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DOMAINS OF OPERATOR SEMI-ATTRACTION OF OPERATOR SEMI-STABLE PROBABILITY MEASURES

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Abstract. In this paper we attempt to describe domains of operator semi-attraction of operator semi-stable probability measures on finite dimensional Euclidean spaces. We give new characterizations of the operator semi-stability, the domains of operator semi-attraction and the domains of operator attraction.

1. Introduction and notation

Let V be a finite dimensional real vector space with an inner product $(.,.)$ and a norm $\|.\|$. For an arbitrary linear operator A acting in V and a *probability measure* (p.m.) p on V, Ap is a p.m. defined by $Ap(E) := p(A^{-1}E)$ for each Borel subset $E \subset V$. Further, we denote the p.m. concentrated at the point $x \in V$ by $\delta(x)$, the convolution of two p.m.'s p and q by pq. Throughout, for natural number n the power p^n is taken in the sense of the convolution. The *characteristic function* (Fourier transform) of a p.m. p is defined by the formula

$$p^\wedge(y) := \int_V e^{i(x;y)} p(dx).$$

It is easy to verify that for linear operators A, B and p.m.'s p, q the formulas

$$(Ap)^\wedge(y) = p^\wedge(A^*y),\ A(pq) = ApAq,\ (AB)p = A(Bp)$$

hold, where A^* denotes the adjoin operator of A. Besides, if p is *infinitely divisible* (inf .div.) p.m. then for every positive $t > 0$ the power $(p^\wedge)^t$ is also the characteristic function of an inf .div. p.m. which is denoted straight by p^t.

Given p.m. p, we define $\bar{p}$ by putting $\bar{p}(E) := p(-E)$ for each Borel subset E of V, where $-E := \{-y : y \in E\}$. Moreover, the p.m. $|p|^2 := p\bar{p}$ is called *symmetrization* of p. The p.m. p is said to be *full* if its support is not contained in any proper subspace of V. Let $\Rightarrow$ mean the weak convergence of measures and Aut$V$ denote the group of all non-singular linear operators acting on V. In the following definitions, the assumption on non-singularity of the operators is not

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essential, it takes place only for convenience in the sequel. Thus, in what follows, (A_k), (b_k) and (n_k), also with other subscripts or indices, will denote a sequence of operators from Aut$V$, a sequence of vectors from V and a strictly increasing sequence of positive integers, respectively. Then if

$$(A_k p^{n_k})\delta(b_k) \Rightarrow q \tag{1}$$

as $k \to \infty$, we say that p belongs to the *domain of operator partial attraction* of q (write $p \in DOPA(q)$). Moreover, if we assume in addition that

$$\frac{n_k}{n_{k+1}} \to r > 0 \tag{2}$$

as $k \to \infty$, then we say that q is operator semi-stable and p belongs to the *domain of operator semi-attraction* of q ($p \in DOSA(q)$), or more exactly, to the *domain of operator r-semi-attraction* of q ($p \in DOSA(r, q)$). Further, we say that q is operator stable and p belongs to the domain of operator attraction of q ($p \in DOA(q)$), if (n_k) in (1) coincides with the sequence of all natural numbers, i.e.

$$(A_k p^k)\delta(b_k) \Rightarrow q \tag{3}$$

as $k \to \infty$.

Following Jajte ([4, Lemma 1]) we have

Lemma 1. *If* (1) *holds and q is full then $A_k \to \Theta$, the zero operator.*

Let $LIM(c_k)$ denote the set of all limit points of a real sequence (c_k). Then under power of Lemma 1, it is easy to see that

Note. If (1) holds for some sequences (A_k), (b_k) and (n_k) with q full then there are new sequences (A^1_k), (b^1_k) and (n^1_k) such that (1) holds for them and $1 \in LIM(n^1_k/n^1_{k+1})$.

