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RELATIVE CAPACITY AND THE RELATIVE EXTREMAL FUNCTIONS UNDER HOLOMORPHIC COVERINGS

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ABSTRACT. In this note we establish formulas on the relative capacity of a subset E in a domain $\Omega \subset \mathbb{C}^n$ for cases either Ω is pseudoconvex or Ω is hyperconvex and $E \subset \Omega$ is a closed subset such that $\overline{E}_{\mathbb{C}^n} \cap \partial\Omega$ is pluripolar. Moreover the relation between relative extremal functions in a generalized holomorphic covering is studied here.

1. INTRODUCTION

As well known, pluripolar sets, i.e. sets on which a certain plurisubharmonic function obtains values $-\infty$, are one of important objects which often are studied in pluripotential theory. Hence, one of essential problems of pluripotential theory is to find characterizations of pluripolar sets in $\mathbb{C}^n$. One knew that every pluripolar set in $\mathbb{C}^n$ has the Lebesgue measure equal to 0 but the converse is not true. For example, the unit circle $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$ has the Lebesgue measure equal to 0 but it is not polar. This shows that to characterize pluripolar sets by the Lebesgue measure is impossible. Hence, for a long time, one tries to find something which characterizes the pluripolarity of sets in $\mathbb{C}^n$ $(n > 1)$. In 1982, after construction of the complex Monge-Ampère operator $(dd^c u)^n$ for u in the class of locally bounded plurisubharmonic functions, Bedford and Taylor introduced the notion about the relative capacity of a Borel subset E in a domain $\Omega \subset \mathbb{C}^n$. Let E be a Borel subset of a domain $\Omega \subset \mathbb{C}^n$. The relative capacity of E to Ω is defined as follows:

$$(1) \qquad \mathrm{C}(E) = \mathrm{C}(E, \Omega) = \sup\{\int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\},$$

(see [1]). They proved that if E is a compact set in a hyperconvex domain Ω then E is pluripolar if and only if $C(E, \Omega) = 0$ (see Proposition 4.7.5 in [3]). However, it is very difficult to show formulas defining the relative capacity of a Borel subset E in a domain $\Omega \subset \mathbb{C}^n$. Under the assumption that Ω is a hyperconvex domain

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in $\mathbb{C}^n$ and $E \subset \Omega$ a relative compact subset one obtained the following formula

$$\mathrm{C}(E) = \int_{\Omega} (dd^c u^*_{E,\Omega})^n \tag{2}$$

(see Proposition 4.7.2 in [3]), where $u^*_{E,\Omega}$ is the upper semi-continuous regularization of the relative extremal function $u_{E,\Omega}$ of E to Ω. The first aim of this paper is to improve the formula (2). Namely we prove that (2) is still true under the assumption that Ω is pseudoconvex (see Theorem 3.1 below). Next we try to remove the hypothesis on the compactness of E in Ω. In Theorem 3.2 we show that if $E \subset \Omega$ is a closed subset and $\overline{E}_{\mathbb{C}^n} \cap \partial\Omega$ is a pluripolar subset in $\mathbb{C}^n$ then the formula (2) is still valid. Moreover, in Example 3.3, we show that if we remove the condition on the pluripolarity of the set $\overline{E}_{\mathbb{C}^n} \cap \partial\Omega$ then (2) is not true.

Next, we investigate the invariance of the relative extremal function $u^*_{E,\Omega}$ under generalized holomorphic coverings (see the detailed definition in Section 2). In 1999, Levenberg and Poletsky proved that if $\mathbb{D}, \mathbb{G}$ are domains in $\mathbb{C}^n$ and $h : \mathbb{D} \to \mathbb{G}$ is a A-covering then for $E \subset \mathbb{G}$ the following equality

$$u_{h^{-1}(E),\mathbb{D}}(z) = u_{E,\mathbb{G}}(h(z)), \forall z \in \mathbb{D}$$

holds [4]. Hence,

$$u^*_{h^{-1}(E),\mathbb{D}}(z) = u^*_{E,\mathbb{G}}(h(z)), \forall z \in \mathbb{D}. \tag{3}$$

In the case if h is a proper holomorphic mapping of $\mathbb{D}$ onto $\mathbb{G}$ and $E \subset \mathbb{G}$ then we always have

$$u_{h^{-1}(E),\mathbb{D}}(z) u_{E,\mathbb{G}}(h(z)), \ \forall z \in \mathbb{D}$$

and hence,

$$u^*_{h^{-1}(E),\mathbb{D}}(z) = u^*_{E,\mathbb{G}}(h(z)), \quad \forall z \in \mathbb{D}.$$

(see Proposition 4.5.14 in [3]).

