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DUAL INTEGRAL EQUATIONS INVOLVING FOURIER TRANSFORMS WITH INCREASING SYMBOLS

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Abstract. The aim of the present work is to propose a method for investigating and solving dual integral equations involving Fourier transfom with increasing symbols.

1. Introduction

Let $\mathbb{R}$ be the real axis, $S(\mathbb{R})$ and $S'(\mathbb{R})$ be the L. Schwartz spaces of test and generalized functions, respectively (see [8, 15]). Denote by F and F^{-1} the direct and inverse Fourier transforms defined on $S'(\mathbb{R})$, respectively. The classical Fourier transforms F and F^{-1} are defined by the formulas

$$F[u](\xi) = \int_{-\infty}^{\infty} u(x)e^{i\xi x}dx, \quad F^{-1}[v](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v(x)e^{-i\xi x}d\xi.$$

The Sobolev-Slobodeskii space $H^s(\mathbb{R})(s \in \mathbb{R})$ is defined as the closure of the set $C_o^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support with respect to the norm (cf. [8])

$$||u||_s = \Big[\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{u}(\xi)|^2d\xi\Big]^{1/2} < \infty, \quad \hat{u} = F[u].$$

For a certain bounded interval $(a, b) \subset \mathbb{R}$, the subspace of $H^s(\mathbb{R})$ consisting of functions $u(x)$ with supp $u \subset [a, b]$ is denoted by $H_o^s(a, b)$, while the space of functions $v(x) = ru(x)$, where $u \in H^s(\mathbb{R})$ and r is the restriction operator to (a, b) is denoted by $H^s(a, b)$. The norm in $H^s(a, b)$ is defined by

$$||v||_{H^s(a,b)} = \inf_l ||lv||_s,$$

where the infimum is taken over all possible extentions $lv \in H^s(\mathbb{R})$.

Let us consider the dual equation

$$\begin{cases} F^{-1}[|\xi|^p A(\xi)\hat{u}(\xi)](x) = f(x), \quad x \in (a, b), \\ F^{-1}[\hat{u}(\xi)](x) = 0, \quad x \in \mathbb{R}\backslash(a, b), \end{cases} \tag{1.1}$$

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where $\hat{u} \in S'(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ is the unknown function, $f(x)$ is a given function in $H^{-p/2}(a,b), p \geq 0$ is an integer. Concerning the function $A(\xi)$ we make the following assumptions:

i) $A(\xi) \in C^{\infty}(\mathbb{R}), \quad A(-\xi) = A(\xi), \quad \mathrm{Re}A(\xi) \geq 0, \quad \forall \xi \in \mathbb{R},$
ii) $L(\xi) = 1 - A(\xi) = O(|\xi|^{-q}), \ |\xi| \to \infty, \quad q \gg 1.$

The dual equation (1.1) is a generalisation of some cases encountered in mixed boundary value problems of mathematical physics and contact problems of elasticity (see for example, [1, 2, 9, 13, 15]). The case $p = 1$ was considered in [10] and the case $p = -m$, where m is positive number was considered in [12].

The aim of the present work is to propose a method for investigating and solving the dual equation (1.1) for an arbitrary non-negative integer p. Depending on whether p is an odd or even number, we shall reduce this dual equation to equivalent Fredholm integral equations of second type.

We get the following result which has been proved in [11].

Theorem 1.1. *If $f(x) \in H^{-p/2}(a,b)$, then under asumptions* i) *and* ii) *the dual equation has a unique solution $u = F^{-1}[\hat{u}] \in H_o^{p/2}(a,b)$.*

2. Some preliminary considerations

Let $J = (a,b)$ be a certain bounded interval, $\varphi(x) \in L^1(a,b)$ and m a positive integer. The differential operator of negative order D_J^{-m} is defined by the following formula (see [4]):

$$D_J^{-m}[\varphi](x) := \frac{1}{\Gamma(m)} \int_a^x (x-t)^{m-1} \varphi(t) dt, \quad x \in J = (a,b), \tag{2.1}$$

where $\Gamma(m)$ is the Gamma-function. It is known that $D_J^{-m}[\varphi] \in C^{m-1}[a,b]$ and

$$D^m D_J^{-m}[\varphi](x) = \varphi(x), \quad \lim_{m \to 0} D_J^{-m}[\varphi](x) = \varphi(x), \tag{2.2}$$

$$D_J^{-m} D^m[\varphi](x) = \varphi(x) + P_{m-1}(x), \tag{2.3}$$

where $D^m = \dfrac{d^m}{dx^m}$, $P_{m-1}(x)$ is an arbitrary polynomial of degree $m-1$.

Extensions of the operator D_J^{-m} $(m > 0)$ for generalized functions can be found in [7, 16].

We introduce the following definition.