Let $G(q) := \{r > 0 : (1)$ and (2) hold with some $p\}$. By virtue of Theorem [4] and Theorem 3.2 [6], the set $G(q)$ is a closed multiplicative subgroup of $R^+ := \{r : r > 0\}$ if $G(q) \neq \emptyset$. Thus either $G(q) = R^+$ and q is operator stable, or $G(q)$ is generated by s, the largest element in R^+ less than 1. In the last case we say that q is *operator* (s)*-semi-stable* and p belongs to the *domain of operator* (s)*-semi-attraction* of q (denote $p \in DOSA((s), q)$).

For more detailed descriptions of operator stable and operator semi-stable p.m.'s, the reader is referred to [3, 4-6, 8]. Here we only concern the problem similar to that of stable and semi-stable p.m.'s discussed in [2]. In particular, we shall show that in the definitions of the operator semi-stability, the domain of operator semi-attraction and the domain of operator attraction, the conditions (2) and (3) can be replace by the weaker ones.

2. Results and demonstration

This section is started with definitions of *type* and *equivalence* of p.m.'s.

Definition 1. Two p.m.'s q_1 and q_2 are said to be of the *same type* if there are $A \in \mathrm{Aut}V$ and $b \in V$ such that $q_2 = (Aq_1)\delta(b)$. If q_1 and q_2 are inf .div. and there is a positive number $s > 0$ such that

$$q_2 = (Aq_1^s)\delta(b) \tag{4}$$

then q_1 and q_2 are called *equivalent*.

Remark. It is well known that an inf .div. p.m. q is operator stable if and only if every power q^s ($s \in R^+$) is of the same type as q (see [3, 8] for example). Therefore, for operator stable p.m.'s, the two concepts of "sample type" and "equivalence" coincide.

Now we state the first main result related with the equivalence of two p.m.'s.

Theorem 1. *Let p, q_1 and q_2 be p.m.'s on V, q_1 and q_2 be full. Suppose that there exist sequences $(A_{i,k})$, $(b_{i,k})$ and $(n_{i,k})$, $i = 1, 2$, and a positive number $c > 0$ such that for $i = 1, 2$*

$$(A_{i,k}p^{n_{i,k}})\delta(b_{i,k}) \Rightarrow q_i \tag{5}$$

as $k \to \infty$, and

$$\frac{n_{1,k}}{n_{1,k+1}} \geq c \tag{6}$$

for all k. Then q_1 and q_2 are equivalent inf. div. p.m.'s.

Proof. The infinite divisibility of the measures q_1 and q_2 is an immediate consequence of Lemma 1. Besides, one can suppose that there is a subsequence of positive integers $(k(m))$ such that

$$n_{1,k(m)-1} \leq n_{2,m} < n_{1,k(m)}.$$

Then by virtue of (6), for all $m = 1, 2, \ldots$, we have

$$c \leq \frac{n_{1,k(m)-1}}{n_{1,k(m)}} \leq \frac{n_{2,m}}{n_{1,k(m)}} < 1.$$

Hence, taking a subsequence if necessary, one can assume in addition that

$$n_{2,m}/n_{1,k(m)} \to s \tag{7}$$

with $c \leq s \leq 1$.

Now, we attempt to show that the sequence of operators

$$(A_m := (A^*_{1,k(m)})^{-1}A^*_{2,m}) \tag{8}$$

is precompact in the space of all linear endomorphisms of V. Let us suppose the contrary, i.e. that the sequence of norms $\|A_m\|$ is unbounded. Choose vectors z_m in V such that $\|z_m\| = 1$ and $\|A_m z_m\| = \|A_m\|$. Let $y_m = A_m z_m$. Taking a subsequence if necessary, we can assume that $\|A_m\| \to \infty$ and $y_m/\|y_m\| \to y \in V$. Then $z_m/\|y_m\| \to 0$ which together with (7) yields

$$\begin{aligned}&|p^\wedge(tA^*_{2,m}(y_m/\|y_m\|))|^{n_{2,m}}\\ &= \{|p^\wedge(tA^*_{1,k(m)}(z_m/\|y_m\|))|^{n_{1,k(m)}}\}^{(n_{2,m}/n_{1,k(m)})} \to \ |q_1^\wedge(0)|^s = 1\end{aligned}$$

for every $t \in R$. Simultaneously, (5) implies

$$|p^\wedge(tA_{2,m}^*(y_m/\|y_m\|))|^{n_{2,m}} \to |q_2^\wedge(ty)|.$$

Therefore, the characteristic function of the symmetrized measure $|q_2|^2$ equals 1 on the subspace $\{ty : t \in R\}$. Consequently, q_2 is not a full measure (see [8, Proposition 1]), which contradicts the assumption.

