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A REMARK ON THE KIM'S THEOREM

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Abstract. In [9], K-T. Kim gave a nice characterization of a domain in $\mathbb{C}^n$ satisfying the Condition (L) at a boundary point. The key point in his proof is the convergence of the certain scaling sequence. Unfortunately, this assertion is not true. The purpose of this article is twofold. The first is to show a counterexample to the convergence of Kim's scaling sequence. The second is to give a correct proof of Kim's characterization theorem.

1. Introduction

Over the recent years, the scaling method introduced by S. Pinchuk [13] has been developed strongly. This method is playing a central role in the study of domains with noncompact automorphism groups (see [1, 2, 3, 4, 9]).

For convex domains, S. Frankel [6] proved the following theorem.

Frankel Theorem. *Let Ω be a convex domain in $\mathbb{C}^n$ $(n \geq 1)$. Suppose that there exist a sequence $\{g_j\} \subset Aut(\Omega)$ and a point $q \in \Omega$ such that $\lim_{j\to\infty} g_j(q) = p \in \partial\Omega$. Then the sequence $\{\omega_j : \Omega \to \mathbb{C}^n\}_{j=1,2,\cdots}$ defined by*

$$\omega_j(z) := [dg_j(q)]^{-1}(g_j(z) - g_j(q))$$

has a subsequence that converges to an one-to-one holomorphic mapping from Ω into $\mathbb{C}^n$.

The Frankel's scaling sequence $\{\omega_j\}_j$ still holds provided that Ω satisfies Condition (L) (see Section 2 in this note). Modifying the Frankel's scaling sequence, in [9], K. T. Kim introduced a new scaling sequence and proved the following

Proposition. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ $(n \geq 1)$ satisfying Condition (L). Suppose that there exist a sequence $\{g_j\} \subset \mathrm{Aut}(\Omega)$ and a point $q \in \Omega$ such that $\lim_{j\to\infty} g_j(q) = p \in \partial\Omega$. Then the sequence $\{\sigma_j : \Omega \to \mathbb{C}^n\}_{j=1,2,\cdots}$ defined by*

$$\sigma_j(z) := [dg_j(q)]^{-1}(g_j(z) - p)$$

has a subsequence that converges to an one-to-one holomorphic mapping from Ω into $\mathbb{C}^n$.

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Unfortunately, the above proposition of Kim is not true.

Basing on the above proposition, he showed the following characterization theorem of a domain in $\mathbb{C}^n$ satisfying the Condition (L) at a boundary point.

Main Theorem. *Let Ω be a domain in $\mathbb{C}^n$ satisfying the Condition (L) at $p \in \partial\Omega$. Then Ω is biholomorphic to the domain represented by the inequality*

$$0 > \mathrm{Re} z_1 + P_{m_2}(z_2) + \cdots + P_{m_n}(z_n) + \sum_{i_2,\cdots,i_n > 0} Q_{i_2,\cdots,i_n}(z_2,\cdots,z_n) \tag{1.1}$$

where

1. *$P_{m_k}(k = 2,\cdots,n)$ are real-valued positive homogeneous polynomials of degree m_k*
2. *$Q_{i_2,\cdots,i_n}(z_2,\cdots,z_n)$ are either identically 0, or real-valued positive homogeneous polynomials of degree $\sum_{l=2}^n i_l$, with fixed degree i_l in variables $z_l, \bar{z}_l$ for each l; and*
3. *$(i_2,\cdots,i_n)$ varies over the set of $(n-1)$-tuples of nonnegative integers, at least two of whose entries are nonzero, satisfying the relation*

$$\frac{i_2}{m_2} + \cdots + \frac{i_n}{m_n} = 1.$$

The aim of this note is to present a counterexample to his proposition and afterwards, to give a correct proof of Main Theorem.

2. Definitions and statements

Definition 1. We say that a domain $\Omega \subset \mathbb{C}^n$ (not necessary bounded) satisfies Condition (L) at $p \in \partial\Omega$, if the following three conditions are satisfied:

1. $\partial\Omega$ is real-analytic near p and is of finite type $2k$ at p.
2. $p \in \partial\Omega$ is convexifiable, i.e., there exist an open neighborhood U of p in $\mathbb{C}^n$ and an one-to-one holomorphic mapping $F : U \to \mathbb{C}^n$ with $F(U \cap \Omega)$ is convex.
3. There exist a point $p_0 \in \Omega$ and a sequence $\{g_j\} \subset \mathrm{Aut}(\Omega)$ such that $\lim_{j\to\infty} g_j(p_0) = p \in \partial\Omega$.

Remark 1. i) By (1) and (2) in Condition (L), there is a local peak holomorphic function of Ω at p. Thus, if Ω satisfies Condition (L), then for each compact subset $K \Subset \Omega$ and each neighborhood U of p, there exists an integer j_0 such that $g_j(K) \subset \Omega \cap U$ for all $j \geq j_0$.

ii) Moreover, since $\partial\Omega$ is smooth and is convexifiable near p, there exists a small ball $B(p)$ centered at p such that $B(p) \cap \Omega$ is hyperconvex and therefore is taut. By Proposition 2.1 in [4], Ω is taut.

We now present a counterexample for the Kim's scaling sequence.

