there exists $y_0 \in K'$ such that $y_n \to y_0$. Thus $f_n(y_n) \to y_0 \in K'$, hence $r \circ f_n(y_n) \to r(y_0)$ since $y_n \to y_0$ implies $r(y_n) \to r(y_0)$. This contradiction proves the claim (*).

Choose δ sufficiently small and less than $\frac{\epsilon}{2}$, then the proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let $\{K_n\}$ be a sequence of non-empty subsets in a complete linear metric space X such that $\cup_{n=1}^{\infty} K_n$ is totally bounded. Then there exists a subsequence $\{K_{nn}\}$ of $\{K_n\}$ such that $\lim^+ K_n = cl \lim^- K_n$ (where clA denotes the closure of A in X).*

Proof. Since $\cup_{n=1}^{\infty} K_n$ is totally bounded and X is complete, then $\lim^+ K_n \neq \emptyset$. Because $\lim^+ K_n \subset cl(\cup_{n=1}^{\infty} K_n)$, $\lim^+ K_n$ is compact.

Choose $z^1 \in \lim^+ K_n$ and a subsequence $\{K_{1n}\}$ of $\{K_n\}$ such that $z^1 \in \lim^- K_{1n}$.

Choose $z^2 \in \lim^+ K_{1n}$ such that $\|z^1 - z^2\| \geqslant 2^{-1}$ (if there exists) and choose a subsequence $\{K_{2n}\}$ of $\{K_{1n}\}$ such that $z^2 \in \lim^- K_{2n}$.

Next, we choose $z^3 \in \lim^+ K_{2n}$ such that $\|z^1 - z^3\| \geqslant 2^{-1}$, $\|z^2 - z^3\| \geqslant 2^{-1}$, (if there exists) and choose a subsequence $\{K_{3n}\}$ of $\{K_{2n}\}$ such that $z^3 \in \lim^- K_{3n}$.

This process has to stop because $\lim^+ K_n \supset \lim^+ K_{1n} \supset \lim^+ K_{2n} \supset \ldots$ and $\lim^+ K_n$ is totally bounded.

Suppose we find $z^1, z^2, \ldots, z^{p_1}$ and a subsequence $\{K_{p_1 n}\}$, we choose $z^{p_1+1} \in \lim^+ K_{p_1 n}$ and $\|z^1 - z^{p_1+1}\| \geqslant 2^{-2}$, $\|z^2 - z^{p_1+1}\| \geqslant 2^{-2}$,...,$\|z^{p_1} - z^{p_1+1}\| \geqslant 2^{-2}$, (if there exists).

Then, we choose a subsequence $\{K_{(p_1+1)n}\}$ of $\{K_{p_1 n}\}$ such that

$$z^{p_1+1} \in \lim{}^- K_{(p_1+1)n}.$$

Continue this process we obtain $p_1 + 1, \ldots, p_2, p_2 + 1, \ldots, p_3, \ldots, p_n, \ldots$.

We consider the two following cases:
(1) The above process stops at a point z^m, then there is no point different from $z^1, z^2, \dots, z^m$ in $\lim^+ K_{mn}$.

We have $z^1 \in \lim^- K_{1n}$, $z^2 \in \lim^- K_{2n}$,...,$z^m \in \lim^- K_{mn}$ and $\lim^- K_{1n} \subset \lim^- K_{2n} \subset \dots \subset \lim^- K_{mn}$.

Since $z^1, z^2, \dots, z^m \in \lim^- K_{mn}$, we have $\mathrm{conv}\{z^1, z^2, \dots, z^m\} \subset \lim^- K_{mn}$ (by the convexity of $\lim^- K_{mn}$). Therefore $\mathrm{conv}\{z^1, z^2, \dots, z^m\} \subset \lim^- K_{mn} \subset \lim^+ K_{mn}$.

