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KALLIN'S LEMMA FOR RATIONAL CONVEXITY

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ABSTRACT. In this paper, we give a version of Kallin's lemma for rationally convex sets. As an application, we give sufficient conditions so that the union of a totally real graphs in $\mathbf{C}^2$ is locally rationally convex.

1. INTRODUCTION

Let K be a compact subset of $\mathbf{C}^n$. By hull(K) we denote the polynomially convex hull of K i.e.,

$$\text{hull}(K) = \{z \in \mathbf{C}^n : |p(z)| \le \max_K |p| \text{ for every polynomial } p \text{ in } \mathbf{C}^n\}.$$

We say that K is polynomially convex if hull$(K) = K$. By definition, R-hull(K) consists of all $z \in \mathbf{C}^n$ such that

$$|g(z)| \le \max_K |g|$$

for every rational function g which is analytic about K. If $K = R$-hull(K), we say that K is rationally convex in $\mathbf{C}^n$. Notice that $K \subset R$-hull$(K) \subset$ hull(K). Moreover, these inclusions may be proper. The interest for studying polynomial convexity and rational convexity stems from the celebrated Oka-Weil approximation theorem (see [1, p. 36]) which states that holomorphic functions near a compact polynomially (resp. rationally) convex subset of $\mathbf{C}^n$ can be uniformly approximated by polynomials (resp. rational functions) in $\mathbf{C}^n$. The reader may consult excellent sources like [1, 2, 8] for more applications of polynomial convexity and rational convexity to function theory of several complex variables. We are also interested in local versions of the above concepts. A closed $F \subset \mathbf{C}^n$ is called locally polynomially convex (resp. locally rationally convex) at $a \in F$ if there exists a closed ball $\overline{B}(a, r)$ centered at a such that $\overline{B}(a, r) \cap F$ is polynomially convex (resp. locally rationally convex).

Observe that union of two polynomially convex sets may even fail to be rationally convex (see [7, p. 272]). On the positive side, the following result due to Kallin gives a sufficient condition for polynomial convexity of union of two polynomially convex compact sets.

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Theorem 1.1. (Kallin's lemma) *Suppose that*
1) X_1 *and* X_2 *are polynomially convex subsets of* $\mathbf{C}^n$*;*
2) Y_1 *and* Y_2 *are polynomially convex subsets of* $\mathbf{C}$ *such that* 0 *is a boundary point of both* Y_1 *and* Y_2 *and* $Y_1 \cap Y_2 = \{0\}$*;*
3) p *is a polynomial such that* $p(X_1) = Y_1$ *and* $p(X_2) = Y_2$*;*
4) $p^{-1}(0) \cap (X_1 \cup X_2)$ *is polynomially convex.*
Then $X_1 \cup X_2$ *is polynomially convex.*

Kallin's lemma is a powerful tool in verifying polynomial convexity of finite union of polynomially convex sets. For a comprehensive survey on Kallin's lemma and its use, we refer the reader to [3] (see also [5] for a recent development).

The purpose of this paper is to provide an analogous result for rational convexity.

Theorem 1.2. *Suppose that*
1) X_1 *and* X_2 *are polynomially convex subsets of* $\mathbf{C}^n$*;*
2) Y_1 *and* Y_2 *are polynomially convex subsets of* $\mathbf{C}$ *such that* Y_2 *is a continuous arc,* ∂Y_1 *is a continuous Jordan curve and* $E = Y_1 \cap Y_2$ *has one-dimensional Hausdorff measure zero;*
3) p *is a rational function with poles off* $X_1 \cup X_2$ *such that* $p(X_1) = Y_1$ *and* $p(X_2) = Y_2$*;*
4) $p^{-1}(\lambda) \cap (X_1 \cup X_2)$ *is polynomially convex for every* $\lambda \in E$*.*
Then $X_1 \cup X_2$ *is rationally convex.*

Recall that by a continuous arc we mean the homeomorphism image of the unit interval $[0, 1]$. Note that there is a continuous arc having positive two-dimension Hausdorff measure (see [1, p. 202]). In comparison with Theorem 1.1, even though the assumptions on p and $Y_1 \cap Y_2$ have been relaxed, we have to impose a stronger restriction on the shape of Y_2. Nevertheless, from Theorem 1.2 we may derive the following consequence which is easy to appreciate.