Counterexample. We consider the unit disc Δ in $\mathbb{C}$. Let $\{a_j\}$ be a sequence in Δ such that $a_j \to 1$ as $j \to \infty$. Let $\{g_j\} \subset \mathrm{Aut}(\Delta)$ defined by

$$g_j(z) = \frac{z + a_j}{1 + \bar{a}_j z}.$$

We see that $g_j(0) = a_j \to 1$ as $j \to \infty$. Thus, Δ satisfies the Condition (L) at $p = 1 \in \partial\Delta$. By a simple computation, the Kim's scaling sequence is given by

$$\sigma_j(z) = \frac{1}{1 - |a_j|^2} \frac{a_j - 1 + z(1 - \bar{a}_j)}{1 + \bar{a}_j z}.$$

If we take $a_j = 1 - \frac{1}{j^2} + \frac{i}{j}$, then the sequence $\{\sigma_j\}$ is not normal.

Let Ω be a domain in $\mathbb{C}^n$, and assume that the boundary $\partial\Omega$ is C^∞ smooth near p. Suppose that there exist a point $q \in \Omega$ and a sequence $\{g_j\} \subset \mathrm{Aut}(\Omega)$ such that $g_j(q) \to p$ as $j \to \infty$. We say that the automorphism orbit $\{g_j(q)\}$ accumulates at p nontangentially to $\partial\Omega$, if there exists a constant $C > 0$ independent of j such that

$$\mathrm{dist}(g_j(q), p) \le C \mathrm{dist}(g_j(q), \partial\Omega), \quad j = 1, 2, \cdots.$$

In the above counterexample, we see that the automorphism orbit $\{g_j(0)\}$ accumulates at 1 tangentially to $\partial\Delta$. However, if automorphism orbit $\{g_j(q)\}$ accumulates at p nontangentially to $\partial\Omega$, then K-T. Kim proved that the Kim's scaling sequence is normal (see [10, Proposition 5, p. 478]). Of course any sequence $\{s_j\} \subset \partial\Omega$ such that the sequence $\{[dg_j(q)]^{-1}(p - s_j)\}$ is bounded, then the sequence $\{\sigma_j(z) = [dg_j(q)]^{-1}(g_j(z) - s_j)\}$ is normal. For instance, we can choose $s_j \in \partial\Omega$ such that $\mathrm{dist}(g_j(q), s_j) = \mathrm{dist}(g_j(q), \partial\Omega)$ $(j = 1, 2, \cdots)$. In fact, for each j if we set $d_j = \|p - g_j(q)\|$, then the open ball $B_{d_j}(g_j(q))$ centered at $g_j(q)$ with radius d_j is contained in Ω. Let $\mathbb{B}$ be the open unit ball centered at the origin in $\mathbb{C}^n$. Then define $f_j : \mathbb{B} \to \Omega$ by

$$f_j(z) = g_j^{-1}(d_j z + g_j(q)).$$

Note that $f_j(0) = q$ for all j and Ω is taut. Therefore, by a normal family argument, there exists a constant $K > 0$ such that

$$\|df_j(0)\| \le K, \quad \forall j,$$

which in turn implies that

$$\|[dg_j(0)]^{-1}\| \le \frac{K}{d_j}, \quad \forall j.$$

Consequently, we get

$$\|[dg_j(0)]^{-1}(p - g_j(q))\| \le K, \quad \forall j.$$

Lemma A in [9] is now replaced by the following lemma.

Lemma 1. *Let Ω be a domain in $\mathbb{C}^n$ satisfying the Condition (L) at $p \in \partial\Omega$. Then there exist a sequence $\{s_j\} \subset \partial\Omega$ and a sequence $\{A_j\{\subset GL_n(\mathbb{C})$ such that*

(1) $\|A_j^{-1}\| \to 0$ *as* $j \to \infty$; *and*

(2) $\lim_{j\to\infty} A_j(\Omega - s_j) = \hat{\Omega}$ *exists and is biholomorphic to* Ω*, where the limit is taken in the sense of local Hausdorff distances in* $\mathbb{C}^n$ *and* $\Omega - s_j = \{z = s_j \in \mathbb{C}^n | z \in \Omega\}$.

We now prove Main Theorem of Kim by using Lemma 1.

3. Proof of Main Theorem

First of all, we prove Main Theorem in complex dimension two.

We may assume that the domain Ω is convex near $0 \in \partial\Omega$. Also, let

$$\Omega = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) < 0\}$$

with $\rho(0, 0) = 0$ and let the tangent plane to $\partial\Omega$ at 0 be defined by $\mathrm{Re}z = 0$. Then we may also assume

$$\rho(z, w) = u + \sum_{i+j=2k} c_{ij} w^i \bar{w}^j + O(v^2, vw, w^{2k+1})$$

near 0, where $u = \mathrm{Re}z$, $v = \mathrm{Im}z$, and $c_{ij} \in \mathbb{C}$.

Then clearly we can find positive numbers C_1, C_2 and domains D_1, D_2, respectively, such that, for some neighborhood U of 0 in $\mathbb{C}^2$, $D_1 \cap U \subset \Omega \cap U \subset D_2 \cap U$ with $\{0\} \subset \partial D_1 \cap \partial\Omega \cap \partial D_2$, where

$$D_1 := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}z < -C_1 |w|^{2k}\},$$
$$D_2 := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}z < -C_2 |w|^{2k}\}.$$

Let $\{A_j^{-1}\} = \{(b^j_{\alpha\beta})_{\alpha,\beta=1,\cdots,n}\}$ be a sequence of complex $n \times n$ matrices in Lemma 1. Let $\Omega_j = A_j(\Omega \cap U - s_j)$, then Ω_j in a large ball $B_R(0)$ may be represented by

$$\mathrm{Re}(b^j_{11} z + b^j_{12} w + a_j) + P_{2k}(b^j_{21} z + b^j_{22} w + b_j) + O(|\mathrm{Im}z^j|^2, |Imz^j||w^j|, |w^j|^{2k+1}) < 0,$$

where $P_{2k}(w) = \sum_{i+j=2k} c_{ij} w^i \bar{w}^j$, $z^j = b^j_{11} z + b^j_{12} w + a_j$, $w^j = b^j_{21} z + b^j_{22} w + b_j$ and $s_j = (a_j, b_j)$.