Definition 2.1. Denote by $\mathcal{O}_m(a,b)$ the class of all functions $\varphi \in L^1(a,b)$, $\mathrm{supp}(\varphi) \subset [a,b]$, satisfying the conditions

$$\int_a^b \varphi(x) x^k dx = 0, \quad (k = 0, 1, \ldots, m-1). \tag{2.4}$$

Obviously, the conditions (2.4) are equivalent to the following

$$\int_a^b \varphi(x)Q_k(x)dx = 0 \quad (k = 0, 1, \dots, m-1), \tag{2.5}$$

where $Q_k(x)$ are arbitrary polynomials of degree k. As an immediate consequence of the formula (2.4), we note the following equalities

$$\int_a^b \varphi(t)(x-t)^k dt = 0 \quad (k = 0, 1, \dots, m-1), \quad -\infty < x < \infty, \tag{2.6}$$

$$\int_a^x \varphi(t)(x-t)^k dt = -\int_x^b \varphi(t)(x-t)^k dt \quad (k = 0, 1, ..., m-1), \quad a \leqslant x \leqslant b. \tag{2.7}$$

For $\varphi \in L^1(a,b)$ we introduce the operator

$$K_m[\varphi](x) = \frac{1}{2\Gamma(m)} \int_a^b \varphi(t)(x-t)^{m-1}\mathrm{sign}(x-t)dt, \quad x \in \mathbb{R}. \tag{2.8}$$

We get the following result.

Lemma 2.1. *If $\varphi \in \mathcal{O}_m(a,b)$, then*

1) $K_m[\varphi](x) = 0 \quad x \notin (a,b)$,
2) $K_m[\varphi](x) \equiv D_J^{-m}[\varphi](x) \quad a \leqslant x \leqslant b$,
3) $F[\varphi](\xi) \in C^\infty(\mathbb{R}), \quad F[\varphi](\xi) = O(|\xi|^k) \quad (\xi \to 0, \quad k \geq m)$,
4) $F[K_m[\varphi]](\xi) = \dfrac{1}{(-i\xi)^m} F[\varphi](\xi) \ (\xi \neq 0)$,
$F[K_m[\varphi]](0) = \dfrac{1}{m\Gamma(m)} \int_a^b \varphi(t)(b-t)^m dt$.

Proof. The assertions 1)-3) hold in virtue of (2.4), (2.6) and (2.7). We prove the assertion 4). The cases $\xi = 0$ is clear. For the cases $\xi \neq 0$, we have

$$F[K_m[\varphi]](\xi) = \frac{1}{\Gamma(m)} \int_a^b \varphi(t) e^{i\xi t} dt \int_0^{b-t} e^{i\xi\lambda} \lambda^{m-1} d\lambda.$$

Using the formula

$$\int_0^{b-t} e^{i\xi\lambda}\lambda^{m-1}d\lambda = \frac{\Gamma(m)}{(-i\xi)^m} + \Gamma(m)e^{i\xi(b-t)} \sum_{k=0}^{m-1} \frac{(-1)^k (b-t)^{m-1-k}}{(i\xi)^{k+1}(m-1-k)!},$$

we have

$$F[K_m[\varphi]](\xi) = \frac{1}{(-i\xi)^m} \int_a^b \varphi(t)e^{i\xi t}dt + \tag{2.9}$$
$$+ e^{i\xi b} \sum_{k=0}^{m-1} \frac{(-1)^k}{(i\xi)^{k+1}(m-1-k)!} \int_a^b \varphi(t)(b-t)^{m-1-k}dt.$$

Since $\varphi \in \mathcal{O}_m^o(a,b)$, from (2.9) the assertion 4) follows. □

Definition 2.2. Denote by $\hat{C}_o^m(a,b)$ the class of continuous functions $u(x) \in S'(\mathbb{R})$, such that $u(x) \in C^{m-1}[a,b]$, $u^{(k)}(x) = 0$ $(k = 0, 1, \ldots, m-1)$, $x \notin (a,b)$, $u^{(m)}(x) \in L^2(a,b)$.

Theorem 2.2. *In order that $u(x)$ belongs to the class $\hat{C}_o^m(a,b)$ it is necessary and sufficient that it is representable in the form* (2.8)*, i.e.*

$$u(x) = K_m[\varphi](x) = \frac{1}{2\Gamma(m)} \int_a^b \varphi(t)(x-t)^{m-1}\text{sign}(x-t)dt, \quad \varphi \in \mathcal{O}_m(a,b). \tag{2.10}$$

Proof. The sufficiency holds in virtue of Lemma 2.1. Now we prove the necessity. Let $u(x) \in \hat{C}_o^m(a,b)$. We put $\varphi(x) = D^m u(x)$. Obviously, $\varphi \in L^1(\mathbb{R})$ and $\text{supp}(\varphi) \subset [a,b]$. Integrating by parts, taking into account that $u(x) \in \hat{C}_o^m(a,b)$, we have

$$\int_a^b x^j \varphi(x)dx = \int_a^b x^j D^m u(x)dx = 0 \quad (j = 0, 1, \ldots, m-1), \tag{2.11}$$

it means that $\varphi \in \mathcal{O}_m^o(a,b)$. Due to (2.2), (2.3) we have

$$u(x) = D_J^{-m}[\varphi](x) + \sum_{j=0}^{m-1} c_j x^j, \quad x \in [a,b], \tag{2.12}$$