Thus $\mathrm{conv}\{z^1, z^2, \dots, z^m\}$ is a singleton; $\lim^+ K_{1n} = \{z^1\}$ and $\lim^- K_{1n} = \{z^1\}$. So $\lim^+ K_{1n} \subset cl(\lim^- K_{1n})$ and $\{K_{1n}\}$ is the sequence which we need.
(2) The above process does not stop. Consider the diagonal sequence $K_{11}, K_{22}, \dots, K_{nn}, \dots$ Now we have to prove that $\lim^+ K_{nn} = cl(\lim^- K_{nn})$, i.e. $\lim^+ K_{nn} \subset cl(\lim^- K_{nn})$. For each $y \in \lim^+ K_{nn}$, for each $m \in \mathbb{N}$ we have $y \in \lim^+ K_{nn} \subset \lim^+ K_{p_m n}$.

By the definition of p_m , there exists $z \in \{z^1, z^2, \dots, z^{p_m}\}$ such that $\|y - z\| < 2^{-m}$.

We have $z^1 \in \lim^- K_{1n}$, $z^2 \in \lim^- K_{2n}$,...,$z^{p_m} \in \lim^- K_{p_m n}$ and $\lim^- K_{1n} \subset \lim^- K_{2n} \subset \dots \subset \lim^- K_{p_m n}$.
It follows that $z \in \lim^- K_{p_m n} \subset \lim^- K_{nn} \subset cl(\lim^- K_{nn})$, hence $y \in cl(\lim^- K_{nn})$. Thus $\lim^+ K_{nn} = cl(\lim^- K_{nn})$. □

Proof of Theorem 2.2. Without loss of generality we can assume X is a complete linear metric space and by Lemma 2.3, we can assume that $\lim^+ K_n = cl(\lim^- K_n)$.

Let $K = \lim^+ K_n = cl(\lim^- K_n)$. Thus K is a compact convex subset of X. Assumse on the contrary that K is an AR. Let $r : X \to K$ be a retraction. Because K is compact then there exists $\delta_0 > 0$ such that
(1) for each $x \in X$ with $d(x, K) < \delta_0$ we have $\|r(x) - x\| < \frac{1}{4}$, where $d(x, K) = \inf\{d(x, a) | a \in K\}$.
We have
(2) there exists $n_0 \in \mathbb{N}$ such that $\sup\{d(x, K) | x \in K_n\} < \frac{\delta_0}{2}$ for each $n > n_0$.
In fact, if (2) does not hold then there exists a subsequence $\{m_n\}$ of $\{n\}$ such that for each $n \in \mathbb{N}$, $\sup\{d(x, K) | x \in K_{m_n}\} \geqslant \frac{\delta_0}{2}$. Thus for each $n \in \mathbb{N}$, there exists $x_{m_n} \in K_{m_n}$ such that $d(x_{m_n}, K) \geqslant \frac{\delta_0}{4}$.

Since $\cup_{n=1}^{\infty} K_n$ is totally bounded then we can assumse that $\{x_{m_n}\}$ converges to an element $x \in X$, hence $x \in K = \lim^+ K_n$.

Since $d(x_{m_n}, K) \geqslant \frac{\delta_0}{4}$ for every $n \in \mathbb{N}$, we get $d(x, K) \geqslant \frac{\delta_0}{4}$, a contradiction.

Because $\lim\limits_{n \to \infty} \epsilon_n = 0$ then
(3) There exists $n_1 \in \mathbb{N}$ such that $\sup\{d(x, K_n) | x \in A_n\} < \frac{\delta_0}{2}$ for each $n > n_1$.
Since K is a convex AR set then K is admissible (Lemma 2.2), therefore there exist a finite dimensional compact convex set $K_0 \subset K$ and a continuous map
(4) $r^+ : K \to K_0$ such that $\|r^+(x) - x\| < \frac{1}{4}$ for each $x \in K$.

