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A NOTE ON THE FINITENESS PROPERTY RELATED TO DERIVED FUNCTORS

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and M, N two finitely generated R-modules. Let t be a non-negative integer. It is shown that for any finitely generated R-module L with $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$, the following statements hold:
(i) $\operatorname{Supp}(\operatorname{Ext}^t_R(L,N)) \subseteq \cup_{i=0}^{t} \operatorname{Supp}(\operatorname{Ext}^i_R(M,N))$;
(ii) $\operatorname{Ass}(\operatorname{Ext}^t_R(L,N)) \subseteq \operatorname{Ass}(\operatorname{Ext}^t_R(M,N)) \cup (\cup_{i=0}^{t-1} \operatorname{Supp}(\operatorname{Ext}^i_R(M,N)))$.

As an immediate consequence, we deduce that if $\operatorname{Supp}(H^i_{\mathfrak{a}}(N))$ or $\operatorname{Supp}(H^i_{\mathfrak{a}}(M,N))$ is finite for all $i < t$, then the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^t_R(M/\mathfrak{a}^n M,N))$ is finite. In particular, if $\operatorname{grade}(\mathfrak{a},N) \geq t$ then the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^t_R(M/\mathfrak{a}^n M,N))$ is finite.

1. Introduction

Throughout this note, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, and that M and N two finitely generated R-modules. We use $\mathbb{N}$ to denote the set of positive integers.
Brodmann [1] proved that the two sequences of associated primes $(\operatorname{Ass}(N/\mathfrak{a}^n N))_{n\in\mathbb{N}}$ and $(\operatorname{Ass}(\mathfrak{a}^n N/\mathfrak{a}^{n+1}N))_{n\in\mathbb{N}}$ eventually become constant for large n. Melkersson and Schenzel [10] showed that, for any given integer $i \geq 0$, the sequences

$$(\operatorname{Ass}(\operatorname{Tor}_i^R(R/\mathfrak{a}^n,N)))_{n\in\mathbb{N}} \quad \text{and} \quad (\operatorname{Att}(\operatorname{Ext}^i_R(R/\mathfrak{a}^n,A)))_{n\in\mathbb{N}}$$

become, for n large, independent of n where A is an Artinian R-module. They also asked whether the set $\operatorname{Ass}(\operatorname{Ext}^i_R(R/\mathfrak{a}^n,N))$ becomes stable for sufficiently large n. Khashyarmanesh and Salarian [7], gave an affirmative answer to the above question in the case $i = 1$. Katzman [5] gave an example of a Noetherian local ring $(R,\mathfrak{m})$ with two elements $x, y \in \mathfrak{m}$ such that the associated prime ideals of local cohomology module $H^2_{(x,y)}(R)$ is an infinite set. Therefore the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^2_R(R/(x,y)^n,R))$ is infinite and so $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^i_R(R/\mathfrak{a}^n,N))$ is not a finite set in general.

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For an integer $i \geq 0$, the i-th generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ of two R-modules M and N with respect to an ideal $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

It is clear that $H^i_{\mathfrak{a}}(R,N)$ is just the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$. We refer the reader to [2] and [4] for the basic properties of local cohomology and generalized local cohomology.

The aim of this note is to prove the following theorems.

Theorem 1.1. *Let t be a non-negative integer. Then for any finitely generated module L with* $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$*, the following statements hold:*
(i) $\operatorname{Supp}(\operatorname{Ext}^t_R(L,N)) \subseteq \cup_{i=0}^t \operatorname{Supp}(\operatorname{Ext}^i_R(M,N))$;
(ii) $\operatorname{Ass}(\operatorname{Ext}^t_R(L,N)) \subseteq \operatorname{Ass}(\operatorname{Ext}^t_R(M,N)) \cup (\cup_{i=0}^{t-1} \operatorname{Supp}(\operatorname{Ext}^i_R(M,N)))$.