where c_j are arbitrary constants. Since $u^{(k)}(x)$ and $D_J^{-(m-k)}[\varphi](x)$ $(k = 0, 1, \ldots, m-1)$ are equal to zero on a and b, from (2.12) it follows that $c_j = 0$ $(j = 0, 1, \ldots, m-1)$. Thus we have

$$u(x) = D_J^{-m}[\varphi](x), \quad x \in [a,b]. \tag{2.13}$$

Using (2.1), (2.7) and (2.13) we get (2.10). ☐

Definition 2.3. By $L^{p\pm 0}(a,b)$ we denote the classes of functions f belonging to $L^{p\pm\varepsilon}$ respectively for sufficiently small $\varepsilon > 0$ $(p - \varepsilon \geq 1)$. If the interval (a,b) is bounded, then the symbol $L^{p-0}(a,b)$ denotes the set of functions f belonging to $L^q(a,b)$, $1 \leqslant q < p$.

Definition 2.4. Let $\rho(x) = \sqrt{(x-a)(b-x)}$ $(a < x < b)$. We denote by $L^2_{\rho^{\pm 1}}(a,b)$ the Hilbert spaces of functions with respect to the scalar product and the norm

$$(u,v)_{L^{\rho^{\pm 1}}} = \int_a^b \rho^{\pm 1}(x)u(x)\overline{v(x)}dx, \quad ||u||_{L_{\rho^{\pm 1}}} = \sqrt{(u,u)_{L_{\rho^{\pm 1}}}} < +\infty.$$

The following lemma holds.

Lemma 2.3. *Let $\varphi \in L^2_\rho(a,b)$. Denote by φ_0 the zero-extension of the function φ on $\mathbb{R}$. Then, $\varphi_0 \in H_o^{-1/2}(a,b)$.*

Proof. Using Holder inequality one can prove that $L^2_\rho(a,b) \subset L^{4/3-0}(a,b)$. Therefore, the function $\varphi_0 \in L^{4/3-0}(\mathbb{R})$. Due to Hausdorff-Young theorem [14] we have $\hat{\varphi}_0(\xi) := F[\varphi_o](\xi) \in L^{4+0}(\mathbb{R})$. Hence

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}_o(\xi)|^2}{1+|\xi|} d\xi \leqslant \left(\int_{-\infty}^{\infty} |\hat{\varphi}_o|^{2q} d\xi\right)^{1/q} \left(\int_{-\infty}^{\infty} \frac{d\xi}{(1+|\xi|)^{q/(q-1)}}\right)^{(q-1)/q}, \tag{2.14}$$

where $q = 2+\varepsilon \quad (\varepsilon > 0)$. From (2.14), we have $\varphi_o \in H^{-1/2}(\mathbb{R})$, hence $\varphi_o \in H_o^{-1/2}(a,b)$. □

In the spaces $L_2^{\rho^{\pm 1}}(a,b)$ we consider the singular integral operator

$$S_J[\varphi](x) = \frac{1}{\pi i} \int_a^b \frac{\varphi(t)}{x-t} dt, \quad x \in J = (a,b),$$

where the integral is taken in the sense of Cauchy principal value. The following theorem is due to Khvedelidze and Duduchava [5].

Theorem 2.4. *The operator S_J is bounded in the spaces $L^2_{\rho^{\pm 1}}(a,b)$.*

In the sequel we shall need the following inverse formula for the Cauchy integral [6].

Theorem 2.5. *Under the assumption that $f(x) \in L^2_\rho(a,b) \cap H^{1/2}(a,b)$ the integral equation*

$$\frac{1}{\pi} \int_a^b \frac{\varphi(t)}{t-x} dt = f(x) \tag{2.15}$$

in the $L^2_\rho(a,b)$ has the solution

$$\varphi(x) = -\frac{1}{\pi\rho(x)} \int_a^b \frac{f(t)\rho(t)}{t-x} dt + \frac{C}{\rho(x)}, \tag{2.16}$$

where C is an arbitrary constant. Besides, if $f(x) \in L^2_{\rho^{-1}}(a,b)$ and the following condition holds

$$\int_a^b \frac{f(x)dx}{\rho(x)} = 0, \tag{2.17}$$

then the integral equation (2.15) *has a unique solution in $L^2_{\rho^{-1}}(a,b)$, defined by the formula*

$$\varphi(x) = -\frac{\rho(x)}{\pi} \int_a^b \frac{f(t)dt}{\rho(t)(t-x)}. \tag{2.18}$$

3. Equation with the symbol $|\xi|^{2m}A(\xi)$

In this section we consider the dual equation (1.1) for the case $p = 2m$, where m is a non-negative integer. Using the formulas

$$|\xi|^{2m} = (-1)^m(-i\xi)^{2m}, \quad F^{-1}[(-i\xi)^k\hat{u}](x) = D_x^k F^{-1}[\hat{u}](x)$$

we can write (1.1) in the form

$$\begin{cases} D^m F^{-1}[(-i\xi)^m A(\xi)\hat{u}(\xi)](x) = (-1)^m f(x), \quad x \in (a,b), \\ u(x) := F^{-1}[\hat{u}](x) = 0, \quad x \notin (a,b). \end{cases} \tag{3.1}$$