Because $K_0 \subset K = cl(\lim^- K_n)$ and K_0 is a finite dimensional compact convex set then there exist $n > \max\{n_0, n_1\}$ and a continuous map
(5) $h : K_0 \to K_n$ such that $\|h(x) - x\| < \frac{1}{4}$ for each $x \in K_0$.
Let $u : A_n \to K_n$ be the restriction of the map $h \circ r^+ \circ r$ on A_n. Then for each $x \in A_n$, $\|u(x) - x\| = \|h \circ r^+ \circ r(x) - x\| \leqslant \|h \circ r^+ \circ r(x) - r^+ \circ r(x)\|$ $+\|r^+ \circ r(x) - r(x)\| + \|r(x) - x\|$. Therefore $\|u(x) - x\| < \frac{1}{4} + \frac{1}{4} + \|r(x) - x\|$.

Because $x \in A_n$ then $d(x, K_n) < \frac{\delta_0}{2}$. Hence there exists an $y \in K_n$ such that $d(x, y) < \frac{\delta_0}{2}$.

By (2), we have $d(y, K) < \frac{\delta_0}{2}$. Thus $d(x, K) \leqslant d(x, y) + d(y, K) < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0$. This implies that $\|r(x) - x\| < \frac{1}{4}$ hence $\|u(x) - x\| < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$. This contradics the assumption on g_n in the statement of Theorem 2.2. □

3. Some remarks on the AR-problem for non-compact sets

In this section we will extend the results of Section 2 for the case of non-compact sets.

The main result of this section is

Theorem 3.1. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of finite - dimensional convex subsets of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$ there exists a continuous map $f_n : K_n \to X$ such that $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$; for each $y \in f_n(K_n)$ there exists $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} f_n(K_n)$ is totally bounded. Then there exists a convex subset which is not an AR set of $\hat{X}$ (Here $\hat{X}$ is the completion of the linear metric space X).*

We note that Theorem 3.1 is an equivalent form of Theorem 1.1.

Without loss of generality we can assume X is a complete linear metric space.

For every $n \in \mathbb{N}$ let $A_n = cl f_n(K_n)$, we see that $\sup\{d(x, K_n) | x \in f_n(K_n)\} = \sup\{d(x, K_n) | x \in cl f_n(K_n)\}$ and A_n is compact. By using the fixed point theorem of Borsuk (Theorem 1.1 (also see [5])), for each continuous map $g_n : A_n \to K_n$ there exists an element $x \in A_n$ such that $\|g_n(x) - x\| \geqslant 1$.

Therefore this theorem is a corollary of the following theorem.

Theorem 3.2. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of convex subsets and $\{A_n\}$ be a sequence of subsets of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$, for each continuous map $g_n : A_n \to K_n$, there exists $x \in A_n$ such that $\|g_n(x) - x\| \geqslant 1$; for every $y \in A_n$ there exists $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} A_n$ is totally bounded. Then there exists a convex subset which is not an AR set of $\hat{X}$.*

Before proving this theorem we have to prove the following lemma:

Lemma 3.1. *Let $\{K_n\}$ be a sequence of convex subsets and $\{A_n\}$ be a sequence of subsets of a complete linear metric space X satisfying the conditions: for* *every $n \in \mathbb{N}$, for every $y \in A_n$ there exists $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} A_n$ is totally bounded. Then there exists a subsequence $\{nn\}$ of $\{n\}$ such that $\lim^+ A_{nn} \subset cl \lim^- K_{nn}$.*

Proof. Since $\cup_{n=1}^{\infty} A_n$ is totally bounded and X is complete, then $\lim^+ A_n \neq \emptyset$. Because $\lim^+ A_n \subset cl \cup_{n=1}^{\infty} A_n$, $\lim^+ A_n$ is compact.
Choose $z^1 \in \lim^+ A_n \subset \lim^+ K_n$ and a subsequence $\{K_{1n}\}$ of $\{K_n\}$ such that $z^1 \in \lim^- K_{1n}$, choose $z^2 \in \lim^+ A_{1n} \subset \lim^+ K_{1n}$ such that $\|z^1 - z^2\| \geqslant 2^{-1}$ (if there exists) and choose a subsequence $\{K_{2n}\}$ of $\{K_{1n}\}$ such that $z^2 \in \lim^- K_{2n}$.

