ACTA MATHEMATICA VIETNAMICA
Volume 34, Number 3, 2009, pp. 319–334

WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

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Abstract. In this paper, we prove the weighted endpoint estimates for multilinear commutator of Marcinkiewicz operator.

1. Introduction and theorems

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see[3]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [2, 5], the boundedness properties of the commutators for the extreme values of p are obtained. In [6], Liu and Wu proved the weighted endpoint estimates for commutator of Marcinkiewicz operator of order one. Following their work, we will prove the weighted boundedness properties of the multilinear commutator of Marcinkiewicz operator for the extreme cases in this paper.

First let us introduce some notations. In this paper, Q will denote a cube of R^n with sides parallel to the axes. For any fixed positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 1. Given a fixed locally integrable function ω, a locally integrable function f is said to belong to $BMO(\omega)$ space if the following inequality

$$\frac{1}{\omega(Q)} \int_Q |f(y) - f_{Q,\omega}| \omega(y) dy \le A.$$

holds for all cubes Q, where $f_{Q,\omega} = \frac{1}{\omega(Q)} \int_Q f(x)\omega(x)dx$ and $\omega(Q) = \int_Q \omega(x)dx$. The smallest bound A satisfying the above inequality is taken to be the norm of

Received February 13, 2008.
2000 *Mathematics Subject Classification.* 42B20, 42B25.
Key words and phrases. Marcinkiewicz operator, multilinear commutator, $BMO(\omega)$, $B^{s,\lambda}(\omega)$, $CMO^{s,\lambda}(\omega)$.

f in this space, i.e.,$||f||_{BMO(\omega)} = \sup_Q \frac{1}{\omega(Q)} \int_Q |f(y) - f_{Q,\omega}|\omega(y)dy$. Following [8, 9], we know the above inequality is equivalent to

$$\frac{1}{\omega(Q)} \int_Q |f(y) - c(Q,\omega)|\omega(y)dy \leq A,$$

for some constant $c(Q,\omega)$. Obviously, if $\omega \equiv 1$, $BMO(\omega)$ is the usual $BMO(R^n)$ space.

Definition 2. Let $\lambda \in R^1$ and $1 \leq s < \infty$, f is said to belong to $B^{s,\lambda}(\omega)$, if the inequality $||f||_{B^{s,\lambda}(\omega)} < \infty$ holds, where

$$||f||_{B^{s,\lambda}(\omega)} = \sup_{r\geq 1} \left(\frac{1}{\omega(Q(0,r))^{1+s\lambda}} \int_{Q(0,r)} |f(x)|^s \omega(x)dx \right)^{1/s}.$$

Remark. The definition is a little difference from usual, $s = 1$ is incorporated the range of s. If $\omega \equiv 1$, then $B^{s,\lambda}(\omega) = B^{s,\lambda}(R^n)$; if $\lambda < -1/s$, then $B^{s,\lambda}(R^n) = \{0\}$; if $\lambda = -1/s$, then $B^{s,-1/s}(R^n) = L^s(R^n)$; if $\lambda = 0$,then $B^{s,0}(R^n) = B^s(\omega)$.

Definition 3. Let $\lambda < 1/n$ and $1 \leq s < \infty$, f is said to belong to $CMO^{s,\lambda}(\omega)$, if the inequality $||f||_{CMO^{s,\lambda}(\omega)} < \infty$ holds, where

$$||f||_{CMO^{s,\lambda}(\omega)} = \sup_{r\geq 1} \left(\frac{1}{\omega(Q(0,r))^{1+s\lambda}} \int_{Q(0,r)} |f(x) - f_{\omega,Q}|^s \omega(x)dx \right)^{1/s}.$$

Remark. The definition is a little difference from usual, $s = 1$ is incorporated the range of s. If $\omega \equiv 1$, then $CMO^{s,\lambda}(\omega) = CMO^{s,\lambda}(R^n)$; if $\lambda < -1/s$, then the space $CMO^{s,\lambda}(R^n)$ reduces to the constant functions; if $\lambda = -1/s$, then $CMO^{s,\lambda}(R^n)$ coincides with $L^s(R^n)$ modulo constants; if $\lambda = 0, s = 1$, then $CMO^{s,\lambda}(\omega) = CMO(\omega)$.

Definition 4. Let $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy,$$

we also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator(see [6, 10]).