Let C be a limit point of the sequence (8). Then by taking a subsequence if necessary, we can suppose that $A_m \to C$. Meanwhile, it follows from (5) that for every $y \in V$ we have

$$\begin{aligned} q_2^\wedge(y) &= \lim_{m\to\infty} (p^\wedge(A_{2,m}^* y))^{n_{2,m}}.e^{i(b_{2,m},y)} \\ &= \lim_{m\to\infty} \{(p^\wedge(A_{1,k(m)}^* A_m y))^{n_{1,k(m)}}.e^{i(b_{1,k(m)},A_m y)}\}^{(n_{2,m}/n_{1,k(m)})}.e^{i(b_m,y)} \end{aligned}$$

with suitable chosen vectors $b_m \in V$ $(m = 1, 2, \ldots)$. Hence, it is evident from (5) and (7) that (b_m) converges to some vector $b \in V$ and

$$q_2^\wedge(y) = q_1^\wedge(Cy)^s.e^{i(b,y)}.$$

Then, by setting $A = C^*$, we have $q_2 = (Aq_1)^s \delta(b)$. Moreover, since the support of Aq_1 is contained in the image $A(V)$ and q_2 is full, the operator A is non-singular. Thus, q_1 and q_2 are equivalent. □

It should be noted that in the above proof we use the technique developed in the fundamental work [9] by K. Urbanik.

Stability, semi-stability and infinite divisibility of p.m.'s are strongly related with their DOPA, DOSA and DOA. The following corollaries show interesting features of DOPA's, DOSA's and DOA's:

Corollary 1. *Let p, q_1 and q_2 be as in Theorem 1. Then*

(i) *The conditions $p \in DOSA(q_1)$ and $p \in DOPA(q_2)$ imply the equivalence of q_1 and q_2;*
(ii) *From the conditions $p \in DOA(q_1)$ and $p \in DOPA(q_2)$ we can conclude the p.m.'s q_1 and q_2 to be of the same type and therefore the both two p.m.'s are operator stable.*

Corollary 2. *Let p, q_1 and q_2 be as in Theorem 1 and $p \in DOSA(r, q_1)$ with $r \in (0, 1]$. Then $p \in DOSA(r, q_2)$ if and only if q_1 and q_2 are equivalent.*

Proof. It is clear that Corollary 1 and the "only if" part of Corollary 2 follow immediately from Theorem 1. We shall prove the "if" part of Corollary 2. Suppose that (4) holds and

$$(A_k p^{n_k})\delta(b_k) \Rightarrow q_1; \qquad \frac{n_k}{n_{k+1}} \to r$$

as $k \to \infty$.

Let us put

$$\begin{aligned} A_k^1 &:= AA_k, \\ n_k^1 &:= [n_k.s], \\ b_k^1 &:= b + \tfrac{n_k^1}{n_k} Ab_k, \end{aligned}$$

where $[t]$ means the largest integer less than or equal to t. Then we have

$$\frac{n_k^1}{n_k} = \frac{[n_k.s]}{n_k} \to s.$$

Therefore

$$(A_k^1 p^{n_k^1})\delta(b_k^1) = A\{(A_k p^{n_k})\delta(b_k)\}^{(n_k^1/n_k)}\delta(b) \Rightarrow (Aq_1)\delta(b) = q_2.$$

On the other hand,

$$\lim_{k\to\infty} \frac{n_k^1}{n_{k+1}^1} = \lim_{k\to\infty} \frac{[n_k.s]}{[n_{k+1}.s]} = \lim_{k\to\infty} \frac{n_k}{n_{k+1}} = r.$$

Thus, $p \in DOSA(r, q_2)$, the proof is complete. □

From the above we see that for two equivalent full p.m.'s, if one of them is operator r-semi-stable then so is the second. Besides, if p belongs to DOSA of some full operator *(r)*-semi-stable p.m. then it does not belong to DOSA of another full operator *(s)*-semi-stable p.m. with $r \neq s$, moreover it does not belong to DOPA of any full p.m. which is not operator semi-stable.