In a fixed large ball, $A_j(D_1 \cap U - s_j) \subset \Omega_j \subset A_j(D_2 \cap U - s_j)$. Let $\hat{\Omega} = \lim_{j\to\infty} \Omega_j$, by scaling lemma, we see that $\hat{\Omega}$ is a taut domain in $\mathbb{C}^n$. Hence, neither can $\lim_{j\to\infty} A_j(D_2 \cap U - s_j)$ be a lower dimensional set, nor $\lim_{j\to\infty} A_j(D_1 \cap U - s_j)$ gets too large to contain a complex line.

Without loss of generality we may assume that $\{b^j_{12}/b^j_{11}\}$ is bounded. Also we may assume that $b^j_{11} > 0$, for all j, replacing A_j by $(b^j_{11}/|b^j_{11}|)A_j$ after extracting a subsequence from A_j if necessary so that the sequence $a^j_{11}/|a^j_{11}|$ is convergent. Then we will have $b^j_{21}/\sqrt[2k]{|b^j_{11}|}$, $\mathrm{Re}a_j/|b^j_{11}|$, $b_j/\sqrt[2k]{|b^j_{11}|}$, and $b^j_{22}/\sqrt[2k]{|a^j_{11}|}$ bounded for all j, since, otherwise, either D_2 collapses to the set with empty interior or D_1 becomes too big to be hyperbolic at the limit, an obvious contradiction to

Lemma 1. Hence there is a "unique and canonical" scaling up to complex linear equivalence, depending only on the local defining function of $\partial\Omega$ at $(0,0)$.

Moreover, we will have, at the limit

$$\operatorname{Re}(z+\alpha w+a) < -P_{2k}(\beta z+\gamma w+b) \tag{3.1}$$

as a defining of $\hat{\Omega}$, where a is a real number, α,β,γ, and b are complex numbers and where $P_{2k}(w)=\sum_{i+j=2k} c_{ij}w^i\bar{w}^j$(Note that all the higher order terms vanish at the limit). Therefore, Ω is biholomorphic to the domain defined by the inequality $\operatorname{Re}\zeta < -P_{2k}(\xi)$. This proves Main Theorem in complex dimension 2.

Now we prove the Main Theorem in complex dimention 3, using the results in complex dimension 2. By Condition (L) in Lemma 1, we may assume that Ω is actually convex near p. Then, since $\partial\Omega$ is of type $2k$ at $p=0$, the defining function ρ near $p=0$ can be written as

$$\begin{aligned}\rho(z_1,z_2,z_3) = \operatorname{Re}z_1 + P_{m_2}(z_2) + P_{m_3}(z_3) + Q_{n_2n_3}(z_2,z_3)+\\ + O(v_1^2, v_1z_2, v_1z_3, z_2^{m_2+1}, z_2^{n_2+1}z_3^{n_3}, z_2^{n_2}z_3^{n_3+1}),\end{aligned} \tag{3.2}$$

where

(a) $v_1=\operatorname{Im}z_1$, $m_2=2k$.
(b) $P_{m_l}(l=1,2)$ is a homogeneous polynomial in $z_l,\bar{z}_l$ with degree m_l.
(c) $Q_{n_1n_2}$ consists of all monomials of degree n_2 in $z_2,\bar{z}_2$ and of degree n_3 in $z_3,\bar{z}_3$, respectively, and,
(d) n_2 and $n_3>0$.

This expression is easily obtained by virtue of the convexity of Ω at $p=0\in\partial\Omega$. Now we try to find $\hat{\Omega}$ explicitly by a direct computation. We introduce the following notations for the later convenience.

$A_j\in GL_n(\mathbb{C})$ the scaling sequence introduced in Lemma 1.
$B_j := A_j^{-1} = (b^j_{\alpha\beta})$,
$\Omega_j = A_j(\Omega\cap U - s_j)$, assuming that $p=0\in\mathbb{C}^n$,
$\hat{\Omega}=\lim_{j\to\infty}\Omega_j$ as before,
$s_j=(s_1^j,s_2^j,s_3^j)$.

Then $\partial\Omega_j$ is defined by

$$\begin{aligned}0 =&\operatorname{Re}(b^j_{11}z_1+b^j_{12}z_2+b^j_{13}z_3+s^j_1)\\ &+P_{m_2}(b^j_{21}z_1+b^j_{22}z_2+b^j_{23}z_3+s^j_2)+P_{m_3}(b^j_{31}z_1+b^j_{32}z_2+b^j_{33}z_3+s^j_3)\\ &+Q_{n_2n_3}(b^j_{21}z_1+b^j_{22}z_2+b^j_{23}z_3+s^j_2, b^j_{31}z_1+b^j_{32}z_2+b^j_{33}z_3+s^j_3)+O^j,\end{aligned} \tag{3.3}$$

in the coordinates (z_1,z_2,z_3), where

$$O^j = O((v_1^j)^2, v_1^jz_2^j, v_1^jz_3^j, (z_2^j)^{m_2+1}, (z_2^j)^{n_2+1}(z_3^j)^{n_3}, (z_2^j)^{n_2}(z_3^j)^{n_3+1}) \tag{3.4}$$

with $z_h^j=\sum_{k=1}^3 b^j_{hk}z_k+s^j_h$ for $h=1,2,3$, and $v_1^j=\operatorname{Im}z_1^j$.