Consider the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = ||F_t(f)(x)|| \text{ and } \mu_\Omega^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Lemma 1. (see [10]) *Let $1 < p < \infty$, $\omega \in A_p$, then μ_Ω is bounded on $L^p(\omega)$.*

Lemma 2. *Let $\omega \in A_p$, $p > 1$, and χ_Q be the characteristic function of the cube Q, then $\omega\chi_Q$ also belongs to A_p.*

Theorem 1. *Let $\omega \in A_p, (p > 1)$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. Then $\mu_\Omega^{\vec{b}}$ is bounded from L^∞ to $BMO(\omega)$.*

Theorem 2. *Let $\omega \in A_p, (p > 1)$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \le j \le m$. If for any $Q = Q(0, R)$, $R > 1$, there is $\omega(Q) \ge 1$, then $\mu_\Omega^{\vec{b}}$ is bounded from $B^{t,\lambda}(\omega)$ to $CMO^{s,\lambda}(\omega)$, where $t > \max\{p, s\}$, $\lambda \le 0$.*

2. Proofs of the lemmas and theorems

Proof of Lemma 2. Let Q_0 be any cube, we now verify the A_p inequalities.

$$\left(\frac{1}{|Q_0|}\int_{Q_0} \omega(x)\chi_Q(x)dx\right)\left(\frac{1}{|Q_0|}\int_{Q_0} (\omega(x)\chi_Q(x))^{-1/(p-1)}dx\right)^{p-1}$$
$$\le \left(\frac{1}{|Q_0|}\int_{Q_0} \omega(x)dx\right)\left(\frac{1}{|Q_0|}\int_{Q_0} \omega(x)^{-1/(p-1)}dx\right)^{p-1}$$
$$\le C.$$

So the lemma is proved. □

Proof of Theorem 1. It suffices to prove that there exists a constant $c(Q, \omega)$ such that

$$\frac{1}{\omega(Q)}\int_Q |\mu_\Omega^{\vec{b}}(f)(x) - c(Q, \omega)|\omega(x)dx \le C||f||_{L^\infty}.$$

For any fixed cube Q, $Q = Q(x_0, r)$,$kQ = Q(x_0, kr), (k \in R)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(2Q)^c}$, where χ_{2Q}, $\chi_{(2Q)^c}$ are characteristic functions of $2Q$ and $(2Q)^c$ separately. We denote $\vec{b}_{2Q}$ by $\vec{b}_{2Q} = ((b_1)_{2Q}, \cdots, (b_m)_{2Q}) \in R^m$, where $(b_j)_{2Q} = \frac{1}{|2Q|}\int_{2Q} b_j(y)dy$, $1 \le j \le m$, then we have

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy$$
$$= \int_{|x-y|\le t} \left[\prod_{j=1}^m ((b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q}))\right] f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy$$

$$= \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma \int_{|x-y| \le t} (b(y) - (b)_{2Q})_{\sigma^c} f(y) \frac{\Omega(x-y)}{|x-y|^{n-1}} dy$$
$$= (b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(x)$$
$$+ (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f)(x)$$
$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x),$$

thus

$$|\mu_\Omega^{\vec{b}}(f)(x) - \mu_\Omega((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})) f_2)(x_0)|$$
$$\le ||F_t^{\vec{b}}(f)(x) - (-1)^m F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})) f_2)(x_0)||$$
$$\le ||(b_1(x) - (b_1)_{2Q}) \cdots (b_m(x) - (b_m)_{2Q}) F_t(f)(x)||$$
$$+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ||(b(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)||$$
$$+ ||F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_1)(x)||$$
$$+ ||F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(x)$$
$$- F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q}) f_2)(x_0)||$$
$$= I_1(x) + I_2(x) + I_3(x) + I_4(x).$$

For $I_1(x)$, by Hölder' s and the reverse of Hölder's inequalities for some $1 < q < \infty$ (see[4]), let $1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q = 1,\ 1/p + 1/p' = 1,\ q_j > 1 (j = 1, 2, \cdots, m)$, $p > 1, p' > 1$, by Lemma 1 and Lemma 2, we get