Note that due to Theorem 1.1 the dual equation (3.1) for $f \in H^{-m}(a,b)$ has a unique solution $u = F^{-1}[\hat{u}] \in H_o^m(a,b)$. According to imbedding theorems [17], we have $u \in \hat{C}_o^m(a,b)$. Then, in virtue of Theorem 2.2 the function $u(x)$ can be represented by the formula (2.10):

$$u(x) = \frac{1}{2\Gamma(m)}\int_a^b \varphi(t)(x-t)^{m-1}\text{sign}(x-t)dt, \quad \varphi \in \mathcal{O}_m^o(a,b)$$

and its Fourier transform has the form

$$\hat{u}(\xi) = F[u](\xi) = \frac{1}{(-i\xi)^m}\int_a^b e^{i\xi t}\varphi(t)dt = \frac{1}{(-i\xi)^m}F[\varphi](\xi). \tag{3.2}$$

We shall find the function $\varphi(t)$ in the space $H_o^0(a,b)$. Note that, the space $H_o^0(a,b)$ consists of functions belonging to the space $L^2(\mathbb{R})$, with supports in $[a,b]$. From Theorem 2.2 we see that, if $\varphi(t) \in H_o^0(a,b) \cap \mathcal{O}_m(a,b)$, then the function $u(x)$ defined by the formula (2.10) belongs to the space $H_o^m(a,b)$.

For convenience, we take the conditions (2.5) in the form

$$\int_a^b \varphi(x)P_k[\xi(x)]dx = 0, \quad (k = 0,1,2,\dots,m-1), \tag{3.3}$$

where $P_k(\xi)$ are Legendre polynomials of order k and

$$\xi(x) = \frac{2x-(a+b)}{b-a}. \tag{3.4}$$

We have

$$\int_a^b P_m[\xi(x)]P_n[\xi(x)]dx = 0 \ (m \neq n), \quad \int_a^b P_m^2[\xi(x)]dx = \frac{b-a}{2m+1}. \tag{3.5}$$

Now we turn to the dual equation (3.1). Since $f \in H^{-m}(a,b)$, there exists $D_J^{-m}f$ introduced in the previous section. Note that the function $F^{-1}[(-i\xi)^m A(\xi)$ $\hat{u}(\xi)](x)$ belongs to $L^2(a,b)$, therefore there exists its extension belonging to the space $S'(\mathbb{R})$ with support in $[a,+\infty)$. Within $J = (a,b)$ the operator D^m can be considered as the operator D_J^m.

Now applying the operator D_J^{-m} to the both sides of the first equation in (3.1), in view of the formula (2.3) we obtain

$$F^{-1}[(-i\xi)^m A(\xi)\hat{u}(\xi)](x) = (-1)^m D_J^{-m} f(x) + \sum_{j=0}^{m-1} a_j P_j[\xi(x)], \tag{3.6}$$

where a_j are arbitrary constants, $P_j(\xi)$ are Legendre polynomials and the function $\xi(x)$ is defined by the formula (3.4). Now in (3.6) we substitute $A(\xi)$ and $\hat{u}(\xi)$ by using the condition ii) in Section 1 and the formula (3.2) respectively. After some transformations we obtain the following integral equation

$$\varphi(x) - \int_a^b l(x-t)\varphi(t)dt = g(x) + \sum_{j=0}^{m-1} a_j P_j[\xi(x)], \tag{3.7}$$

where

$$l(x) = \frac{1}{\pi}\int_0^\infty L(\xi)\cos(x\xi)d\xi, \quad g(x) = D_J^{-m}[(-1)^m f](x).$$

Using (3.7) and (3.5), fulfiling the conditions (3.3) we have

$$a_j = -\frac{2j+1}{b-a}\Big(\int_a^b g(y)P_j[\xi(y)]dy + \int_a^b \varphi(t)dt \int_a^b l(y-t)P_j[\xi(y)]dy\Big). \tag{3.8}$$

From (3.7) and (3.8) we have the following integral equation

$$\varphi(x) - \int_a^b K(x,t)\varphi(t)dt = h(x), \quad x \in (a,b), \tag{3.9}$$

where

$$h(x) = g(x) - \sum_{j=0}^{m-1}\Big(\frac{2j+1}{b-a}\int_a^b g(y)P_j[\xi(y)]dy\Big)P_j[\xi(x)], \tag{3.10}$$

$$K(x,t) = l(x-t) - \sum_{j=0}^{m-1}\Big(\frac{2j+1}{b-a}\int_a^b l(y-t)P_j[\xi(y)]dy\Big)P_j[\xi(x)]. \tag{3.11}$$