In Section 4 below we extend the formula (3) to the situation where $h : \mathbb{D} \to \mathbb{G}$ is a generalized holomorphic covering outside a complex subvariety $A \subset \mathbb{G}$.

The paper is organized as follows. In Section 2 we recall some basic notions and results of pluripotential theory which will be used in the paper. Section 3 is devoted to prove the improvement of the formula (2) in the cases explained above. Section 4 deals with the proof of (3) for generalized holomorphic coverings.

2. Backgrounds

In this section we recall some elements of pluripotential theory that will be used throughout the paper. All these can be found in [1, 2, 3].

2.1. In this paper by $\mathbb{D}, \mathbb{G}, \Omega$ we always mean domains in $\mathbb{C}^n$.

2.2. As well known that the Monge-Ampère operator $(dd^c.)^n$ is well defined on $\mathrm{PSH} \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{G})$ and if $u \in \mathrm{PSH}(\mathbb{G}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{G})$ then $(dd^c u)^n$ is a positive Borel measure. Moreover, it is continuous under monotone sequences. Namely, if $\{u_j\}_{j\geq 1} \subset \mathrm{PSH}(\mathbb{G}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{G})$ is a sequence either increasing or decreasing which converges pointwise to a function $u \in \mathrm{PSH}(\mathbb{G}) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{G})$ then $(dd^c u_j)^n$ is weak*-convergent to $(dd^c u)^n$ (see [1]).

2.3. Let Ω be an open subset in $\mathbb{C}^n$ and E a Borel subset of Ω. The relative capacity in the sense of Bedford-Taylor of E to Ω is given by

$$C(E) = C(E,\Omega) = \sup\left\{\int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega), -1 \leq u \leq 0\right\}.$$

Some following results on the relative capacity can be found in [1, 2, 3].

2.3.1. Proposition.

i) *If* $E_1 \subset E_2 \subset \Omega$ *then* $C(E_1,\Omega) \leq C(E_2,\Omega)$.
ii) *If* $E \subset \Omega \subset \widetilde{\Omega}$ *then* $C(E,\Omega) \geq C(E,\widetilde{\Omega})$.
iii) *If* $E_j \uparrow E$ *then* $\lim\limits_{j\to\infty} C(E_j) = C(E)$.

2.4. Let Ω be a domain in $\mathbb{C}^n$ and E a subset of Ω. The relative extremal function of E in Ω is defined by

$$u_E(z) = u_{E,\Omega}(z) = \sup\left\{v(z) : \quad v \in \mathrm{PSH}^-(\Omega), v|_E \leq -1\right\},$$

where $\mathrm{PSH}^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. By $u^*_{E,\Omega}$ we denote the upper semi-continuous regularization of $u_{E,\Omega}$. Below we give its basic properties which can be found in [1, 2] or [3].

2.4.1. Proposition.

i) $u^*_{E,\Omega}$ *is maximal in* $\Omega \setminus \overline{E}$.
ii) $u^*_{E\cup F,\Omega} = u^*_{E,\Omega}$ *if there exists* $v \in PSH^-(\Omega)$ *such that* $F \subset \{v = -\infty\}$.
iii) *If* $\{K_j\}$ *is a sequence of compact subsets of* Ω *decreasing to* K *then* $u_{K_j,\Omega} \uparrow u_{K,\Omega}$.

The following results which will be used in the proof of Section 3 of this paper come from Theorem 3.1.7 and Proposition 3.1.9 in [2].