$$\frac{1}{\omega(Q)} \int_Q |I_1(x)| \omega(x) dx$$
$$\le \frac{1}{\omega(Q)} \left(\int_Q |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q})|^{p'} \omega(x) dx \right)^{1/p'} \left(\int_{R^n} |\mu_\Omega(f)(x)|^p \omega(x) \chi_Q dx \right)^{1/p}$$
$$\le \frac{1}{\omega(Q)} \left(\int_Q |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q})|^{p'} \omega(x) dx \right)^{1/p'} \left(\int_{R^n} |f(x)|^p \omega(x) \chi_Q dx \right)^{1/p}$$
$$\le \frac{1}{\omega(Q)} \left(\int_Q |\prod_{j=1}^{m} (b_j(x) - (b_j)_{2Q})|^{p'} \omega(x) dx \right)^{1/p'} ||f||_{L^\infty} \left(\int_Q \omega(x) dx \right)^{1/p}$$
$$\le \frac{1}{\omega(Q)} ||f||_{L^\infty} [\omega(Q)]^{1/p} \times$$
$$\times \left[\prod_{j=1}^{m} \left(\int_{2Q} |b_j(x) - (b_j)_{2Q}|^{p' q_j} dx \right)^{1/q_j} \left(\int_Q \omega(x)^q dx \right)^{1/q} \right]^{1/p'}$$

$$\leq \frac{C}{\omega(Q)}||f||_{L^\infty}[\omega(Q)]^{1/p}|2Q|^{1/p'q_1+\cdots+1/p'q_m}||\vec{b}||_{BMO}\left[|Q|^{1/q-1}\int_Q \omega(x)dx\right]^{1/p'}$$
$$\leq C||\vec{b}||_{BMO}||f||_{L^\infty}[\omega(Q)]^{1/p'+1/p-1}|Q|^{1/p'[1/q_1+\cdots+1/q_m+1/q-1]}$$
$$\leq C||\vec{b}||_{BMO}||f||_{L^\infty}.$$

For $I_2(x)$, by Hölder's inequality with $1/p+1/p'=1$, $p>1, p'>1$, we have

$$\frac{1}{\omega(Q)}\int_Q |I_2(x)|\omega(x)dx$$
$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)}\int_Q |(b(\vec{x})-\vec{b}_{2Q})_\sigma||\mu_\Omega((\vec{b}-\vec{b}_{2Q})_{\sigma^c}f)(x)|\omega(x)dx$$
$$\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{\omega(Q)}\int_Q |(\vec{b}(x)-\vec{b}_{2Q})_\sigma|^{p'}\omega(x)dx\right)^{1/p'}\times$$
$$\times\left(\frac{1}{\omega(Q)}\int_Q |\mu_\Omega((\vec{b}-\vec{b}_{2Q})_{\sigma^c}f)(x)|^p\omega(x)dx\right)^{1/p}$$
$$=\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}K_1K_2.$$

For K_1, for some $1<q<\infty$, $\sum_{j\in\sigma}1/q_j+1/q=1$, $q_j>1(j\in\sigma)$, by Hölder's inequality and the reverse of Hölder's inequality, we have

$$K_1=\left(\frac{1}{\omega(Q)}\int_Q |(\vec{b}(x)-\vec{b}_{2Q})_\sigma|^{p'}\omega(x)dx\right)^{1/p'}$$
$$\leq \omega(Q)^{-1/p'}\prod_{j\in\sigma}\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{p'q_j}dx\right)^{1/p'q_j}\left[\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/p'}$$
$$\leq C\omega(Q)^{-1/p'}|2Q|^{\sum_{j\in\sigma}1/p'q_j}||\vec{b}_\sigma||_{BMO}\left[|Q|^{1/q-1}\int_Q \omega(x)dx\right]^{1/p'}$$
$$\leq C||\vec{b}_\sigma||_{BMO}.$$

For K_2, for some $1<q<\infty$,$\sum_{j\in\sigma}1/q_j+1/q=1$, $q_j>1(j\in\sigma)$, by Hölder's inequality and the reverse of Hölder's inequality, we have

$$K_2=\omega(Q)^{-1/p}\left(\int_{R^n}|\mu_\Omega((\vec{b}(x)-\vec{b}_{2Q})_{\sigma^c}f)(x)|^p\omega(x)\chi_Q(x)dx\right)^{1/p}$$
$$\leq \omega(Q)^{-1/p}\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_{2Q})_{\sigma^c}f(x)|^p\omega(x)\chi_Q(x)dx\right)^{1/p}$$
$$\leq \omega(Q)^{-1/p}||f||_{L^\infty}\left[\prod_{j\in\sigma^c}\left(\int_Q |b_j(x)-(b_j)_{2Q}|^{pq_j}dx\right)^{1/q_j}\left(\int_Q \omega(x)^q dx\right)^{1/q}\right]^{1/p}$$

$$\leq \omega(Q)^{-1/p}||f||_{L^\infty}\times$$

$$\times\prod_{j\in\sigma^c}\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{pq_j}dx\right)^{1/pq_j}\left[\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/p}$$