Theorem 1.2. *Let t be a non-negative integer such that* $\operatorname{Supp}(H^i_{\mathfrak{a}}(N))$ *or* $\operatorname{Supp}(H^i_{\mathfrak{a}}(M,N))$ *is finite for all $i < t$. Then the set* $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^t_R(M/\mathfrak{a}^n M,N))$ *is finite. In particular, if* $\operatorname{grade}(\mathfrak{a},N) \geq t$ *then* $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^t_R(M/\mathfrak{a}^n M,N))$ *is finite.*

2. The results

Theorem 2.1. *Let t be a non-negative integer. Then for any finitely generated module L with* $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$*, the following statements hold:*
(i) $\operatorname{Supp}(\operatorname{Ext}^t_R(L,N)) \subseteq \cup_{i=0}^t \operatorname{Supp}(\operatorname{Ext}^i_R(M,N))$;
(ii) $\operatorname{Ass}_R(\operatorname{Ext}^t_R(L,N)) \subseteq \operatorname{Ass}(\operatorname{Ext}^t_R(M,N)) \cup (\cup_{i=0}^{t-1} \operatorname{Supp}(\operatorname{Ext}^i_R(M,N)))$.

Proof. We will only prove the first part, and the proof of the second part is similar. We use induction on t. Let $t = 0$. Since $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$, we get by Gruson's Theorem [11, Theorem 4.1] that there exists a finite filtration

$$0 = L_0 \subset L_1 \subset \ldots \subset L_k = L$$

such that the factors L_i/L_{i-1} are homomorphic images of a direct sum of finitely copies of M. By using short exact sequences, we may reduce the situation to the case $k = 1$. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow M^m \longrightarrow L \longrightarrow 0,$$

for some $m \in \mathbb{N}$ and some finitely generated R-module K. This induces an exact sequence $0 \longrightarrow \operatorname{Hom}_R(L,N) \longrightarrow \operatorname{Hom}_R(M^m,N)$ and so the result for $t = 0$ is complete. Now suppose, inductively, that $t > 0$ and we have established that $\operatorname{Supp}(\operatorname{Ext}^j_R(L,N)) \subseteq \cup_{i=0}^j \operatorname{Supp}(\operatorname{Ext}^i_R(M,N))$ for all $j < t$ and all finitely generated modules L with $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$. Again, by using Gruson's Theorem, we have an exact sequence $0 \longrightarrow K \longrightarrow M^m \longrightarrow L \longrightarrow 0$ that induces a long exact sequence

$$\ldots \longrightarrow \operatorname{Ext}^{t-1}_R(K,N) \longrightarrow \operatorname{Ext}^t_R(L,N) \longrightarrow \operatorname{Ext}^t_R(M^m,N) \longrightarrow \ldots.$$

Therefore $\operatorname{Supp}(\operatorname{Ext}_R^t(L,N)) \subseteq \operatorname{Supp}(\operatorname{Ext}_R^t(M,N)) \cup \operatorname{Supp}(\operatorname{Ext}_R^{t-1}(K,N))$ and so by inductive hypothesis the result follows. □

The following corollary immediately follows by Theorem 2.1.

Corollary 2.2. *Let t be a non-negative integer. Then for any finitely generated module L with $\operatorname{Supp}(L) = \operatorname{Supp}(M)$, the following statements hold:*
(i) $\cup_{i=0}^t \operatorname{Supp}(\operatorname{Ext}_R^i(L,N)) = \cup_{i=0}^t \operatorname{Supp}(\operatorname{Ext}_R^i(M,N))$;
(ii)

$$\begin{aligned}&\operatorname{Ass}(\operatorname{Ext}_R^t(L,N)) \cup (\cup_{i=0}^{t-1} \operatorname{Supp}(\operatorname{Ext}_R^i(M,N)))\\ &= \operatorname{Ass}(\operatorname{Ext}_R^t(M,N)) \cup (\cup_{i=0}^{t-1} \operatorname{Supp}(\operatorname{Ext}_R^i(M,N))).\end{aligned}$$