2.4.2. Proposition.

i) *Assume that* $E_j \subset \Omega_j, \quad j = 1, 2, \cdots$ *are such that* $E_j \uparrow E, \quad \Omega_j \uparrow \Omega$ *and* Ω *is bounded. Then* $u^*_{E_j,\Omega_j} \downarrow u^*_{E,\Omega}$.
ii) *Let* Ω *be a bounded domain in* $\mathbb{C}^n$ *and* $E \subset \Omega$ *a Borel subset. Then there is an increasing sequence of compact sets* $K_j \subset E$ *such that* $u^*_{K_j,\Omega} \downarrow u^*_{E,\Omega}$.

2.5. Now we give some definitions on generalized holomorphic coverings and the property $(\mathbb{P})$ on a domain in $\mathbb{C}^n$.

Let $\mathbb{D} \subset \mathbb{C}^n$ and $\mathbb{G} \subset \mathbb{C}^m, m \leq n$ be domains and $h : \mathbb{D} \to \mathbb{G}$ a holomorphic surjection. h is said to be a generalized holomorphic covering if for every $a \in \mathbb{G}$ there exists a neighborhood V_a of a in $\mathbb{G}$ and an index set I (may be, non-countable) such that

$$h^{-1}(V_a) = \coprod_{i\in \mathrm{I}} W_i,$$

where $W_i \subset \mathbb{D}$ are open such that $V_a \cong W_i$ for all $i \in \mathrm{I}$.

The following example shows that there are such generalized holomorphic coverings.

Let $\mathbb{D} = \Delta^2 = \{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$ be the bidisc in $\mathbb{C}^2$ and $\mathbb{G} = \Delta = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc in $\mathbb{C}$. It is easy to check that the map $h : \mathbb{D} \to \mathbb{G} : (z,w) \mapsto z$ is a generalized holomorphic covering.

2.6. Let Ω be a domain in $\mathbb{C}^n$. Ω is said to have the property $(\mathbb{P})$ if for every pluripolar subset $E \subset \Omega$ there exists $u \in \mathrm{PSH}^-(\Omega), u \neq -\infty$ such that $E \subset \{z \in \Omega : u(z) = -\infty\}$.

The following proposition gives some results on the property $(\mathbb{P})$.

2.6.1. Proposition. *Let $\mathbb{G}_1, \mathbb{G}_2$ be domains in $\mathbb{C}^n$. Then*

i) *If $\mathbb{G}_1 \subset \mathbb{G}_2 \subset \mathbb{C}^n$ and $\mathbb{G}_2$ has the property $(\mathbb{P})$ then so is $\mathbb{G}_1$.*
ii) *If $h : \mathbb{G}_1 \to \mathbb{G}_2$ is a proper holomorphic map and $\mathbb{G}_1$ has the property $(\mathbb{P})$ then so is $\mathbb{G}_2$. In particular, the property $(\mathbb{P})$ is invariant under biholomorphisms.*
iii) *If $\mathbb{G}_1 \subset \mathbb{C}$ and $\mathbb{C} \setminus \mathbb{G}_1$ has the non-empty interior then $\mathbb{G}_1$ has the property $(\mathbb{P})$.*
iv) *If $\mathbb{G} \subset \mathbb{C}^n$ is bounded then $\mathbb{G}$ has the property $(\mathbb{P})$.*

Proof. i) It is obvious.

ii) Let $E \subset \mathbb{G}_2$ be a pluripolar set, then so is $Fh^{-1}(E) \subset \mathbb{G}_1$. Hence, there exists $u \in \mathrm{PSH}^-(\mathbb{G}_1)$ such that $h^{-1}(E) \subset \{z : u(z) = -\infty\}$. Put

$$v(w) = \max\{u(z) : z \in h^{-1}(w)\}, w \in \mathbb{G}_2.$$

Proposition 2.9.26 in [3] implies that $v \in \mathrm{PSH}^-(\mathbb{G}_2)$. Obviously, $E \subset \{w \in \mathbb{G}_2 : v(w) = -\infty\}$ and we are done.