$$\leq C\omega(Q)^{-1/p}||f||_{L^\infty}|2Q|^{\sum_{j\in\sigma^c}1/pq_j}||\vec{b}_{\sigma^c}||_{BMO}\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/p}$$

$$\leq C||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^\infty}.$$

So

$$\frac{1}{\omega(Q)}\int_Q|I_2(x)|\omega(x)dx$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^\infty}$$

$$\leq C||\vec{b}||_{BMO}||f||_{L^\infty}.$$

For $I_3(x)$, for some $q>1$, $\sum_{j=1}^m 1/q_j+1/q=1$, $q_j>1(j=1,2,\cdots,m)$, taking $p>1$, using Lemma 1, Lemma 2 and Hölder's inequality, we have

$$\frac{1}{\omega(Q)}\int_Q|I_3(x)|\omega(x)dx$$

$$=\frac{1}{\omega(Q)}\int_Q|\mu_\Omega(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_1)(x)|\omega(x)dx$$

$$\leq\left(\frac{1}{\omega(Q)}\int_{R^n}|\mu_\Omega(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}f_1)(x)|^p\omega(x)\chi_Q(x)dx\right)^{1/p}$$

$$\leq\omega(Q)^{-1/p}\left(\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}f_1(x)|^p\omega(x)dx\right)^{1/p}$$

$$\leq\omega(Q)^{-1/p}||f||_{L^\infty}\left(\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}|^p\omega(x)dx\right)^{1/p}$$

$$\leq\omega(Q)^{-1/p}||f||_{L^\infty}\left[\prod_{j=1}^m\left(\int_Q|b_j(x)-(b_j)_{2Q}|^{pq_j}dx\right)^{1/q_j}\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/p}$$

$$\leq\omega(Q)^{-1/p}||f||_{L^\infty}\prod_{j=1}^m\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{pq_j}dx\right)^{1/pq_j}\left[\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/p}$$

$$\leq C\omega(Q)^{-1/p}||f||_{L^\infty}|2Q|^{\sum_{j=1}^m 1/pq_j}||\vec{b}||_{BMO}\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/p}$$

$\leq C||\vec{b}||_{BMO}||f||_{L^\infty}.$

For $I_4(x)$, by Minkowski's inequality and noting that $|x_0 - y| \approx |x - y|$ for $y \in (2Q)^c$, we have

$$
\begin{aligned}
I_4(x) &= ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x) \\
&\quad - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|| \\
&= \left(\int_0^\infty \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1}} \left[\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right] dy \right. \\
&\quad \left. - \int_{|x_0-y|\leq t} \frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1}} \left[\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right] dy\Big|^2 \frac{dt}{t^3}\right)^{1/2} \\
&\leq \left(\int_0^\infty \left[\int_{|x_0-y|\leq t, |x_0-y|>t} \frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1}} \times \right.\right. \\
&\quad \left.\left. \times \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y|>t, |x_0-y|\leq t} \frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1}} \times \right.\right. \\
&\quad \left.\left. \times \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y|\leq t, |x_0-y|\leq t} \left|\frac{|\Omega(x-y)|}{|x-y|^{n-1}} - \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\right| \times \right.\right. \\
&\quad \left.\left. \times \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| |f_2(y)| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&= I_4^{(1)} + I_4^{(2)} + I_4^{(3)},
\end{aligned}
$$

then by Hölder's inequality with exponent $\sum_{j=1}^m 1/r_j = 1$, $r_j > 1$,$j = 1, 2, \cdots, m$,

$$
\begin{aligned}
I_4^{(1)} &\leq C \int_{(2Q)^c} \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\leq t<|x_0-y|} \frac{dt}{t^3}\right)^{1/2} dy \\
&\leq C \int_{(2Q)^c} \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|}{|x-y|^{n-1}} \left|\frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2}\right|^{1/2} dy \\
&\leq C \int_{(2Q)^c} \left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|}{|x-y|^{n-1}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy||f||_{L^\infty}\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\prod_{j=1}^{m}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_j(y)-(b_j)_{2Q})|^{r_j}\,dy\right)^{1/r_j}||f||_{L^\infty}\\
&\leq C\sum_{k=1}^{\infty}k2^{-k/2}\prod_{j=1}^{m}||b_j||_{BMO}||f||_{L^\infty}\\
&\leq C||\vec{b}||_{BMO}||f||_{L^\infty};
\end{aligned}
$$

similarly, we have $I_4^{(2)} \leq C||\vec{b}||_{BMO}||f||_{L^\infty}$.