We now verify that the solution φ of the integral equation (3.9) satisfies the conditions (3.3). Indeed, from (3.9)-(3.11) for $k = 0, 1, \dots, m-1$, we get

$$\begin{aligned}&\int_a^b \varphi(x)P_k[\xi(x)]dx - \int_a^b \varphi(t)dt\int_a^b l(x-t)P_k[\xi(x)]dx+\\ &\quad + \int_a^b \varphi(t)dt \sum_{j=0}^{m-1}\frac{2j+1}{b-a}\int_a^b l(y-t)P_j[\xi(y)]dy.\int_a^b P_j[\xi(x)]P_k[\xi(x)]dx\\ &= \int_a^b g(x)P_k[\xi(x)]dx - \sum_{j=0}^{m-1}\frac{2j+1}{b-a}\int_a^b g(y)P_j[\xi(y)]dy.\int_a^b P_j[\xi(x)]P_k[\xi(x)]dx.\end{aligned} \tag{3.12}$$

Using (3.5), from (3.12) we have

$$\int_a^b \varphi(x)P_k[\xi(x)]dx = 0 \ (k = 0, 1, 2, \dots, m-1),$$

that is, the conditions (3.3) are fulfilled.

Thus, we have proved the following result.

Theorem 3.1. *The dual equation* (3.1) *considered in* $H_o^m(a,b)$ *with respect to* $u = F^{-1}[\hat{u}]$ *is equivalent to the Fredholm integral equation* (3.9) *with respect to* $\varphi(x) \in L^2(a,b)$. *Hence, according to the theory of Fredholm equations, this equation has a unique solution in* $L^2(a,b)$, *if* $g(x) = D_J^{-m}[(-1)^m f](x) \in L^2(a,b)$. *In this case, the solution with respect to* $u = F^{-1}[\hat{u}]$ *of the dual integral equation* (3.1) *in* $H_o^m(a,b)$ *is given by the formula* (2.10):

$$u(x) = \frac{1}{2\Gamma(m)} \int_a^b \varphi(t)(x-t)^{m-1}\text{sign}(x-t)dt, \quad \varphi \in \mathcal{O}_m(a,b).$$

Let us consider the following example. For simplicity, let $A(\xi) \equiv 1$.

Example 3.1. Let $J = (-1,1)$, $|\xi|^{2m}A(\xi) \equiv |\xi|^4$, $f(x) = Q\delta_J(x)$, where $Q =$ *const*, $\delta_J(x)$ is the restriction of the δ-function on J. In this case the dual integral equation (3.1) is equivalent to the following boundary value problem

$$\frac{d^4u(x)}{dx^4} = Q\delta_J(x), \quad (-1 < x < 1), \tag{3.13}$$

$$u(x) = u'(x) = 0, \quad x \notin (-1,1). \tag{3.14}$$

Note that, in mechanics sense, the problem (3.13)- (3.14) represents the bend equation of a beam with clamped ends, under a force concentrated at center of the beam. We shall find the solution of the problem (3.13)-(3.14) in the space $H_o^2(-1,1)$. It is well-known that $\delta \in H^{-1/2-\varepsilon}(\mathbb{R}) \subset H^{-2}(\mathbb{R})$, $\forall \varepsilon > 0$, therefore $\delta_J \in H^{-2}(-1,1)$ and the function $\delta(x)$ is an extension of the function $\delta_J(x)$. Further, $D_J^{-2}[\delta_J] \in L^2(-1,1)$.

According to (2.10) and (3.9), we have

$$u(x) = \begin{cases} \int_{-1}^x \varphi(t)(x-t)dt, & |x| < 1, \\ 0, \quad |x| \geq 1, \end{cases} \tag{3.15}$$

where

$$\varphi(t) = QD_J^{-2}[\delta_J(t)] + a_0 + a_1 t = \begin{cases} \dfrac{-Q}{4} - \dfrac{Qt}{2}, & -1 < t < 0, \\ \dfrac{-Q}{4} + \dfrac{Qt}{2}, & 0 \leqslant t < 1. \end{cases} \tag{3.16}$$

Putting (3.16) into (3.15), after some transforms, we have

$$u(x) = \begin{cases} \dfrac{Q}{24}(-2x^3 - 3x^2 + 1), & -1 < x \leqslant 0, \\ \dfrac{Q}{24}(2x^3 - 3x^2 + 1), & 0 \leqslant x < 1. \end{cases} \tag{3.17}$$

It is easy to verify that the function $u(x)$ defined by the formula (3.17) satisfies the differential equation (3.13) and the boundary conditions (3.14).

4. Equation with the symbol $|\xi|^{2m+1}A(\xi)$

For the case $p = 2m+1$ the dual equation (1.1) can be written in the following form

$$\begin{cases} D^m F^{-1}[(-i\xi)^{m+1} i\mathrm{sign}(\xi)A(\xi)\hat{u}(\xi)](x) = (-1)^m f(x), \quad x \in (a,b), \\ u(x) := F^{-1}[\hat{u}](x) = 0, x \notin (a,b). \end{cases} \tag{4.1}$$