iii) Without loss of generality we may assume that $\overline{\Delta}(0,1) \subset (\mathbb{C} \setminus \mathbb{G}_1)$ where $\overline{\Delta}(0,1) = \{z \in \mathbb{C} : |z| \leq 1\}$. Thus $\mathbb{G}_1 \subset \mathbb{C} \setminus \overline{\Delta}(0,1)$. The map $f : \mathbb{C} \setminus \overline{\Delta}(0,1) \to \Delta^*(0,1)$ given by $f(z) = \frac{1}{z}$, is a biholomorphism, where $\Delta^*(0,1) = \Delta(0,1) \setminus \{0\}$. Since $\Delta^*(0,1)$ has the property $(\mathbb{P})$ then i) and ii) give the desired conclusion.

iv) Obviously. □

2.6.2. Remark. Using the extended maximum principle in [5] it is easy to see that $\mathbb{G} = \mathbb{C} \setminus P$, where $P \subset \mathbb{C}$ is a closed polar set, has not the property $(\mathbb{P})$.

3. Capacity and relative extremal functions

We begin this section with the following result.

3.1. Theorem. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with the property $(\mathbb{P})$ and E a relatively compact Borel subset in Ω. Then*

$$\mathrm{C}(E,\Omega) = \int_\Omega (dd^c u^*_{E,\Omega})^n. \tag{4}$$

Proof. First we show that

$$\mathrm{C}(E,\Omega) \le \int_{\Omega} (dd^c u^*_{E,\Omega})^n. \tag{5}$$

Indeed, suppose that $\{\Omega_j\}_{j\ge 1}$ is an exhaustion increasing sequence of hyperconvex domains in Ω satisfying: $\overline{E} \subset \Omega_1$, $\overline{\Omega}_j \subset \mathrm{int}(\Omega_{j+1})$, $j \ge 1$, $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Theorem 3.1.4 in [2] implies that for every $j \ge 1$ the equality

$$\mathrm{C}(E,\Omega_j) = \int_{\Omega_j} (dd^c u^*_{E,\Omega_j})^n$$

holds. Since $\Omega_j \uparrow \Omega$ and Ω has the property $(\mathbb{P})$, Proposition 2.4.2 i) implies that

$$u^*_{E,\Omega_j} \downarrow u^*_{E,\Omega}.$$

Hence, $(dd^c u^*_{E,\Omega_j})^n$ is weakly convergent to $(dd^c u^*_{E,\Omega})^n$. Now we have

$$\mathrm{C}(E,\Omega) \le \mathrm{C}(E,\Omega_j) = \int_{\Omega_j} (dd^c u^*_{E,\Omega_j})^n = \int_{\overline{E}} (dd^c u^*_{E,\Omega_j})^n, \quad \text{for} \quad j \ge 1$$

because $\mathrm{supp}(dd^c u^*_{E,\Omega_j})^n \subset \overline{E}$.

Thus

$$\begin{aligned}\mathrm{C}(E,\Omega) &\le \limsup_j \mathrm{C}(E,\Omega_j) = \limsup_j \int_{\overline{E}} (dd^c u^*_{E,\Omega_j})^n \\ &\le \int_{\overline{E}} (dd^c u^*_{E,\Omega})^n = \int_{\Omega} (dd^c u^*_{E,\Omega})^n\end{aligned}$$

and (5) follows.

It remains to prove that

$$\int_{\Omega} (dd^c u^*_{E,\Omega})^n \le \mathrm{C}(E,\Omega). \tag{6}$$

If E is compact then (6) follows from the definition of $\mathrm{C}(E,\Omega)$. From the hypothesis and Proposition 2.4.2 ii) it follows that there is an increasing sequence of compact subsets $K_j \Subset E$ such that $u^*_{K_j,\Omega} \downarrow u^*_{E,\Omega}$. Consequently, $(dd^c u^*_{K_j,\Omega})^n$ weakly converges to $(dd^c u^*_{E,\Omega})^n$. Therefore,

$$\begin{aligned}\int_{\Omega} (dd^c u^*_{E,\Omega})^n &\le \liminf_j \int_{\Omega} (dd^c u^*_{K_j,\Omega})^n = \liminf_j \int_{K_j} (dd^c u^*_{K_j,\Omega})^n \\ &\le \liminf_j \mathrm{C}(K_j,\Omega) \le \mathrm{C}(E,\Omega)\end{aligned}$$

and we are done. □

Next we prove the equality (2) without the assumption on the relative compactness of E in Ω. Namely, we give the following