We now estimate $I_4^{(3)}$. By the following inequality (see [10]) :

$$
\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \leq \left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),
$$

we gain

$$
\begin{aligned}
I_4^{(3)} \leq\ & C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|}{|x_0-y|^n}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
&+C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\times\\
&\times\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy\\
\leq\ & C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\times\\
&\times\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\right)|f(y)|dy\\
\leq\ & C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
\leq\ & C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy||f||_{L^\infty}
\end{aligned}
$$

$$\le C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})\prod_{j=1}^{m}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_j(y)-(b_j)_{2Q})|^{r_j}\,dy\right)^{1/r_j}||f||_{L^\infty}$$
$$\le C\sum_{k=1}^{\infty}k(2^{-k}+2^{-k\gamma})\prod_{j=1}^{m}||b_j||_{BMO}||f||_{L^\infty}$$
$$\le C||\vec{b}||_{BMO}||f||_{L^\infty};$$

so

$$\frac{1}{\omega(Q)}\int_Q|I_4(x)|\omega(x)dx\le C||\vec{b}||_{BMO}||f||_{L^\infty}.$$

Combining the inequalities above, we complete the proof of Theorem 1. □

Proof of Theorem 2. Because for any $Q=Q(0,R)$, $R>1$, there is $\omega(Q)\ge 1$ and $\lambda\le 0$, it suffices to prove that there exists a constant $c(Q,\omega)$ such that

$$\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q|\mu_\Omega^{\vec{b}}(f)(x)-c(Q,\omega)|^s\omega(x)dx\right]^{\frac{1}{s}}\le C||f||_{B^{t,\lambda}(\omega)},$$

where $Q=Q(0,R)$, $R\ge 1$. For any fixed cube Q, $kQ=Q(0,kR),(k\in R_+)$, we decompose f into $f=f_1+f_2$ with $f_1=f\chi_{2Q},\ f_2=f\chi_{(2Q)^c}$ and denote $\vec{b}_{2Q}$ by $\vec{b}_{2Q}=((b_1)_{2Q},\cdots,(b_m)_{2Q})\in R^m$, where $(b_j)_{2Q}=\frac{1}{|2Q|}\int_{2Q}b_j(y)dy,\ 1\le j\le m$, then we have

$$F_t^{\vec{b}}(f)(x)=\int_{|x-y|\le t}\left[\prod_{j=1}^{m}(b_j(x)-b_j(y))\right]f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy$$
$$=\int_{|x-y|\le t}\left[\prod_{j=1}^{m}((b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q}))\right]f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy$$
$$=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{|x-y|\le t}(b(y)-(b)_{2Q})_{\sigma^c}f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy$$
$$=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)$$
$$+(-1)^mF_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x),$$

thus

$$|\mu_\Omega^{\vec{b}}(f)(x)-\mu_\Omega((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)|$$
$$\le||F_t^{\vec{b}}(f)(x)-(-1)^mF_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)||$$
$$\le||(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)||$$

$$
\begin{aligned}
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)|| \\
&+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|| \\
&+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x) \\
&-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|| \\
=\ & J_1(x)+J_2(x)+J_3(x)+J_4(x).
\end{aligned}
$$

For $J_1(x)$, for some $1<q<\infty$, let $1/q_1+1/q_2+\cdots+1/q_m+1/q=1$, $q_j>1\,(j=1,2,\cdots,m)$. Choosing p so that $sp<t$, $1/p+1/p'=1$, $p>1$, $p'>1$, by Hölder's and the reverse of Hölder's inequalities and Lemma 1, Lemma 2, we get