Corollary 1.3. *Let* φ *be a smooth complex valued function of class* $\mathcal{C}^1$ *defined on a neighborhood* U *of* $0 \in \mathbf{C}$*. Assume that*
1) $\varphi(0) = \frac{\partial \varphi}{\partial z}(0) = \frac{\partial \varphi}{\partial \overline{z}}(0) = 0.$
2) $\{z : \operatorname{Im} \varphi(z) = 0\}$ *is countable.*
Let M *be the graph* $\{(z, \overline{z} + \varphi(z)) : z \in U\}$*. Then* $\mathbf{R}^2 \cup M$ *is locally rationally convex at the origin.*

There are a lot of functions φ verifying the assumptions of Corollary 1.3. Indeed, let $\{a_j\}_{j\geq 1}$ be a sequence of real numbers decreasing to 0. Let φ_1 be a real valued $\mathcal{C}^1$ smooth function on $\mathbf{R}$ such that $\varphi_1 \geq 0, \varphi_1 = 0$ precisely on the set $\{a_j\}_{j\geq 1} \cup \{0\}$. Then the function $\varphi(z) := \varphi_1(x) + i|z|^2$ satisfies the conditions of Corollary 1.3.

2. Preliminaries

For a compact set $K \subset \mathbf{C}^n$, let $C(K)$ denote the algebra of all continuous complex valued functions on K, with the norm

$$\|g\|_K = \max\{|g(z)| : z \in K\}, \text{ for every } g \in C(K),$$

and let $P(K)$ denote the closure of the set of polynomials in $C(K)$; let $A(K)$ be the subalgebra of $C(K)$ of functions which are holomorphic on the interior $\text{int}(K)$ of K; let $R(K)$ be the closure in $C(K)$ of rational functions with poles off K. It is well-known that $K, \hat{K}$ and R-hull(K) respectively can be identified with the space of maximal ideals of $C(K), P(K)$ and $R(K)$ (see [1, 2]). In the special case that K is a compact subset of the complex plane, we will make use of Mergelyan's theorem (see [2, p. 48]) which states that if $\hat{K} = K$ then $A(K) = P(K)$. We will also use well-known results concerning the algebra $R(K)$ for K a compact subset of the complex plane. The following result is Hartogs-Rosenthal's theorem (see [1, p. 10]).

Theorem 2.1. (Hartogs-Rosenthal) *If K is a compact subset of the complex plane which has two-dimensional Lebesgue measure zero, then $R(K) = C(K)$.*

Let E be a subset of the complex plane. By $AC(E)$ will be denoted the family of functions f such that f is continuous on the Riemann sphere S^2, f is analytic off some compact subset of E, $\|f\|_{S^2} \leq 1$ and $f(\infty) = 0$. The continuous analytic capacity of E is

$$\alpha(E) = \sup\{|f'(\infty)| : f \in AC(E)\}.$$

For basic materials on continuous analytic capacity the readers may consult [2]. The following result is Vituskhin's characterization of K for which $R(K) = A(K)$ in terms of continuous analytic capacity (see [2, p. 217]) .

Theorem 2.2. (Vituskhin) *Let K be a compact subset of the complex plane. The following are equivalent*
1) *$R(K) = A(K)$.*
2) *For every bounded open set D, $\alpha(D \setminus K) = \alpha(D \setminus \text{int } K)$.*

The next lemma is an easy consequence of Vitushkin's theorem.

Lemma 2.3. *Let K_1 be a compact subset of $\mathbf{C}$ such that $R(K_1) = A(K_1)$. Let $K_2 \subset \mathbf{C}$ be a compact subset having two-dimensional Lebesgue measure zero. Then*

$$R(K_1 \cup K_2) = A(K_1 \cup K_2).$$

Proof. Let D be any open bounded subset of $\mathbf{C}$. By virtue of Vitushkin's theorem, it suffices to show

$$\alpha\big(D \setminus (K_1 \cup K_2)\big) = \alpha\big(D \setminus \text{int}(K_1 \cup K_2)\big)$$

where α is continuous analytic capacity. Since K_2 has two-dimensional Lebesgue measure zero and K_1 is compact, we infer that $\text{int}(K_1 \cup K_2) = \text{int}K_1$.