3.2. Theorem. *Let Ω be a bounded hyperconvex domain, $E \subset \Omega$ a closed subset such that $\overline{E}_{\mathbb{C}^n} \cap \partial\Omega = K$ is pluripolar. Then*

$$\mathrm{C}(E,\Omega) = \int_E (dd^c u^*_{E,\Omega})^n. \tag{7}$$

Proof. We have to show that if $\mathrm{C}(K,\Omega) > \alpha$ then $\int_E (dd^c u^*_{E,\Omega})^n > \alpha$, where $\alpha > 0$ is arbitrary. Fix $\varepsilon > 0$ such that $\mathrm{C}(E,\Omega) > \alpha > \varepsilon > 0$. Choose $u \in \mathrm{PSH}(\Omega)$, $-1 \leq u \leq 0$ such that $\int_E (dd^c u)^n \geq \alpha$. Put

$$U_\varepsilon = \Big\{ z \in \Omega : u^*_{E,\Omega}(z) < -1 + \varepsilon \Big\}.$$

Then U_ε is open in Ω. Theorem 7.1 in [1] implies that $E \setminus U_\varepsilon$ is pluripolar. Since $E \subset U_\varepsilon \cup (E \setminus U_\varepsilon)$ it follows that

$$\int_{U_\varepsilon} (dd^c u)^n = \int_{U_\varepsilon \cup (E\setminus U_\varepsilon)} (dd^c u)^n \geq \int_E (dd^c u)^n \geq \alpha.$$

Pick $E_\varepsilon \Subset U_\varepsilon$ such that

$$\int_{E_\varepsilon} (dd^c u)^n > \alpha - \frac{\varepsilon}{2}. \tag{8}$$

Assume that $\varphi \in \mathrm{PSH}(\Omega)$ such that $K \subset \{\varphi = -\infty\}$. Since Ω is bounded then we may assume that $\varphi < 0$ on Ω. We prove that for $m \geq 1$ sufficiently large the inequality

$$\int_{L_\varepsilon} (dd^c u)^n \geq \int_{E_\varepsilon} (dd^c u)^n > \alpha - \varepsilon \tag{9}$$

holds, where $L_\varepsilon = E_\varepsilon \cap \{\varphi \geq -m\}$. Indeed, take $E_\varepsilon \Subset \omega \Subset U_\varepsilon$. Then

$$\int_{E_\varepsilon \cap \{\varphi < -m\}} (dd^c u)^n \leq \frac{1}{m} \int_{E_\varepsilon} (-\varphi)(dd^c u)^n \leq \frac{C_{E_\varepsilon,\omega}}{m} \|\varphi\|_{\mathrm{L}^1(\omega)} \longrightarrow 0$$

as $m \to +\infty$, where the second inequality follows from Theorem 2.1.7 in [2]. Hence, for m large enough we have

$$\int_{E_\varepsilon \cap \{\varphi < -m\}} (dd^c u)^n < \frac{\varepsilon}{2}. \tag{10}$$

From (10) it follows that

$$\int_{E_\varepsilon} (dd^c u)^n = \int_{L_\varepsilon} (dd^c u)^n + \int_{E_\varepsilon \setminus L_\varepsilon} (dd^c u)^n \leq \int_{L_\varepsilon} (dd^c u)^n + \frac{\varepsilon}{2}$$

and hence, (8) implies that (9) is true. Fix m such that (9) is true. We claim that there exists $\varepsilon' > 0$ so small that the function

$$v_{\varepsilon'} = \max\left\{u^*_{E,\Omega}, (1-6\varepsilon)u - 3\varepsilon + \frac{\varepsilon\varphi * \varrho_{\varepsilon'}}{m}\right\}$$

satisfies

(a) $v_{\varepsilon'} = u^*_{E,\Omega}$ on a neighborhood of $\partial\Omega$.
(b) $v_{\varepsilon'} = (1-6\varepsilon)u - 3\varepsilon + \frac{\varepsilon\varphi*\varrho_{\varepsilon'}}{m}$ on a neighborhood V of L_ε.