We may assume, without loss of generality, that $\{b^j_{1k}/b^j_{11}\}$, $\{\mathrm{Re}s^j_1/b^j_{11}\}$ are bounded for any $j = 1, 2, 3, \cdots$, for each $k = 1, 2, 3$. Also we may assume that $b^j_{11} > 0$ for all j. Replacing A_j by $b^j_{11}/|b^j_{11}|A_j$ after extracting a subsequence from $\{A_j\}$ if necessary so that the sequence $b^j_{11}/|b^j_{11}|$ is convergent. We now prove

Lemma 2. *The sequences* $\{b^j_{lk}/\sqrt[m_2]{b^j_{11}}\}$ *and* $\{s^j_h/\sqrt[m_2]{b^j_{11}}\}$ *are bounded for* $h = 2, 3$ *and all possible* l, k *and* j.

Proof. Since Ω is convex at $p = 0$ and is of finite type m_2, there is an open neighborhood N of 0 in $\mathbb{C}^3$ such that $\Omega \cap N \subset E_c \cap N$, where $E_c = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | \mathrm{Re}z_1 < -C(|z_2|^{m_2} + |z_3|^{m_2})\}$ and C is independent of j. Therefore, considering the fact that, for fixed $R > 0$, $\Omega_j \cap B_R \subset A_j(E_c - s_j) \cap B_R$ for any j large, it is clear that $\lim_{j\to\infty} A_j(E_c - s_j)$ must contain some open set. But then $A_j(E_c - s_j)$ is given by the inequality

$$\begin{aligned}\mathrm{Re}(b^j_{11}z_1 + b^j_{12}z_2 + b^j_{13}z_3 + \mathrm{Re}s^j_1) < -C(&|b^j_{21}z_1 + b^j_{22}z_2 + b^j_{23}z_3 + s^j_2|^{m_2}\\ &+ |b^j_{31}z_1 + b^j_{32}z_2 + b^j_{33}z_3 + s^j_3|^{m_2}\end{aligned} \tag{3.5}$$

i.e.,

$$\begin{aligned}&\mathrm{Re}\left(z_1 + \frac{b^j_{12}}{b^j_{11}}z_2 + \frac{b^j_{13}}{b^j_{11}}z_3 + \frac{\mathrm{Re}s^j_1}{b^j_{11}}\right)\\ &< -C\left(\left|\frac{b^j_{21}}{\sqrt[m_2]{b^j_{11}}}z_1 + \frac{b^j_{22}}{\sqrt[m_2]{b^j_{11}}}z_2 + \frac{b^j_{23}}{\sqrt[m_2]{b^j_{11}}}z_3 + \frac{s^j_2}{b^j_{11}}\right|^{m_2}\right.\\ &\left.+\left|\frac{b^j_{31}}{\sqrt[m_2]{b^j_{11}}}z_1 + \frac{b^j_{32}}{\sqrt[m_2]{b^j_{11}}}z_2 + \frac{b^j_{33}}{\sqrt[m_2]{b^j_{11}}}z_3 + \frac{s^j_3}{\sqrt[m_2]{b^j_{11}}}\right|^{m_2}\right).\end{aligned} \tag{3.6}$$

Therefore, to have $\lim_{j\to\infty} A_j(E_c - s_j)$ contain some open set, we must have the right-hand side bounded. So the lemma is proved. ☐

To understand the proof more geometrically and intuitively, we assume that the tangent planes $T_0(\partial\Omega_j)$ converge in the sense of the local Hausdorff distances in $\mathbb{C}^n$. By the lemma above, we assume that $b^j_{lk}/\sqrt[m_2]{b^j_{11}}$ converges for any $l = 2, 3$, $k = 1, 2, 3$, and that b^j_{1l}/b^j_{11} converges for $l = 2, 3$. Then consider

$$\Omega' := \{(z_1, z_2, z_3) \in \Omega | z_3 = 0\} \tag{3.7}$$

which is a complex two dimensional section of Ω, repesented by

$$0 > \mathrm{Re}z_1 + P_{m_2}(z_2) + 0(v_1^2, v_1 z_2, z_2^{m_2+1}), \tag{3.8}$$

Then it is clear that $\hat{\Omega}$ contains $\lim_j A_j(\Omega' - s_j)$ in its closure. Note that $\lim_j A_j(\Omega' - s_j)$ will be defined by

$$\mathrm{Re}(z_1 + az_2 + bz_3 + s_1) < -P_{m_2}(\alpha z_1 + \beta z_2 + \gamma z_3 + s_2), \tag{3.9}$$

as in (3.5), where s_1 is a real number and $a, b, \alpha, \beta, \gamma$, and s_2 are complex numbers. Since $\hat{\Omega}$ is a convex domain in $\mathbb{C}^3$, if the vector $(1, a, b)$ and (α, β, γ) are $\mathbb{C}$-linearly dependent, $\hat{\Omega}$ will contain a complex line. This violates the fact that $\hat{\Omega}$ has to be hyperbolic in Kobayashi's sense. Therefore, the vectors

$$(1, a, b) = \lim_j \left(1, b_{12}^j / b_{11}^j, b_{13}^j / b_{11}^j\right) \tag{3.10}$$

and

$$(\alpha, \beta, \gamma) = \lim_j \left(\frac{b_{21}^j}{\sqrt[m_2]{b_{11}^j}}, \frac{b_{22}^j}{\sqrt[m_2]{b_{11}^j}}, \frac{b_{23}^j}{\sqrt[m_2]{b_{11}^j}} \right) \tag{3.11}$$

are linearly independent over $\mathbb{C}$.