Next, we choose $z^3 \in \lim^+ A_{2n} \subset \lim^+ K_{2n}$ such that $\|z^1 - z^3\| \geqslant 2^{-1}$, $\|z^2 - z^3\| \geqslant 2^{-1}$, (if there exists) and choose a subsequence $\{K_{3n}\}$ of $\{K_{2n}\}$ such that $z^3 \in \lim^- K_{3n}$.

This process has to stop because $\lim^+ A_n \supset \lim^+ A_{1n} \supset \lim^+ A_{2n} \supset \dots$ and $\lim^+ A_n$ is totally bounded.

Suppose we find $z^1, z^2, \dots, z^{p_1}$ and a subsequence $\{A_{p_1 n}\}$, we choose $z^{p_1+1} \in \lim^+ A_{p_1 n} \subset \lim^+ K_{p_1 n}$ and $\|z^1 - z^{p_1+1}\| \geqslant 2^{-2}$, $\|z^2 - z^{p_1+1}\| \geqslant 2^{-2}$,...,$\|z^{p_1} - z^{p_1+1}\| \geqslant 2^{-2}$, (if there exists).

Now, we choose a subsequence $\{K_{(p_1+1)n}\}$ of $\{K_{p_1 n}\}$ such that

$$z^{p_1+1} \in \lim{}^- K_{(p_1+1)n}.$$

Continue this process we obtain $p_1 + 1, \dots, p_2, p_2 + 1, \dots, p_3, \dots, p_n, \dots$.

We consider two following cases:
(a) The above process stops at a point z^m. Then there is no point different from $z^1, z^2, \dots, z^m$ in $\lim^+ A_{mn}$. We have $z^1 \in \lim^- K_{1n}$, $z^2 \in \lim^- K_{2n}, \dots, z^m \in \lim^- K_{mn}$ and $\lim^- K_{1n} \subset \lim^- K_{2n} \subset \dots \subset \lim^- K_{mn}$. So $z^1, z^2, \dots, z^m \in \lim^- K_{mn}$. Thus $\lim^+ A_{mn} \subset \lim^+ K_{mn} \subset cl \lim^- K_{mn}$
(b) The above process does not stop. Consider the diagonal sequence $K_{11}, K_{22}, \dots, K_{nn}, \dots$ and we have to prove that $\lim^+ A_{nn} \subset cl \lim^- K_{nn}$.

For each $y \in \lim^+ A_{nn}$ and for each $\epsilon > 0$, choose $m \in \mathbb{N}$ such that $2^{-m} < \epsilon$, $y \in \lim^+ A_{nn} \subset \lim^+ A_{p_m n}$.

By the definition of p_m, there exists $z \in \{z^1, z^2, \dots, z^{p_m}\}$ such that $\|y - z\| < 2^{-m}$.

We see $z^1 \in \lim^- K_{1n}$, $z^2 \in \lim^- K_{2n}$,...,$z^{p_m} \in \lim^- K_{p_m n}$ and $\lim^- K_{1n} \subset \lim^- K_{2n} \subset \dots \subset \lim^- K_{p_m n}$. So $z \in \lim^- K_{p_m n} \subset \lim^- K_{nn} \subset cl \lim^- K_{nn}$. Thus $y \in cl \lim^- K_{nn}$ hence $\lim^+ A_{nn} \subset cl \lim^- K_{nn}$. □

Proof of Theorem 3.2. Without loss of generality we can assume X is a complete linear metric space and by Lemma 3.1, we can assume that $\lim^+ A_n \subset cl \lim^- K_n$.