Due to Theorem 1.1 the dual equation (4.1) for $f \in H^{-(m+1/2)}(a,b)$ has a unique solution $u = F^{-1}[\hat{u}] \in \overset{\circ}{H}{}^{m+1/2}(a,b)$. According to imbedding theorem [17], we have $u(x) \in \hat{\overset{\circ}{C}}{}^{m+1}(a,b)$. In virtue of Theorem 2.1 this function can be represented in the form

$$u(x) = \frac{1}{2\Gamma(m+1)} \int_a^b \varphi(t)(x-t)^m \mathrm{sign}(x-t)dt, \tag{4.2}$$

where $\varphi(t) \in \mathcal{O}_{m+1}(a,b)$. For convenience we take the conditions (2.4) in the form

$$\int_a^b \varphi(x) T_k[\xi(x)]dx = 0, \quad (k = 0, 1, \dots, m), \tag{4.3}$$

where $T_k(\xi)$ are Chebyshev polynomials of first order and the function $\xi(x)$ is defined by the formula (3.4). As we know, the Fourier transform $\hat{u}(\xi)$ of the function $u(x)$ is defined by the formula (see (2.10)):

$$\hat{u}(\xi) = \frac{1}{(-i\xi)^{m+1}} \int_a^b \varphi(t) e^{i\xi t} dt = \frac{1}{(-i\xi)^{m+1}} F[\varphi](\xi). \tag{4.4}$$

We shall find the function $\varphi(t)$ in the space $L^2_\rho(a,b)$, $\rho(x) = \sqrt{(x-a)(b-x)}$. Using Lemma 2.4 and the formula (4.4) one can show that the function $u(x)$ belongs to the space $\overset{\circ}{H}{}^{m+1/2}(a,b)$. By the same argument as in the previous section we can apply the operator D_J^{-m}, $J = (a,b)$ to the first equality in (4.1) and get

$$F^{-1}[(-i\xi)^{m+1} i \ \mathrm{sign}(\xi) A(\xi)\hat{u}(\xi)](x) = g(x) + \sum_{j=0}^{m-1} c_j U_j[\xi(x)], \ x \in (a,b), \tag{4.5}$$

where $c_j = const$, $g(x) = (-1)^m D_J^{-m} f(x), U_j(\xi)$ are Chebyshev polynomials of second order. Now in (4.5) we substitute $A(\xi)$ and $\hat{u}(\xi)$ by using the condition ii) in section 1 and the formula (4.4) respectively. Using the formula

$$F^{-1}[\mathrm{sign}(\xi) F[\varphi](\xi)](x) = \frac{1}{\pi i} \int_a^b \frac{\varphi(t)dt}{x-t}, \quad \varphi \in L^2_{\rho^{\pm 1}}(a,b),$$

after some transformations we have

$$\frac{1}{\pi}\int_a^b \varphi(t)\frac{dt}{t-x} = -\frac{1}{\pi}\int_a^b \varphi(\tau)k(x-\tau)d\tau - g(x) - \sum_{j=0}^{m-1} c_j U_j[\xi(x)], \tag{4.6}$$

where

$$k(x) = \int_0^{+\infty} L(\xi)\sin(x\xi)d\xi = \int_0^{+\infty}[1-A(\xi)]\sin(x\xi)d\xi.$$

Applying the formula (2.16) to the equation (4.6), we have the following integral equation of second kind

$$\begin{aligned}\varphi(x) = {} & \frac{1}{\pi^2\rho(x)}\int_a^b \varphi(\tau)d\tau\int_a^b \rho(t)k(t-\tau)\frac{dt}{t-x} + \frac{1}{\pi\rho(x)}\int_a^b \rho(t)g(t)\frac{dt}{t-x} \\ & + \frac{1}{\pi\rho(x)}\sum_{j=0}^{m-1} c_j\int_a^b \rho(t)U_j[\xi(t)]\frac{dt}{t-x} + \frac{C}{\rho(x)}, \quad a < x < b, \end{aligned} \tag{4.7}$$

where C is an arbitrary constant. Using the condition

$$\int_a^b \varphi(t)T_0[\xi(t)]dt = \int_a^b \varphi(t)dt = 0$$

and the formula

$$\int_a^b \frac{1}{\rho(t)}\frac{dt}{t-x} = 0$$

from (4.7) we obtain $C = 0$.

For the determination of the coefficients c_j, $(j = 0, 1, \ldots, m-1)$ we shall use the following formulas which were obtained from [13] by the corresponding substitutions of variables:

$$\frac{1}{\pi}\int_a^b \rho(t)U_j[\xi(t)]\frac{dt}{t-x} = -\frac{b-a}{2}T_{j+1}[\xi(x)], \tag{4.8}$$

$$\frac{1}{\pi}\int_a^b \frac{T_j[\xi(x)]dx}{\rho(x)(t-x)} = -\frac{2}{b-a}U_{j-1}[\xi(t)], \tag{4.9}$$

$$\frac{1}{\pi}\int_a^b T_k[\xi(x)]T_j[\xi(x)]\frac{dx}{\rho(x)} = \sigma_k\delta_{kj}, \ \ (\sigma_0 = 1, \ \sigma_k = \frac{1}{2}, k = 1, 2, \ldots), \tag{4.10}$$

where δ_{kj} is the Kronecker symbol. Thus, from (4.7) and (4.8) we have

$$\begin{aligned}\varphi(x) = {} & \frac{1}{\pi^2\rho(x)}\int_a^b \varphi(\tau)d\tau\int_a^b \rho(t)k(t-\tau)\frac{dt}{t-x} + \frac{1}{\pi\rho(x)}\int_a^b \rho(t)g(t)\frac{dt}{t-x} \\ & - \frac{b-a}{2\pi\rho(x)}\sum_{j=1}^{m} c_j^* T_j[\xi(x)], \quad c_j^* = c_{j-1}, \quad a < x < b. \end{aligned} \tag{4.11}$$