So we have two cases to consider, knowing that $\hat{\Omega}$ exists

Case 1. $P_{m_3}(\sum_{k=1}^3 b_{3k}^j z_k + s_3^j)/b_{11}^j$ is bounded.

Case 2. $P_{m_3}(\sum_{k=1}^3 b_{3k}^j z_k + s_3^j)/b_{11}^j$ tends to ∞ as $j \to \infty$.

We will show that, in the Case 1, there is a unique $\hat{\Omega}$, up to biholomorphic equivalence, determined entirely by the local defining function of Ω at $p \in \partial\Omega$ which is the boundary point satisfying Condition (L). Also, we will show that Case 2 does not occur. This will complete the proof of Main Theorem.

Proof of Main Theorem.

Case 1. In this case, as we pointed out before, $\{b_{3k}^j / \sqrt[m_3]{b_{11}^j}\}$ has to be a bounded sequence and hence may be assumed to be convergent by extracting a subsequence for each $k = 1, 2, 3$. Then expression (3.3) gives us, at the limit, the defining equation of $\partial\hat{\Omega}$

$$\begin{aligned} 0 = &\operatorname{Re}(z_1 + a z_2 + b z_3 + s_1) \\ &+ P_{m_2}(\alpha z_1 + \beta z_2 + \gamma z_3 + s_2) + P_{m_3}(s z_1 + t z_2 + r z_3 + s_3) \\ &+ \lim_{j \to \infty} \left\{ Q_{n_2, n_3} \left(\sum_{k=1}^3 b_{2k}^j z_k + s_2^j, \sum_{k=1}^3 b_{3k}^j z_k + s_3^j \right) / b_{11}^j + O^j / b_{11}^j \right\}, \end{aligned} \tag{3.12}$$

where O^j and $a, b, \alpha, \beta, \gamma$ are as in (3.4), (3.10) and (3.11), respectively, and where

$$(s, t, r) = \lim_j \left(\frac{b_{31}^j}{\sqrt[m_3]{b_{11}^j}}, \frac{b_{32}^j}{\sqrt[m_3]{b_{11}^j}}, \frac{b_{33}^j}{\sqrt[m_3]{b_{11}^j}} \right). \tag{3.13}$$

Since the right-hand side of (3.12) has to be finite as a whole, we must have $Q_{n_2,n_3}/b_{11}^j$ bounded, because it is the terms which grows fastest, if it diverges in this case. But then

(3.14)

$$Q_{n_2,n_3}\left(\sum_{k=1}^{3} b^j_{2k}z_k + s^j_2, \sum_{k=1}^{3} b^j_{3k}z_k + s^j_3\right)/b^j_{11}$$
$$= Q_{n_2,n_3}\left(\sum_{k=1}^{3} \frac{b^j_{2k}}{\sqrt[m_2]{b^j_{11}}}z_k + \frac{s^j_2}{\sqrt[m_2]{b^j_{11}}}, \sum_{k=1}^{3} \frac{b^j_{3k}}{\sqrt[m_3]{b^j_{11}}}z_k + \frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right)\left(b^j_{11}\right)^{\frac{n_2}{m_2}+\frac{n_3}{m_3}-1}$$

and hence we must have $\frac{n_2}{m_2} + \frac{n_3}{m_3} \geq 1$ and the Q term will converge either to 0 or to $Q_{n_2,n_3}(\alpha z_1 + \beta z_2 + \gamma z_3 + s_2, sz_1 + tz_2 + rz_3 + s_3)$. This convergence depends on the defining function ρ. The $\hat{\Omega}$ will be defined by, up to a holomorphic change of coordinates

$$\tilde{\rho}(\xi_1, \xi_2, \xi_3) = \mathrm{Re}\xi_1 + P_{m_2}(\xi_2) + P_{m_3}(\xi_3) + \tilde{Q}(\xi_2, \xi_3) < 0,$$

where $\tilde{Q}$ is either 0 or Q_{n_2,n_3} with $\frac{n_2}{m_2} + \frac{n_3}{m_3} = 1$ depending on ρ. So Case 1 is now completely understood.

Remark 2. If the vectors $(1, a, b)$, (α, β, γ), (s, t, r) are linearly dependent over $\mathbb{C}$, then $\hat{\Omega}$ would contain a complex line, which is not allowed.

Case 2. We now prove that no scaling by $\{A_j\}$ is possible in this case.