For a moment assume that (a) and (b) are satisfied. Then by the Stoke's theorem we have

$$\begin{aligned}
\int\limits_\Omega (dd^c u^*_{E,\Omega})^n &= \int\limits_\Omega (dd^c v_{\varepsilon'})^n \\
&\geq \int\limits_V (dd^c v_{\varepsilon'})^n \\
&= (1-6\varepsilon)^n \int\limits_V (dd^c u)^n + \frac{\varepsilon^n}{m^n}\int\limits_V (dd^c \varphi * \varrho_{\varepsilon'})^n \\
&\geq (1-6\varepsilon)^n \int\limits_V (dd^c u)^n \\
&\geq (1-6\varepsilon)^n \int\limits_{\mathrm{L}_\varepsilon} (dd^c u)^n \\
&> (1-6\varepsilon)^n(\alpha - \varepsilon). \qquad (11)
\end{aligned}$$

Tending ε to 0 in (11) we get

$$\int\limits_\Omega (dd^c u^*_{E,\Omega})^n \geq \alpha$$

and the desired conclusion follows. Thus it remains to prove (a) and (b). First we prove (b). Pick $\varepsilon' > 0$ so small and set

$$V = \{z \in U_\varepsilon : \varphi * \varrho_{\varepsilon'}(z) > -m\}$$

where $\varrho_{\varepsilon'}$ is the canonical smooth kernel. Then V is an open neighborhood of L_ε. Indeed, if $z \in L_\varepsilon$ then $-m \leq \varphi(z) < \varphi * \varrho_{\varepsilon'}(z)$. Hence, $z \in V$. On the other hand, on Ω we have

$$(1-6\varepsilon)u - 3\varepsilon + \frac{\varepsilon\varphi * \varrho_{\varepsilon'}}{m} \geq -1 + 3\varepsilon + \frac{\varepsilon\varphi * \varrho_{\varepsilon'}}{m}. \qquad (12)$$

From the definition of V it follows that

$$-1 + 3\varepsilon + \frac{\varepsilon\varphi * \varrho_{\varepsilon'}}{m} \geq -1 + 3\varepsilon - \varepsilon = -1 + 2\varepsilon > -1 + \varepsilon > u^*_{E,\Omega}$$

on V. Hence, (b) is satisfied.

Now we show that (a) is valid. Assume that (a) is false. Then there exist sequences $\{x_j\} \subset \Omega$ and $\{\varepsilon_j\}$, $\varepsilon_j \downarrow 0$ such that

(i) $x_j \to \xi \in \partial\Omega$.

(ii)$u^*_{E,\Omega}(x_j) < (1-6\varepsilon)u(x_j) - 3\varepsilon + \frac{\varepsilon\varphi*\varrho_{\varepsilon_j}(x_j)}{m}$.

Since the right hand side of (ii) $< -3\varepsilon$ and, by the hypothesis on the hyperconvexity of Ω it is easy to see that $u^*_{E,\Omega}(\xi) = 0$ if $\varphi(\xi) > -\infty$, then we must have $\varphi(\xi) = -\infty$. Hence, $\limsup\limits_{j\to\infty} \varphi * \varrho_{\varepsilon_j}(x_j) = -\infty$ and we get a contradiction because the left hand side ≥ -1. $\square$

3.3. Example. Now we give an example which shows that if the hypothesis on the pluripolarity of the set $\overline{E}_{\mathbb{C}^n} \cap \partial\Omega$ is removed then the formula (7) in Theorem 3.2 is not true.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$.For each $r > 0$ we denote $\Delta(0,r) = \{z \in \mathbb{C} : |z| < r\}$ and $\overline{\Delta}(0,r) = \{z \in \mathbb{C} : |z| \leq r\}$. Let $\Omega = \Delta^2 = \{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$ be the bidisc in $\mathbb{C}^2$ and $E = \overline{\Delta}(0,\frac{1}{2}) \times \Delta \subset \Omega$. It is easy to see that E is closed in Ω and $\overline{E} \cap \partial\Omega = \overline{\Delta}(0,\frac{1}{2}) \times \partial\Delta$, where $\partial\Delta = \{z \in \mathbb{C} : |z| = 1\}$. First we prove $\overline{E} \cap \partial\Omega$ is not pluripolar in $\mathbb{C}^2$. To get a contradiction we assume that $\overline{E} \cap \partial\Omega$ is pluripolar. Then there exists $\varphi(z,w) \in \mathrm{PSH}(\mathbb{C}^2)$, $\varphi \neq -\infty$ and $\varphi|_{\overline{\Delta}(0,\frac{1}{2})\times\partial\Delta} = -\infty$. For each $w \in \partial\Delta$, the function $z \mapsto \varphi(z,w)$ is subharmonic on $\mathbb{C}$ and $= -\infty$ on $\overline{\Delta}(0,\frac{1}{2})$. Hence, $\varphi(z,w) = -\infty$ on $\mathbb{C}$. Thus $\varphi|_{\mathbb{C}\times\partial\Delta} = -\infty$. By the maximum principle it follows that $\varphi = -\infty$ on $\mathbb{C} \times \Delta$ which is impossible. Now for each $j \geq 2$ set