For the determination of coefficients c_j^* in (4.11) we shall use the conditions (4.3) and the formula (4.4). We have

$$c_j^* = -\frac{8}{(b-a)^2\pi}\int_a^b \varphi(\tau)d\tau\int_a^b \rho(t)k(t-\tau)U_{j-1}[\xi(t)]dt$$

$$-\frac{8}{(b-a)^2}\int_a^b \rho(t)g(t)U_{j-1}[\xi(t)]dt, \quad (j=1,2,...,m). \tag{4.12}$$

From (4.11) and (4.12) we have the integral equation with respect to $\sqrt{\rho(x)}\varphi(x) \in L^2(a,b)$:

$$\sqrt{\rho(x)}\varphi(x) - \int_a^b H(x,\tau)\sqrt{\rho(\tau)}\varphi(\tau)d\tau = h(x), \quad a<x<b, \tag{4.13}$$

where

(4.14)

$$H(x,\tau) = \frac{1}{\sqrt{\rho(x)}\sqrt{\rho(\tau)}\pi^2}\int_a^b \rho(t)k(t-\tau)\Big\{\frac{1}{t-x}+\frac{4}{b-a}\sum_{j=1}^m T_j[\xi(x)]U_{j-1}[\xi(t)]\Big\}dt,$$

$$h(x) = \frac{1}{\sqrt{\rho(x)}\pi}\int_a^b \rho(t)g(t)\Big\{\frac{1}{t-x}+\frac{4}{b-a}\sum_{j=1}^m T_j[\xi(x)]U_{j-1}[\xi(t)]\Big\}dt. \tag{4.15}$$

Note that, under the condition ii) in the section 1 for the function $A(\xi)$, in virtue of Theorem 2.5, from (4.14) we have $H(x,\tau) \in L^2((a,b)\times(a,b))$. Analogously, if $g(x) = (-1)^m D_J^{-m} f(x) \in L^2_\rho(a,b)$, then the function $h(x)$ given by the formula (4.15) belongs to $L^2(a,b)$.

We now verify that the solution φ of the integral equation (4.13) satisfies the conditions (4.3). Indeed, from (4.13) - (4.15) for $k=0,1,...,m$, it follows that

$$\int_a^b \varphi(x)T_k[\xi(x)]dx = \int_a^b \varphi(\tau)d\tau\int_a^b H(x,\tau)\sqrt{\rho(\tau)}T_k[\xi(x)]\frac{dx}{\sqrt{\rho(x)}}$$

$$= \int_a^b h(x)T_k[\xi(x)]\frac{dx}{\sqrt{\rho(x)}}. \tag{4.16}$$

By virtue of (4.9) - (4.10), we have

$$\int_a^b H(x,\tau)\sqrt{\rho(\tau)}T_k[\xi(x)]\frac{dx}{\sqrt{\rho(x)}}$$

$$= \frac{1}{\pi^2}\int_a^b k(t-\tau)\rho(t)dt\Big\{\int_a^b \frac{T_k[\xi(x)]dx}{\rho(x)(t-x)}$$

$$+\frac{4}{b-a}\sum_{j=1}^m U_{j-1}[\xi(t)]\int_a^b T_j[\xi(x)]T_k[\xi(x)]\frac{dx}{\rho(x)}\Big\}$$

$$= \frac{1}{\pi^2}\int_a^b k(t-\tau)\rho(t)dt\Big\{-\frac{2\pi}{b-a}U_{k-1}[\xi(t)]+\frac{4}{b-a}U_{k-1}[\xi(t)].\frac{\pi}{2}\Big\} \equiv 0. \tag{4.17}$$

Analogously, we have

$$\int_a^b h(x)T_k[\xi(x)]\frac{dx}{\sqrt{\rho(x)}} = \frac{1}{\pi}\int_a^b \rho(t)g(t)dt\Big\{\int_a^b \frac{T_k[\xi(x)]dx}{\rho(x)(t-x)} + \frac{4}{b-a}\sum_{j=1}^m U_{j-1}[\xi(t)]\int_a^b T_j[\xi(x)]T_k[\xi(x)]\frac{dx}{\rho(x)}\Big\} = 0. \tag{4.18}$$

From (4.16) - (4.18) it follows that the conditions (4.3) are fulfilled.