Since $\hat{\Omega}$ exists, the following expression for $\partial\Omega_j$ must have the right-hand side bounded

$$\begin{aligned}0 =&\mathrm{Re}\left(z_1 + \frac{b^j_{12}}{b^j_{11}}z_2 + \frac{b^j_{13}}{b^j_{11}}z_3 + \frac{s^j_1}{b^j_{11}}\right) + P_{m_2}\left(\sum_{k=1}^{3} \frac{b^j_{2k}}{\sqrt[m_2]{b^j_{11}}}z_k + \frac{s^j_2}{\sqrt[m_2]{b^j_{11}}}\right)\\ &+ P_{m_2}\left(\sum_{k=1}^{3} \frac{b^j_{3k}}{\sqrt[m_2]{b^j_{11}}}z_k + \frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right)\\ &+ Q_{n_2,n_3}\left(\sum_{k=1}^{3} b^j_{2k}z_k + s^j_2, \sum_{k=1}^{3} b^j_{3k}z_k + s^j_3\right)/b^j_{11} + O^j/b^j_{11},\end{aligned} \tag{3.15}$$

where O^j is as in (3.4).

Since $|\mathrm{Re}s^j_1| \approx |s^j_2|^{m_2} + |s^j_3|^{m_3}$, the sequence $\left\{\frac{s^j_3}{\sqrt[m_3]{b^j_{11}}})\right\}$ is bounded. Thus, in this case, the sequence $\left\{\left(b^j_{31}/\sqrt[m_3]{b^j_{11}}, \cdots, b^j_{33}/\sqrt[m_3]{b^j_{11}}\right)\right\}$ is not bounded. Notice that we have shown that the first terms on the right-hand side are bounded. Again by considering the rate of divergence, we can just ignore the O part, because it is lower growth rate, even though it goes to infinity. Hence, we must have

$$|Q_{n_2,n_3}/b^j_{11}| \to \infty, \tag{3.16}$$

and

$$(P_{m_3} + Q_{n_2,n_3})/b^j_{11} \quad \text{is stable.} \tag{3.17}$$

So we let

$$\begin{aligned} T^j(z) :=& P_{m_3}\left(\sum_{k=1}^{3} b^j_{3k} z_k + s^j_3\right)/b^j_{11} \\ &+ Q_{n_2,n_3}\left(\sum_{k=1}^{3} b^j_{2k} z_k + s^j_2, \sum_{k=1}^{3} b^j_{3k} z_k + s^j_3\right)/b^j_{11}. \end{aligned} \tag{3.18}$$

Let $z^0 = (z^0_1, z^0_2, z^0_3) \in \hat{\Omega}$. Then there will be $\epsilon_0 > 0$ such that $(1+\epsilon_0)z^0 \in \hat{\Omega}$ since $\hat{\Omega}$ is open. Then we have

(3.19)

$$\begin{aligned} T^j((1+\epsilon_0)z^0) = (1+\epsilon_0)^{m_3}&\Bigg[P_{m_3}\left(\sum_{k=1}^{3} b^j_{3k} z_k + \frac{s^j_3}{1+\epsilon_0}\right)/b^j_{11} \\ &+ Q_{n_2,n_3}\left(\sum_{k=1}^{3} b^j_{2k} z_k + \frac{s^j_2}{1+\epsilon_0}, \sum_{k=1}^{3} b^j_{3k} z_k + \frac{s^j_3}{1+\epsilon_0}\right)/b^j_{11}\Bigg] \\ &+ \left[(1+\epsilon_0)^{n_2+n_3} - (1+\epsilon_0)^{m_3}\right] \times \\ &\times Q_{n_2,n_3}\left(\sum_{k=1}^{3} b^j_{2k} z_k + \frac{s^j_2}{1+\epsilon_0}, \sum_{k=1}^{3} b^j_{3k} z_k + \frac{s^j_3}{1+\epsilon_0}\right)/b^j_{11} \end{aligned}$$

bounded. Thus we must have $m_3 = n_2 + n_3$. But then

$$\begin{aligned} T^j(z) =& P_{m_3}\left(\sum_{k=1}^{3} \frac{b^j_{3k}}{\sqrt[m_3]{b^j_{11}}} z_k + \frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right) \\ &+ Q_{n_2,n_3}\left(\sum_{k=1}^{3} \frac{b^j_{2k}}{\sqrt[m_2]{b^j_{11}}} z_k + \frac{s^j_2}{\sqrt[m_3]{b^j_{11}}}, \sum_{k=1}^{3} \frac{b^j_{3k}}{\sqrt[m_3]{b^j_{11}}} z_k + \frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right). \end{aligned} \tag{3.20}$$

Note that $m_3 \leq m_2$, since $m_2 = \tau(\partial\Omega, 0)$. Hence the sequence

$$\left\{\left(b^j_{21}/\sqrt[m_3]{b^j_{11}}, \cdots, b^j_{23}/\sqrt[m_3]{b^j_{11}}\right)\right\} \tag{3.21}$$

is never bounded unless $m_3 = m_2$. But in Case 2, because of Lemma 1, we know that $m_3 < m_2$. As we observed before

$$\begin{aligned} &\left(b^j_{21}/\sqrt[m_3]{b^j_{11}}, \cdots, b^j_{23}/\sqrt[m_3]{b^j_{11}}\right) \\ &= \left(b^j_{21}/\sqrt[m_2]{b^j_{11}}, \cdots, b^j_{23}/\sqrt[m_3]{b^j_{11}}\right).(b^j_{11})^{\frac{1}{m_2}-\frac{1}{m_3}} \to \infty \end{aligned} \tag{3.22}$$

as $j\to\infty$. So by (3.21) and (3.22), we have

$$\begin{aligned}T^j(z) =&P_{m_3}\left(\sum_{k=1}^{3}\frac{b^j_{3k}}{\sqrt[m_3]{b^j_{11}}}z_k+\frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right)+\left(b^j_{11}\right)^{n_1(\frac{1}{m_2}-\frac{1}{m_3})}\times\\ &\times Q_{n_2,n_3}\left(\sum_{k=1}^{3}\frac{b^j_{2k}}{\sqrt[m_2]{b^j_{11}}}z_k+\frac{s^j_2}{\sqrt[m_2]{b^j_{11}}},\sum_{k=1}^{3}\frac{b^j_{3k}}{\sqrt[m_3]{b^j_{11}}}z_k+\frac{s^j_3}{\sqrt[m_3]{b^j_{11}}}\right).\end{aligned}\tag{3.23}$$