Now because $R(K_1) = A(K_1)$, we apply Theorem 2.2 to get

$$\alpha(D \setminus K_1) = \alpha\big(D \setminus \text{int}K_1\big).$$

Thus

$$\alpha(D \setminus K_1) = \alpha\big(D \setminus \text{int}K_1\big) = \alpha\big(D \setminus \text{int}(K_1 \cup K_2)\big). \tag{1}$$

On the other hand, since K_2 has two-dimensional Lebesgue measure zero, from Hartog-Rosenthal's theorem we deduce $R(K_2) = C(K_2) = A(K_2)$. This implies that

$$\alpha\Big((D \setminus K_1) \setminus K_2\Big) = \alpha\Big((D \setminus K_1) \setminus \mathrm{int}K_2\Big) = \alpha\big(D \setminus K_1\big).$$

Hence

$$\alpha(D \setminus (K_1 \cup K_2)\big) = \alpha(D \setminus K_1). \tag{2}$$

Combining (1) and (2) we get

$$\alpha\big(D \setminus (K_1 \cup K_2)\big) = \alpha\big(D \setminus \mathrm{int}(K_1 \cup K_2)\big).$$

The lemma is proved. □

Let K be a compact subset of $\mathbf{C}^n$ and let $\mathcal{A}$ be a uniform algebra on K. A point $x \in K$ is a peak point for $\mathcal{A}$ if there is a function $f \in \mathcal{A}$ such that $f(x) = 1$ while $|f(y)| < 1$ for $y \in K$ and $y \neq x$. The function f which satisfies this condition is called to be peak at x. The well-known lemma below (see [8, p. 62]) is a simple observation that certain points are peak point for $P(K)$.

Lemma 2.4. *If K is a compact, polynomially convex subset of the complex plane, then every point of ∂K is a peak point of $P(K)$.*

Proof. Without loss of generality, assume that the origin is a point of ∂K and that K is a subset of the open unit disk. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{C} \setminus K$ that converges to the origin, and for each n, let γ_n be an arc in the Riemann sphere from z_n to infinity that misses K. Fix a point $z_0 \in K \setminus \{0\}$, and for each n, let θ_n be a branch of $\log(z - z_n)$ defined on $\mathbf{C} \setminus \gamma_n$, the θ_n is chosen so that the sequence $\theta_n(z_0)$ converges. The sequence $\{\theta_n\}_{n=1}^{\infty}$ converges pointwise on $K \setminus \{0\}$ to a continuous branch of $\log z$. We shall denote the limit function by $\log z$. The function $\varphi(z)$ defined by $\varphi(z) = \frac{\log z}{\log z - 1}$, $z \in K \setminus \{0\}$, $\varphi(0) = 1$, is continuous on K and is holomorphic on the interior of K. Moreover , $\varphi(0) = 1 > |\varphi(z)|$ for every $z \in K \setminus \{0\}$. From Mergelyan's theorem we get that $\varphi \in P(K)$, so 0 is a peak point for the algebra $P(K)$. The lemma is proved. □

3. Proof of the main results

Proof of Theorem 1.2. We follow the lines in the proof of the classical Kallin's lemma (Theorem 1.1). Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. First we claim that every point $x \in \partial Y$ is a peak point for $R(Y)$. Indeed, since Y_1 is polynomially convex, by Mergelyan's theorem we have $P(Y_1) = A(Y_1)$. In particular $R(Y_1) = A(Y_1)$. Since Y_2 is a continuous arc, it follows that Y_2 has two-dimensional Lebesgue zero. Applying Lemma 2.3 we get $R(Y) = A(Y)$. If $x \in \partial Y_1$, by Lemma 2.4, there exists $p \in A(Y_1)$ peaking at x. By extending p to a continuous function on Y_2, we infer that x is a peak point for $A(Y) = R(Y)$. The case where $x \in Y_2$ can be treated analogously. The claim follows.

Now we let $x \in R\text{-hull}(X)$. Let μ be a representing measure for $R(X)$ on X, representing the point x, that is, μ is a positive regular Borel measure on X such that $f(x) = \int f d\mu$ for all $f \in R(X)$ (see [2, p. 31]). By the global description of rational convex hulls (see [7, p. 262]) we have $p(x) \in p(X) = Y$. There are two cases to be considered.