$$E_j = \overline{\Delta}\left(0,\frac{1}{2}\right) \times \Delta\left(0, 1-\frac{1}{j}\right).$$

Notice that $\{E_j\}$ is an increasing sequence of subsets of E and $\bigcup\limits_{j\geq 2}^{\infty} E_j = E$. Proposition 2.2.1 in [2] implies that

$$\mathrm{C}(E,\Omega) = \lim_{j\to\infty} \mathrm{C}(E_j,\Omega) = \lim_{j\to\infty} \frac{2\pi}{\log 2}.\frac{2\pi}{-\log(1-\frac{1}{j})} = +\infty$$

where the second equality follows from Theorem 3.1.11 in [2] and the formula $\mathrm{C}(\overline{\Delta}(0,r),\Delta(0,R)) = \frac{2\pi}{\log R - \log r}$. On the other hand, Theorem 3.1.11 in [2] shows that

$$\begin{aligned} u^*_{E,\Omega} &= \max\{u^*_{\overline{\Delta}(0,\frac{1}{2}),\Delta}, u^*_{\Delta,\Delta}\} = \max\{u^*_{\overline{\Delta}(0,\frac{1}{2}),\Delta}, -1\} \\ &= u^*_{\overline{\Delta}(0,\frac{1}{2}),\Delta} = \max\{\frac{\log|z|}{\log 2}, -1\} \end{aligned}$$

Thus $\int\limits_E (dd^c u^*_{E,\Omega})^2 = \frac{2\pi}{\log 2}$ and the desired conclusion follows.

4. Relative extremal functions and generalized holomorphic coverings

In the end of this paper we investigate the relation between the relative extremal functions through generalized holomorphic coverings. Namely, we prove the following.

4.1. Theorem. *Let $\mathbb{D} \subset \mathbb{C}^n$ and $\mathbb{G} \subset \mathbb{C}^m$ be domains, $m \leq n$ and $h : \mathbb{D} \to \mathbb{G}$ be a generalized holomorphic covering outside a complex subvariety A of $\mathbb{G}$. Assume that $\mathbb{G}$ has the property ($\mathbb{P}$). Then for every subset $F \subset \mathbb{G}$ we have*

$$u^*_{h^{-1}(F),\mathbb{D}}(z) = u^*_{F,\mathbb{G}}(h(z)),\ \forall z \in \mathbb{D}. \tag{13}$$

Proof. Put $E = h^{-1}(F)$. First we show that

$$u^*_{F,\mathbb{G}}(h(z)) \leq u^*_{E,\mathbb{D}}(z),\ \forall z \in \mathbb{D}. \tag{14}$$

Indeed, it is easy to see that $h(E) = F$. Let $v \in \mathrm{PSH}^-(\mathbb{G}), v|_F \leq -1$. Then $v \circ h \in \mathrm{PSH}^-(\mathbb{D})$, $v \circ h|_E \leq -1$. This implies that

$$(v \circ h)(z) \leq u^*_{E,\mathbb{D}}(z),\quad \forall z \in \mathbb{D}.$$

Hence,

$$\sup\{v(h(z)) : \quad v \in \mathrm{PSH}^-(\mathbb{G}),\quad v|_F \leq -1\} \leq u^*_{E,\mathbb{D}}(z),\quad z \in \mathbb{D}.$$