Corollary 2.3. *Let t be a non-negative integer. Then the following equalities holds.*

$$\begin{aligned}\cup_{i=0}^t(\cup_{n\in\mathbb{N}} \operatorname{Supp}(\operatorname{Ext}_R^i(M/\mathfrak{a}^n M,N))) &= \cup_{i=0}^t \operatorname{Supp}(\operatorname{Ext}_R^i(M/\mathfrak{a}M,N))\\ &= \cup_{i=0}^t \operatorname{Supp}(H_{\mathfrak{a}}^i(M,N)).\end{aligned}$$

Proof. The first equality follows from Corollary 2.2 and the second equality follows from [3, Lemma 2.8]. □

The following theorem extends [6, Theorem 2.4], [8, Theorem B] and [9, Theorem 2.12].

Theorem 2.4. *Let t be a non-negative integer. Then the following statements hold:*
(i) *If $\operatorname{Supp}(H_{\mathfrak{a}}^i(N))$ is finite for all $i < t$, then the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^t(M/\mathfrak{a}^n M,N))$ is finite. In particular, $\operatorname{Ass}(H_{\mathfrak{a}}^t(N))$ and $\operatorname{Ass}(H_{\mathfrak{a}}^t(M,N))$ are finite.*
(ii) *If $\operatorname{Supp}(H_{\mathfrak{a}}^i(M,N))$ is finite for all $i < t$, then the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^t(M/\mathfrak{a}^n M, N))$ is finite. In particular, $\operatorname{Ass}(H_{\mathfrak{a}}^t(M,N))$ is finite.*

Proof. Apply Theorem 2.1 and Corollary 2.3. □

The following corollary immediately follows by Theorem 2.4.

Corollary 2.5. *Let $\operatorname{grade}(\mathfrak{a},N) \geq t$. Then the set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^t(M/\mathfrak{a}^n M,N))$ is finite. In particular, $\operatorname{Ass}(H_{\mathfrak{a}}^t(N))$ and $\operatorname{Ass}(H_{\mathfrak{a}}^t(M,N))$ are finite.*

The following result extends [7, Corollary 2.3].

Corollary 2.6. *The set $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^1(M/\mathfrak{a}^n M,N))$ is finite. In particular, $\operatorname{Ass}(H_{\mathfrak{a}}^1(N))$ and $\operatorname{Ass}(H_{\mathfrak{a}}^1(M,N))$ are finite.*

Proof. The exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0$ provides an exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1(M/\mathfrak{a}^n M,\Gamma_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Ext}_R^1(M/\mathfrak{a}^n M,N) \longrightarrow \operatorname{Ext}_R^1(M/\mathfrak{a}^n M,N/\Gamma_{\mathfrak{a}}(N)).$$

Thus $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^1(M/\mathfrak{a}^n M,N)) \subseteq (\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^1(M/\mathfrak{a}^n M,\Gamma_{\mathfrak{a}}(N)))) \cup (\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}_R^1(M/\mathfrak{a}^n M,N/\Gamma_{\mathfrak{a}}(N))))$. On the other hand, by Corollary 2.5

$\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^1_R(M/\mathfrak{a}^n M, N/\Gamma_{\mathfrak{a}}(N)))$ is finite and so it is enough to prove that $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^1_R(M/\mathfrak{a}^n M, \Gamma_{\mathfrak{a}}(N)))$ is finite. The exact sequence

$$0 \longrightarrow \mathfrak{a}^n M \longrightarrow M \longrightarrow M/\mathfrak{a}^n M \longrightarrow 0$$

induces, for large n, the following exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(\mathfrak{a}^n M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Ext}^1_R(M/\mathfrak{a}^n M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Ext}^1_R(M, \Gamma_{\mathfrak{a}}(N)).$$

This proves that $\cup_{n\in\mathbb{N}} \operatorname{Ass}(\operatorname{Ext}^1_R(M/\mathfrak{a}^n M, \Gamma_{\mathfrak{a}}(N)))$ is finite, as required. □

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