Now we write

$$b^j_k=(b^j_{k1},\cdots,b^j_{k3}),\ B^j_k=\frac{b^j_k}{|b^j_k|}$$

for $k=2$, 3 and assume, again choosing subsequences of $\{A_j\}$ if necessary, that

$$\lim_{j\to\infty}B^j_2=B_2\quad\text{and}\quad\lim_{j\to\infty}B^j_3=B_3.$$

Then if B_2 and B_3 are linearly independent over $\mathbb{C}$, we can choose $z^1,z^2\in\hat{\Omega}$ such that $B_3.z^1=B_3.z^2,\ B_2.z^1\neq B_2.z^2$ and

$$Q_{n_2,n_3}\left(B_2.z^1+\frac{s^j_2}{|b^j_2|},B_3.z^1+\frac{s^j_3}{|b^j_3|}\right)\neq Q_{n_2,n_3}\left(B_2.z^2+\frac{s^j_2}{|b^j_2|},B_3.z^2+\frac{s^j_3}{|b^j_3|}\right).$$

Notice that both $\{\frac{s^j_2}{|b^j_2|}\}$ and $\{\frac{s^j_3}{|b^j_3|}\}$ are bounded. Then either $|T^j(z^1)|\to\infty$ or $|T^j(z^2)|\to\infty$, which is clearly a contradiction. Therefore, we have only to check the last remaining possibility that the vectors B_2 and B_3 are linearly dependent over $\mathbb{C}$.

Let us assume that $B_2=\lambda B_3$ for some $\lambda\in\mathbb{C},\ |\lambda|=1$. Then (3.23) becomes

$$T^j(z)=\Lambda^j_1.P_{m_3}\left(B_3.z+\frac{s^j_3}{|b^j_3|}\right)+\Lambda^j_2Q_{n_2,n_3}\left(cB^j_2.z+\frac{s^j_2}{|b^j_2|},B^j_3.z+\frac{s^j_3}{|b^j_3|}\right),\tag{3.24}$$

which must converge, where c is a complex number. Therefore, they must cancel out completely, because both Λ^j_1 and Λ^j_2 tend to infinity.

Now we repeat the same process on the homogeneous parts of next higher degree like $P_{m_3}+Q_{n_2,n_3}$, and end up with the limit domain defined by the inequality

$$\mathrm{Re}(B_1.z+s_1)+P_{m_3}(B_2.z+s_2)+H(B_2.z+s_2,B_3.z+s_3)<0,$$

for some real valued homogeneous polynomial H. But since $B_2=\lambda B_3$, this domain will process a complex line sitting inside, which cannot be allowed by the hyperbolicity of Ω and Lemma 1. This completes the proof in the case of complex dimension three.

Finally, we specify the induction step on n = complex dimension of (Ω) to finish the proof.

(H_n) *With defining function*

$$\rho(z_1, \cdots, z_n) = 2Rez_1 + P_{m_2}(z_2) + \cdots + P_{m_n}(z_n)$$
$$\sum_{i_2,\cdots,i_n} Q_{i_2,\cdots,i_n}(z_2, \cdots, z_n) + \textit{higher order terms,}$$

where the sum is taken over the $(n-1)$*-tuples* $(i_2, \cdots, i_n)$ *of nonnegative integers satisfying*

$$\frac{i_2}{m_2} + \cdots + \frac{i_n}{m_n} = 1$$

and where P_{m_k} *and* Q *are as described in Main Theorem, the scaling sequence satisfies the following conditions*

(1) b^j_{1l}/b^j_{11} *and* $b^j_{kl}/\sqrt[m_k]{b^j_{11}}$ *are bounded regardless of* j*, for each* $k = 2, 3$ *and* $l = 1, 2, 3$.
(2) $b^j_{11} > 0$.
(3) *The vectors* $B_1, \cdots, B_n$ *defined by*

$$B_1 = \lim_j (1, b^j_{12}/b^j_{11}, \cdots, b^j_{1n}/b^j_{11}),$$

$$B_k = \lim_j (1, b^j_{k2}/\sqrt[m_k]{b^j_{11}}, \cdots, b^j_{kn}/\sqrt[m_k]{b^j_{11}}), \quad (k = 2, \cdots, n)$$

are linearly independent over $\mathbb{C}$.

Now the induction step is completed by the same argument we have used to prove (H_3) after assuming (H_2). This completes the proof. □

4. Examples

In this section we consider some examples which show that the Kim's scaling sequence is not normal but the new one is.