$$
\begin{aligned}
&\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q|J_1(x)|^s\omega(x)dx\right]^{1/s} \\
&\le\frac{1}{\omega(Q)^{1/s+\lambda}}\left(\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^{sp'}\omega(x)dx\right)^{1/sp'}\times \\
&\quad\times\left(\int_{R^n}|\mu_\Omega(f)(x)|^{sp}\omega(x)\chi_Q(x)dx\right)^{1/sp} \\
&\le\frac{1}{\omega(Q)^{1/s+\lambda}}\left(\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^{sp'}\omega(x)dx\right)^{1/sp'}\times \\
&\quad\times\left(\int_{R^n}|f(x)|^{sp}\omega(x)\chi_Q(x)dx\right)^{1/sp} \\
&\le\frac{1}{\omega(Q)^{1/s+\lambda}}\left(\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^{sp'}\omega(x)dx\right)^{1/sp'}\left(\int_Q|f(x)|^{sp}\omega(x)dx\right)^{1/sp} \\
&\le\frac{1}{\omega(Q)^{1/s+\lambda}}||f||_{B^{sp,\lambda}(\omega)}[\omega(Q)]^{1/sp+\lambda}\times \\
&\quad\times\left[\prod_{j=1}^m\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{sp'q_j}dx\right)^{1/q_j}\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/sp'} \\
&\le\frac{C}{\omega(Q)^{1/s+\lambda}}||f||_{B^{sp,\lambda}(\omega)}[\omega(Q)]^{1/sp+\lambda}|2Q|^{\Sigma_{j=1}^m 1/sp'q_j}||\vec{b}||_{BMO}\times \\
&\quad\times\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/sp'} \\
&\le C||\vec{b}||_{BMO}||f||_{B^{sp,\lambda}(\omega)} \\
&\le C||\vec{b}||_{BMO}||f||_{B^{t,\lambda}(\omega)}.
\end{aligned}
$$

For $J_2(x)$, choosing p so that $sp < t$, by Hölder's inequality with $1/p + 1/p' = 1, p > 1, p' > 1$, we have

$$\begin{aligned}
&\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q |J_2(x)|^s\omega(x)dx\right]^{1/s}\\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q |(b(\vec{x})-\vec{b}_{2Q})_\sigma|^s|\mu_\Omega((\vec{b}-\vec{b}_{2Q})_{\sigma^c}f)(x)|^s\omega(x)dx\right]^{1/s}\\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)^{1/s+\lambda}}\left[\int_Q |(\vec{b}(x)-\vec{b}_{2Q})_\sigma|^{sp'}\omega(x)dx\right]^{1/sp'}\times\\
&\times\left[\int_Q |\mu_\Omega((\vec{b}-\vec{b}_{2Q})_{\sigma^c}f)(x)|^{sp}\omega(x)dx\right]^{1/sp}\\
&= \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)^{1/s+\lambda}}L_1L_2.
\end{aligned}$$

For L_1, by Hölder's inequality and the reverse of Hölder's inequality for some $1 < q < \infty$, $\sum_{j\in\sigma} 1/q_j + 1/q = 1$, $q_j > 1(j \in \sigma)$, we have

$$\begin{aligned}
L_1 &= \left[\int_Q |(\vec{b}(x)-\vec{b}_{2Q})_\sigma|^{sp'}\omega(x)dx\right]^{1/sp'}\\
&\le \prod_{j\in\sigma}\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{sp'q_j}dx\right)^{1/sp'q_j}\left[\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/sp'}\\
&\le C|2Q|^{\sum_{j\in\sigma}1/sp'q_j}||\vec{b}_\sigma||_{BMO}\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/sp'}\\
&\le C||\vec{b}_\sigma||_{BMO}\omega(Q)^{1/sp'}.
\end{aligned}$$

For L_2, choosing r so that $rsp < t$, $1/r + 1/r' = 1$, $r > 1, r' > 1$, by Hölder's inequality and the reverse of Hölder's inequality for some $q > 1$, $\sum_{j\in\sigma} 1/q_j+1/q = 1$, $q_j > 1(j \in \sigma)$, we have

$$\begin{aligned}
L_2 &= \left[\int_{R^n}|\mu_\Omega((\vec{b}-\vec{b}_{2Q})_{\sigma^c}f)(x)|^{sp}\omega(x)\chi_Q(x)dx\right]^{1/sp}\\
&\le \left[\int_{R^n}|(\vec{b}(x)-\vec{b}_{2Q})_{\sigma^c}f(x)|^{sp}\omega(x)\chi_Q(x)dx\right]^{1/sp}\\
&\le \left[\int_Q|(\vec{b}(x)-\vec{b}_{2Q})_{\sigma^c}|^{r'sp}\omega(x)dx\right]^{1/r'sp}\left[\int_Q|f(x)|^{rsp}\omega(x)dx\right]^{1/rsp}\\
&\le \prod_{j\in\sigma^c}\left(\int_Q|b_j(x)-(b_j)_{2Q}|^{r'spq_j}dx\right)^{1/r'spq_j}\left[\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/r'sp/}\times
\end{aligned}$$

$$\times ||f||_{B^{rsp,\lambda}(\omega)}\omega(Q)^{\lambda+1/rsp}$$
$$\leq C||f||_{B^{rsp,\lambda}(\omega)}\omega(Q)^{\lambda+1/rsp}|2Q|^{\sum_{j\in\sigma^c}1/r'spq_j}||\vec{b}_{\sigma^c}||_{BMO}\times$$
$$\times\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/r'sp}$$
$$\leq C||f||_{B^{t,\lambda}(\omega)}||\vec{b}_{\sigma^c}||_{BMO}\omega(Q)^{\lambda+1/sp}.$$