Case 1. $p(x) \in \partial Y$. Let h be a peak function for $R(Y)$, peaking at $p(x)$. Let $H = h \circ p$. Clearly $H \in R(X)$. For every polynomial f and positive integers k we have $f.H^k \in R(X)$. We obtain

$$f(x) = f(x)H^k(x) = \int f H^k d\mu \tag{3}$$

for all positive integers k. Let ν be the restriction of μ to $p^{-1}\big(p(x)\big) \cap X$. Passing to limit as $k \to \infty$ in (3), by Lebesgue's dominated convergence theorem we obtain

$$f(x) = \int f d\nu.$$

It follows that

$$|f(x)| = |\int f d\nu| \leq ||f||_{p^{-1}\left(p(x)\right) \cap X} \int d\nu \leq ||f||_{p^{-1}\left(p(x)\right) \cap X}.$$

This implies that

$$x \in \text{hull}\Big(p^{-1}\big(p(x)\big) \cap X\Big).$$

We will show that $x \in X$. If $p(x) \in E$ then $p^{-1}\big(p(x)\big) \cap X$ is polynomially convex. It follows that

$$x \in \text{hull}\Big(p^{-1}\big(p(x)\big) \cap X\Big) = p^{-1}\big(p(x)\big) \cap X \subset X.$$

If $p(x) \in \partial Y \setminus E$ then $p(x) \in Y_1 \setminus E$ or $p(x) \in Y_2 \setminus E$. This implies that

$$p^{-1}\big(p(x)\big) \cap X \subset X_1 \text{ or } p^{-1}\big(p(x)\big) \cap X \subset X_2.$$

Since X_1, X_2 are polynomially convex we get that

$$x \in \text{hull}\Big(p^{-1}\big(p(x)\big) \cap X\Big) \subset \text{hull}(X_1) = X_1$$

or

$$x \in \text{hull}\Big(p^{-1}\big(p(x)\big) \cap X\Big) \subset \text{hull}(X_2) = X_2.$$

Hence $x \in X$.

Case 2. $p(x) \in \text{int } Y$. We let φ be a complex valued continuous function on ∂Y_1 such that $\varphi = 0$ on E and $\varphi \not\equiv 0$. By Rudin-Carleson interpolation's theorem (see [6]), we can find $g \in A(Y_1)$ such that $g = \varphi$ on ∂Y_2. Clearly $g \not\equiv 0$ on $\text{int}(Y_1)$. Thus there exists a sequence $\{x_j\}_{j \geq 1} \in R\text{-hull}(X), x_j \to x$ such that $p(x_j) \in \text{int}(Y_1)$ and $g(p(x_j)) \neq 0$. By setting $g = 0$ on Y_2, in view of the relation $R(Y) = A(Y)$, we have $g \in R(Y)$.

Fix a polynomial f. For $j, k \geq 1$ we define

$$f_{j,k}(z) = \frac{g\big(p(z)\big)}{g\big(p(x_j)\big)} f^k(z).$$

Clearly $f_{j,k} \in R(X)$. It follows that

$$(4) \qquad |f^k(x_j)| = |f_{j,k}(x_j)| = \Big| \int_X f_{j,k} d\mu \Big| \le \|f^k\|_{X_1} \int_{X_1} \Big| \frac{g\big(p(z)\big)}{g\big(p(x_j)\big)} \Big| d\mu(z).$$

Taking kth roots and letting $k \to \infty$ in (4), we obtain $|f(x_j)| \le \|f\|_{X_1}$ $\forall j \ge 1$. Letting $j \to \infty$ we infer $|f(x)| \le \|f\|_{X_1}$. This means that $x \in \mathrm{hull}(X_1) = X_1 \subset X$. The proof is thereby complete. □

Proof of Corollary 3. By a well-known result of Wermer on local polynomial convexity of graphs (see [1, p. 102]) and by the first condition on φ we deduce that $\mathbf{R}^2$ and M are locally polynomially convex at the origin. Choose $r > 0$ small enough such that $M_r := M \cap \overline{B}(0, r)$ is polynomially convex.

Now we set $p(z, w) = z + w$. Clearly $p(\mathbf{R}^2) = \mathbf{R}$ and for every $r > 0$ small enough $N_r := p(M_r)$ is a compact polynomially convex subset of $\mathbf{C}$. Moreover, by the second assumption on φ, the set $E := N_r \cap \mathbf{R}$ is countable. Notice that for $\lambda \in E$, the set $p^{-1}(\lambda) \cap (\mathbf{R}^2 \cup M_r) \cap \overline{B}(0, r)$ is the union of two smooth arcs. It follows that this set is polynomially convex (see [1, p. 84]). Thus with the choice of p, we may apply Theorem 1.2 to get the desired conclusion. □

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