Let $K = cl \lim^- K_n$. We see that K is a closed convex subset of X. Assume on the contrary that K is an AR. Let $r : X \to K$ be a retraction. We will show that for every $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for each $n > N_0$, for each $x_n \in A_n$, $\|r(x_n) - x_n\| < \delta$ (*).

In fact, assume on the contrary that there exist $\delta_0 > 0$ and a subsequence $\{m_n\}$ of $\{n\}$ such that there exists $x_{m_n} \in A_{m_n}$ with $\|r(x_{m_n}) - x_{m_n}\| \geqslant \delta_0$.

Since $\cup_{n=1}^{\infty} K_n$ is totally bounded then we can assumse that $\{x_{m_n}\}$ converges to an element $x \in X$.

Since $\lim_{n\to\infty} x_{m_n} = x$ we have $\lim_{n\to\infty} r(x_{m_n}) = r(x) = x = \lim_{n\to\infty} x_{m_n}$. This contradits $\|r(x_{m_n}) - x_{m_n}\| \geqslant \delta_0$, for each $n \in \mathbb{N}$.

Thus

(1) There exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, for each $x \in A_n$, $\|r(x) - x\| < \frac{1}{4}$.
We see that $r(cl \cup_{n=1}^{\infty} A_n) \subset K$, $r(cl \cup_{n=1}^{\infty} A_n)$ is compact, K is an AR. By Lemma 2.2, K is admissible. Therefore there exist a finite dimensional convex set $H \subset K$ and a continuous map $\gamma : r(cl \cup_{n=1}^{\infty} A_n) \to H$ such that
(2) $\|\gamma(x) - x\| < \frac{1}{4}$ for all $x \in r(cl \cup_{n=1}^{\infty} A_n)$.
By Mazur's Lemma and the closedness of K, we can assume that H is compact.

Because $K = cl \lim^{-} K_n$ and H is a finite dimensional compact convex subset of K then there exist n sufficiently large and a continuous map

$\xi_n : H \to K_n$ such that
(3) $\|\xi_n(x) - x\| < \frac{1}{4}$ for each $x \in H$.
Consider the composite map $\xi_n \circ \gamma \circ r|_{A_n} : A_n \to K_n$. Then for each $x \in A_n$, from (1), (2), (3) we have $\|\xi_n \circ \gamma \circ r(x) - x\| \leqslant \|\xi_n \circ \gamma \circ r(x) - \gamma \circ r(x)\| + \|\gamma \circ r(x) - r(x)\| + \|r(x) - x\| < \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < 1$.

This is a contradiction to our assumption. □

As an immediate consequence, we obtain

Corollary 3.1. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of finite - dimensional subspaces of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$ there exists a continuous map $f_n : K_n \to X$ such that $\|f_n(x) - x\| \geqslant 1$ for each $x \in K_n$; for every $y \in f_n(K_n)$ there exists $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} f_n(K_n)$ is totally bounded. Then there exists a linear subspace which is not an AR set of $\hat{X}$ (where $\hat{X}$ is the completion of the linear metric space X).*

Theorem 3.3. *Let $\{\epsilon_n\}$ be an arbitrary sequence of positive numbers which tends to zero, $\{K_n\}$ be a sequence of linear subspaces and $\{A_n\}$ be a sequence of subsets of a linear metric space X satisfying the condition: for every $n \in \mathbb{N}$, for each continuous map $g_n : A_n \to K_n$, there exists $x \in A_n$ such that $\|g_n(x) - x\| \geqslant 1$; for every $y \in A_n$ there exists $x \in K_n$ such that $\|y - x\| < \epsilon_n$ and $\cup_{n=1}^{\infty} A_n$ is totally bounded. Then there exists a linear subspace which is not an AR set of $\hat{X}$.*

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Faculty of Mathematics
College of Education
University of Danang, Vietnam

E-mail address: lhtri@dce.udn.vn

E-mail address: nh_thanh@ud.edu.vn