So

$$\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q|J_2(x)|^s\omega(x)dx\right]^{1/s}$$
$$\leq\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{\omega(Q)^{1/s+\lambda}}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{B^{t,\lambda}(\omega)}\omega(Q)^{1/sp'}\omega(Q)^{\lambda+1/sp}$$
$$\leq C||\vec{b}||_{BMO}||f||_{B^{t,\lambda}(\omega)}.$$

For $J_3(x)$, choosing r,p so that $rsp<t$, $p>1$, $1/r+1/r'=1$, $r>1,r'>1$; for some $q>1$, $\sum_{j=1}^m 1/q_j/+1/q=1$, $q_j>1(j=1,2,\cdots,m)$, using Lemma 1, Lemma 2 and Hölder's inequality, we have

$$\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q|J_3(x)|^s\omega(x)dx\right]^{1/s}$$
$$=\frac{1}{\omega(Q)^{\lambda}}\left[\frac{1}{\omega(Q)}\int_Q|\mu_\Omega(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_1)(x)|^s\omega(x)dx\right]^{1/s}$$
$$\leq\frac{1}{\omega(Q)^{\lambda}}\left[\frac{1}{\omega(Q)}\int_{R^n}|\mu_\Omega(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}f_1)(x)|^{sp}\omega(x)\chi_Q(x)dx\right]^{1/sp}$$
$$\leq\frac{1}{\omega(Q)^{1/sp+\lambda}}\left[\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}f_1(x)|^{sp}\omega(x)dx\right]^{1/sp}$$
$$\leq\frac{1}{\omega(Q)^{1/sp+\lambda}}\left[\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q}|^{r'sp}\omega(x)dx\right]^{1/r'sp}\times$$
$$\times\left[\int_Q|f_1(x)|^{rsp}\omega(x)dx\right]^{1/rsp}$$
$$\leq\frac{1}{\omega(Q)^{1/sp+\lambda}}||f||_{B^{rsp,\lambda}(\omega)}\omega(Q)^{1/rsp+\lambda}\prod_{j=1}^m\left(\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{r'spq_j}dx\right)^{1/r'spq_j}\times$$
$$\times\left[\left(\int_Q\omega(x)^q dx\right)^{1/q}\right]^{1/r'sp}$$

$$\leq C\frac{1}{\omega(Q)^{1/sp+\lambda}}||f||_{B^{rsp,\lambda}(\omega)}\omega(Q)^{1/rsp+\lambda}|2Q|^{\sum_{j=1}^m 1/r'spq_j}||\vec{b}||_{BMO}\times$$
$$\times\left[|Q|^{1/q-1}\int_Q\omega(x)dx\right]^{1/r'sp}$$
$$\leq C||\vec{b}||_{BMO}||f||_{B^{rsp,\lambda}(\omega)}$$
$$\leq C||\vec{b}||_{BMO}||f||_{B^{t,\lambda}(\omega)}.$$

For $J_4(x)$, by Minkowski's inequality and noting that $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, we have

$$\begin{aligned}
J_4(x) &= ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)\\
&\quad -F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)||\\
&=\left(\int_0^\infty\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1}}\left[\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right]dy\right.\\
&\quad\left.-\int_{|x_0-y|\leq t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1}}\left[\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right]dy\Big|^2\frac{dt}{t^3}\right)^{1/2}\\
&\leq\left(\int_0^\infty\left[\int_{|x_0-y|\leq t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1}}\times\right.\right.\\
&\quad\left.\left.\times\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\leq t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1}}\times\right.\right.\\
&\quad\left.\left.\times\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|\leq t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\right|\times\right.\right.\\
&\quad\left.\left.\times\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&=J_4^{(1)}+J_4^{(2)}+J_4^{(3)}.
\end{aligned}$$