Example 1. Take $\Omega = \Delta = \{|z| < 1\}$ and let $\{a_j\} \subset \Delta$ be a sequence such that $a_j \to 1 \in \partial\Delta$ as $j \to \infty$. Let $\{g_j := \frac{z + a_j}{1 + \bar{a}_j z}\} \subset \mathrm{Aut}(\Omega)$. Then as we pointed out before the Kim's scaling sequence given by

$$\sigma_j(z) = \frac{1}{1 - |a_j|^2} \frac{a_j - 1 + z(1 - \bar{a}_j)}{1 + \bar{a}_j z}$$

is not normal in general. But if we choose the sequence $\left\{s_j := \frac{a_j}{|a_j|}\right\} \subset \partial\Delta$, then the new scaling sequence given by

$$\tilde{\sigma}_j(z) = \frac{1}{1 - |a_j|^2} \left[\frac{z + a_j}{1 + \bar{a}_j z} - \frac{a_j}{|a_j|}\right]$$

is normal. In fact, by a simple computation we have

$$\tilde{\sigma}_j(z) = \frac{1}{1 + |a_j|} \frac{z}{1 + z} - \frac{1}{1 + |a_j|} \frac{a_j}{|a_j|} \frac{1}{1 + z}.$$

It is easy to see that this sequence converges to the biholomorphic mapping $\tilde{\sigma}(z) = \frac{1}{2} \cdot \frac{z-1}{1+z}$ and the unit disc is biholomorphically equivalent to the left-half plane $D := \{w \in \mathbb{C} : \mathrm{Re} w < 0\}$.

Example 2. Consider the ellipsoid $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$. Let $\{a_j\} \subset \Delta$ be a sequence such that $a_j \to 1 \in \partial\Delta$ as $j \to \infty$. Let $\{g_j\} \subset \mathrm{Aut}(\Omega)$ be a sequence defined by

$$\begin{cases} z' = \dfrac{z + a_j}{1 + \bar{a}_j z} \\ w' = \dfrac{\sqrt[4]{1 - |a_j|^2}}{\sqrt{1 + \bar{a}_j z}} w. \end{cases}$$

It is easy to see that $g_j(0, \frac{1}{2}) = (a_j, \frac{\sqrt[4]{1-|a_j|^2}}{2}) \to (1, 0) \in \partial\Omega$ as $j \to \infty$ and $\partial\Omega$ satisfies the Condition (L) at $p = (1, 0) \in \partial\Omega$. By a simple computation, we have

$$dg_j\left(0, \frac{1}{2}\right) = \begin{bmatrix} 1 - |a_j|^2 & 0 \\ -\dfrac{\bar{a}_j \sqrt[4]{1-|a_j|^2}}{4} & \sqrt[4]{1-|a_j|^2} \end{bmatrix},$$

$$\left[dg_j\left(0, \frac{1}{2}\right)\right]^{-1} = \begin{bmatrix} \dfrac{1}{1 - |a_j|^2} & 0 \\ \dfrac{\bar{a}_j}{4(1-|a_j|^2)} & \dfrac{1}{\sqrt[4]{1-|a_j|^2}} \end{bmatrix}.$$

Note that the Kim's scaling sequence $\sigma_j(z, w) := [dg_j(0, \frac{1}{2})]^{-1}(g_j(z, w) - (1, 0))$ given by

$$\begin{cases} z' = \dfrac{1}{1 - |a_j|^2}\left(\dfrac{z + a_j}{1 + \bar{a}_j z} - 1\right) \\ w' = \dfrac{\bar{a}_j}{4(1-|a_j|^2)}\left(\dfrac{z + a_j}{1 + \bar{a}_j z} - 1\right) + \dfrac{1}{\sqrt[4]{1-|a_j|^2}} \dfrac{\sqrt[4]{1-|a_j|^2}}{\sqrt{1 + \bar{a}_j z}}.w \end{cases}$$

is not normal in general. But if we choose the sequence $\{s_j\} \subset \partial\Omega$, where $s_j := \left(\frac{\sqrt{15 + |a_j|^2}}{4} \frac{a_j}{|a_j|}, \frac{\sqrt[4]{1-|a_j|^2}}{2}\right)$, then the new scaling sequence $\{\tilde{\sigma}_j\}$ given by

$$\begin{cases} z' = \dfrac{1}{1 - |a_j|^2}\left(\dfrac{z + a_j}{1 + \bar{a}_j z} - \dfrac{\sqrt{15 + |a_j|^2}}{4} \dfrac{a_j}{|a_j|}\right) \\ w' = \dfrac{\bar{a}_j}{4(1-|a_j|^2)}\left(\dfrac{z + a_j}{1 + \bar{a}_j z} - \dfrac{\sqrt{15 + |a_j|^2}}{4} \dfrac{a_j}{|a_j|}\right) + \dfrac{1}{\sqrt[4]{1-|a_j|^2}} \times \\ \qquad \times \left(\dfrac{\sqrt[4]{1-|a_j|^2}}{\sqrt{1 + \bar{a}_j z}} w - \dfrac{\sqrt[4]{1-|a_j|^2}}{2}\right) \end{cases}$$

is normal. By a simple computation, we can conclude that the above sequence converges to the biholomorphic mapping $\tilde{\sigma}$ which is given by

$$\begin{cases} z' = \dfrac{17z - 15}{32(1+z)} \\ w' = \dfrac{1}{4}\dfrac{17z - 15}{32(1+z)} + \dfrac{w}{\sqrt{1+z}} - \dfrac{1}{2} \end{cases}$$

and thus the domain Ω is biholomorphically equivalent to the domain

$$\tilde{\Omega} = \left\{ (z, w) \in \mathbb{C}^2 : 2\mathrm{Re}(z' - \frac{1}{32}) + |w' - \frac{z'}{4} + \frac{1}{2}|^4 < 0 \right\}.$$

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