Thus, we obtain the following result

Theorem 4.1. *The dual equation* (4.1) *with respect to*

$$u(x) = F[\hat{u}](x) \in H_\circ^{m+1/2}(a,b)$$

is equivalent to the Fredholm integral equation (4.13) *with respect to* $\sqrt{\rho(x)}\varphi(x) \in L^2(a,b)$. *If* $g(x) = (-1)^m D_J^{-m} f(x) \in L^2_\rho(a,b)$, *then the equation* (4.13) *has a unique solution in* $L^2(a,b)$. *In this case, the solution with respect to* $u(x) = F[\hat{u}](x) \in H_\circ^{m+1/2}(a,b)$ *of the dual equation* (4.1) *is given by the formula:*

$$u(x) = \frac{1}{2\Gamma(m+1)}\int_a^b (x-t)^m \text{sign}(x-t)\varphi(t)dt, \quad x \in \mathbb{R}. \tag{4.19}$$

Example 4.1. As an illustration of the proposed method we consider the following dual equation

$$\begin{cases} F^{-1}[|\xi|^3\hat{u}(\xi)](x) = f_0 = const, & x \in (-1,1),\\ u(x) := F^{-1}[\hat{u}(\xi)](x) = 0, & x \notin (-1,1). \end{cases}$$

This equation can be written in the form

$$\begin{cases} \dfrac{d}{dx}F^{-1}[(-i\xi)^2 i.\text{sign}(\xi)\hat{u}(\xi)](x) = -f_0, & x \in (-1,1),\\ u(x) = 0, & x \notin (-1,1). \end{cases} \tag{4.20}$$

In this case we have

$$u(x) = \begin{cases} \int_{-1}^x (x-t)\varphi(t)dt, & |x| < 1,\\ 0, & |x| \ge 1, \end{cases} \tag{4.21}$$

(4.22)

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\int_{-1}^1 \sqrt{1-t^2}[-f_0(t+1)]\Big\{\frac{1}{t-x} + 2T_1(x)U_o(t)\Big\}dt, \quad -1 < x < 1.$$

Using formulas (4.8), (4.9) and taking into account that

$$T_o(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1,$$

$$U_o(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1$$

we can reduce the formula (4.22) to the following one

$$\varphi(x) = \frac{f_0}{2\sqrt{1-x^2}} - f_0\sqrt{1-x^2}, \quad |x| < 1. \tag{4.23}$$

Putting (4.23) into (4.21), we have

$$u(x) = \begin{cases} \frac{f_0}{6}(1-x^2)^{3/2}, & |x| < 1. \\ 0, & |x| \geq 1. \end{cases} \tag{4.24}$$

Putting $\varphi(x) = 0$ when $|x| \geq 1$ we have the following formulas for Fourier transforms of $\varphi(x)$ and $u(x)$ [3]:

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = -\frac{\pi f_0}{2} J_2(\xi), \quad \hat{u}(\xi) = F[u](\xi) = \frac{\pi f_0}{2}\frac{J_2(\xi)}{\xi^2}, \tag{4.25}$$

where $J_n(\xi)$ is the Bessel function of first kind:

$$J_n(\xi) = \Big(\frac{\xi}{2}\Big)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\Big(\frac{\xi}{2}\Big)^{2k}.$$

From (4.25) it is clear that

$$\hat{u}(\xi) = \frac{1}{(-i\xi)^2}\hat{\varphi}(\xi). \tag{4.26}$$

Using the asymptotic expansion of Bessel functions

$$J_n(\xi) = O\Big(\frac{1}{\sqrt{|\xi|}}\Big), \quad \xi \to \infty$$

we can show that $u(x) \in H_{\circ}^{3/2}(-1,1)$.

Now we verify the fulfilment of the dual equation (4.20). Putting (4.25) into (4.20) and taking into account (4.26), we have

$$\begin{aligned} \frac{d}{dx}F^{-1}[(-i\xi)^2\hat{u}(\xi)i \text{ sign}(\xi)](x) &= \frac{d}{dx}F^{-1}[\hat{\varphi}(\xi)i.\text{sign}(\xi)](x) \\ &= -\frac{f_0}{2}\frac{d}{dx}\int_0^\infty J_2(\xi)\sin(x\xi)d\xi \\ &= -\frac{f_0}{2}\frac{d}{dx}\int_0^\infty \Big[\frac{2J_1(\xi)}{\xi} - J_0(\xi)\Big]\sin(x\xi)d\xi. \end{aligned}$$

Using the identities (cf. [15]):

$$\int_0^\infty J_0(\xi)\sin(x\xi)d\xi = \begin{cases} 0, & |x| \leqslant 1, \\ \frac{\pm 1}{\sqrt{x^2-1}}, & \pm x > 1, \end{cases}$$

$$\int_0^\infty \frac{J_1(\xi)}{\xi}\sin(x\xi)d\xi = \begin{cases} 0, & |x| < 1, \\ \frac{\pm 1}{|x| + \sqrt{x^2-1}}, & \pm x \geq 1, \end{cases}$$

when $|x| < 1$ we have

$$\frac{d}{dx}F^{-1}[(-i\xi)^2\hat{u}(\xi)i\text{sign}(\xi)](x) = F^{-1}[|\xi|^3\hat{u}(\xi)](x) = -\ f_0, \quad |x| < 1.$$

Thus the given dual integral equation is fulfilled.

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