Consequently,

$$u^*_{F,\mathbb{G}}(h(z)) \leq u^*_{E,\mathbb{D}}(z),\quad z \in \mathbb{D}$$

and (14) is proved. Now we show that the reverse inequality

$$u^*_{E,\mathbb{D}}(z) \leq u^*_{F,\mathbb{G}}(h(z)),\ \forall z \in \mathbb{D}$$

holds. Assume that $v \in \mathrm{PSH}^-(\mathbb{D}), v|_E \leq -1$. For each $a \in (\mathbb{G} \setminus A)$ we can find a neighborhood $V_a \subset (\mathbb{G} \setminus A)$ of a such that

$$h^{-1}(V_a) = \coprod_\alpha U_\alpha$$

where U_α is a neighborhood of x_α with $h(x_\alpha) = a$ and $h_\alpha = h|_{V_a} : V_a \to U_\alpha$ is biholomorphism for all $\alpha \in \mathrm{I}$. Then $v \circ h_\alpha \in \mathrm{PSH}(V_a)$ for all $\alpha \in \mathrm{I}$. For $x \in V_a$, set

$$\widetilde{v}(x) = \Big(\sup\{v \circ h_\alpha(x) : \alpha \in \mathrm{I}\}\Big)^*.$$

Then $\widetilde{v} \in \mathrm{PSH}(V_a)$. Thus we may define a plurisubharmonic function $\widetilde{v} \in \mathrm{PSH}^-(\mathbb{G} \setminus A)$ given by

$$\widetilde{v}(w) = \Big(\sup\{v(t) : t \in h^{-1}(w)\}\Big)^*$$

for all $w \in (\mathbb{G} \setminus A)$. Since A is a closed pluripolar set in $\mathbb{G}$, Theorem 2.7.1 in [3] implies that there exists $\widetilde{u} \in \mathrm{PSH}^-(\mathbb{G})$ such that $\widetilde{u}|_{(\mathbb{G}\setminus A)} = \widetilde{v}$. We show that $\widetilde{u}|_{(h(E)\setminus A)\setminus Z} \leq -1$, where $Z \subset \mathbb{G}$ is a pluripolar set. Indeed, let $x \in (E \setminus h^{-1}(A))$.

Then $h(x) \in (F \setminus A)$ and for all $t \in h^{-1}(h(x))$ we observe that $v(t) \le -1$. Thus $\widetilde{u}(w) \le -1$ for $w \in (h(E) \setminus A) \setminus Z$, $Z \subset \mathbb{G}$ is pluripolar. It follows that

$$\widetilde{u}(h(z)) \le u^*_{(h(E)\setminus A)\setminus Z, \mathbb{G}}(h(z)),\ z \in \mathbb{D}.$$

Since $\mathbb{G}$ has the property $(\mathbb{P})$ then by repeating the arguments presented in the proof of Theorem 3.1.7 in [2] we deduce that

$$u^*_{(h(E)\setminus A)\setminus Z, \mathbb{G}}(w) = u^*_{h(E),\mathbb{G}}(w), \quad w \in \mathbb{G}.$$

Therefore,

$$\widetilde{u}(h(z)) \le u^*_{h(E),\mathbb{G}}(h(z)) u^*_{F,\mathbb{G}}(h(z)), \quad z \in \mathbb{D}.$$

Obviously,

$$\widetilde{u}(h(z)) \ge v(z), \quad z \in \mathbb{D} \setminus h^{-1}(A).$$

However, $h^{-1}(A)$ is a complex subvariety of $\mathbb{D}$ and hence, it is a pluripolar set in $\mathbb{D}$. Hence,

$$\widetilde{u}(h(z)) \ge v(z), \quad z \in \mathbb{D}.$$

From the above arguments we arrive at

$$v(z) \le u^*_{F,\mathbb{G}}(h(z)), \quad z \in \mathbb{D}$$

and consequently,

$$u^*_{E,\mathbb{D}}(z) \le u^*_{F,G}(h(z)), \quad z \in \mathbb{D}.$$

Thus (13) follows and the proof of Theorem 4.1 is complete. □

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