For $J_4^{(1)}$, because $\omega\in A_p$, $\omega^{-p'/p}\in A_{p'}$, we exploit the definition of A_p weight; for some $q>1$, let$\sum_{j=1}^m 1/q_j+1/q=1$, $q_j>1$, $j=1,2,\cdots,m$, $1/p+1/p'=1$, using Hölder's inequality and the reverse of Hölder's inequality; at the end, we use the double measure of ωdx,i.e.,$\omega(2Q)\leq a\omega(Q)$ for some $a>1$, we obtain

$$
\begin{aligned}
J_4^{(1)} &\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left(\int_{|x-y|\leq t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}\omega(y)^{-1/p}}{|x_0-y|^{n+1/2}}|f(y)|\omega(y)^{1/p}dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p'}\omega(y)^{-p'/p}dy\right]^{1/p'}\\
&\quad\times\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^p\,\omega(y)dy\right]^{1/p}\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\prod_{j=1}^m\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_j(y)-(b_j)_{2Q})|^{p'q_j}\,dy\right]^{1/p'q_j}\\
&\quad\times\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\omega(y)^{-qp'/p}dy\right]^{1/qp'}\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^p\,\omega(y)dy\right]^{1/p}\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}k^m\prod_{j=1}^m||b_j||_{BMO}\left[\frac{C}{|2^{k+1}Q|}\int_{2^{k+1}Q}\omega(y)^{-p'/p}dy\right]^{1/p'}\times\\
&\quad\times\omega(2^{k+1}Q)^{1/p+\lambda}|2^{k+1}Q|^{-1/p}||f||_{B^{p,\lambda}(\omega)}\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}k^m||\vec{b}||_{BMO}\left(\frac{|2^{k+1}Q|}{\omega(2^{k+1}Q)}\right)^{1/p}\times\\
&\quad\times\omega(2^{k+1}Q)^{1/p+\lambda}|2^{k+1}Q|^{-1/p}||f||_{B^{p,\lambda}(\omega)}\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}k^m||\vec{b}||_{BMO}\omega(2^{k+1}Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}k^m a^{k\lambda}||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}\\
&\leq C||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)};
\end{aligned}
$$

similarly, we have $J_4^{(2)}\leq C||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}$.

We now estimate $J_4^{(3)}$. Using the properties in J_1 and the inequality

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \leq \left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),$$

we gain

$$\begin{aligned}
J_3 &\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|}{|x_0-y|^n}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy \\
&\quad + C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\left(\int_{|x_0-y|\leq t,|x-y|\leq t}\frac{dt}{t^3}\right)^{1/2}dy \\
&\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\right)|f(y)|dy \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\omega(y)^{-1/p}\times \\
&\quad \times|f(y)|\omega(y)^{1/p}dy \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{p'}\omega(y)^{-p'/p}dy\right]^{1/p'} \\
&\quad \times\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^p\,\omega(y)dy\right]^{1/p} \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})\prod_{j=1}^m\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_j(y)-(b_j)_{2Q})|^{p'q_j}\,dy\right]^{1/p'q_j} \\
&\quad \times\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\omega(y)^{-qp'/p}dy\right]^{1/qp'}\left[\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^p\,\omega(y)dy\right]^{1/p} \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})k^m\prod_{j=1}^m||b_j||_{BMO}\left[\frac{C}{|2^{k+1}Q|}\int_{2^{k+1}Q}\omega(y)^{-p'/p}dy\right]^{1/p'}\times \\
&\quad \times\omega(2^{k+1}Q)^{\frac{1}{p}+\lambda}|2^{k+1}Q|^{-1/p}||f||_{B^{p,\lambda}(\omega)} \\
&\leq C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})k^m||\vec{b}||_{BMO}\left(\frac{|2^{k+1}Q|}{\omega(2^{k+1}Q)}\right)^{1/p}\times \\
&\quad \times\omega(2^{k+1}Q)^{1/p+\lambda}|2^{k+1}Q|^{-1/p}||f||_{B^{p,\lambda}(\omega)}
\end{aligned}$$

$$\leq C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})k^m||\vec{b}||_{BMO}\omega(2^{k+1}Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}$$
$$\leq C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})k^m a^{k\lambda}||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}$$
$$\leq C||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{p,\lambda}(\omega)}$$
$$\leq C||\vec{b}||_{BMO}\omega(Q)^{\lambda}||f||_{B^{t,\lambda}(\omega)},$$

so

$$\left[\frac{1}{\omega(Q)^{1+s\lambda}}\int_Q |J_4(x)|^s\omega(x)dx\right]^{1/s}\leq C||\vec{b}||_{BMO}||f||_{B^{t,\lambda}(\omega)}.$$

Combining the inequalities above, Theorem 2 is therefore proved. □

Acknowledgement

The author would like to express his gratitude to the referee for his comments and suggestions.

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