The following theorem gives a new characterization of the operator semi-stability:

Theorem 2. *Let p and q be p.m.'s on V, q be full. Assume that* (1) *holds and*

$$LIM\left(\frac{n_k}{n_{k+1}}\right) \cap (0,1) \neq \emptyset. \tag{9}$$

Then q is operator semi-stable.

Proof. Under the power of (9) we can choose a subsequence $(k(m))$ of natural numbers and a positive number c such that

$$\frac{n_{k(m)}}{n_{k(m)+1}} \to c$$

as $m \to \infty$. Then using the same technique as in the proof of Theorem 1 we can infer from (1) that there exist $A \in \mathrm{Aut}V$, $b \in V$ such that $q = (Aq^c)\delta(b)$ which together with Theorem in [4] implies the operator semi-stability of q. □

After this theorem one can put the following question: Does $p \in DOSA(q)$ if p and q satisfy the condition in Theorem 2? The answer will be given partially in the following theorem:

Theorem 3. *Let p and q be as in Theorem 2. Suppose that* (1) *holds and there exists a positive number $c > 0$ such that*

$$\frac{n_k}{n_{k+1}} \geq c \tag{6*}$$

for all k. Then

(a) *If q is operator (r)-semi-stable then $p \in DOSA((r), q)$,*
(b) *If q is operator stable then $p \in DOA(q)$.*

The proof of the theorem will be preceded by the following two lemmas:

Lemma 2. *Let $0 < r < 1$ and q be a full operator (r)-semi-stable p.m. Suppose that there exist sequences (A_k), (b_k), (n_k) and a real number c, $0 < c < 1$, such that* (1) *and* ($6^\star$) *hold. Then there exist sequences (A_k^1), (b_k^1) and (n_k^1) such that*

$$(A_k^1 p^{n_k^1})\delta(b_k^1) \Rightarrow q \tag{1'}$$

and

$$LIM(n_k^1/n_{k+1}^1) = \{r, 1\}. \tag{2'}$$

Proof. Let $A \in \mathrm{Aut}V$ and $b \in V$ such that

$$q^r = (Aq)\delta(b) \tag{10}$$

(see Theorem [4]) and let N be the natural number satisfying $r^N \geq c > r^{N+1}$. We define sequences $(A_k^{(m)})$, $(b_k^{(m)})$ and $(n_k^{(m)})$, $m = 1, 2, ..., N+1$, by

$$\begin{aligned} A_k^{(m)} &:= A^{m-1}A_k, \\ b_k^{(m)} &:= \left(\frac{[n_k/r^{(m-1)}]}{n_k}\right)A^{m-1}b_k, \\ n_k^{(m)} &:= [n_k/r^{(m-1)}], \end{aligned}$$

with $A^0 := Id$, the identity operator. Then for $m = 1, 2, \ldots, N+1$ there is an element $b^{(m)} \in V$ such that

$$(A_k^{(m)} p^{n_k^{(m)}})\delta(b_k^{(m)}) \Rightarrow q\delta(b^{(m)}). \tag{11}$$

Indeed, the left sides of (11) can be written as

$$(A^{m-1}((A_k p^{n_k})\delta(b_k)))^{[n_k/r^{(m-1)}]/n_k} \Rightarrow A^{m-1}(q^{1/r^{(m-1)}})$$

as $k \to \infty$ because of (1) and of

$$\frac{[n_k/r^{(m-1)}]}{n_k} \to \frac{1}{r^{(m-1)}}$$

as $k \to \infty$. On the other hand, (10) implies

$$A^{m-1}(q^{1/r^{(m-1)}}) = q\delta(b^{(m)})$$

with $b^{(m)} \in V$. Thus (11) is true.

For $k = 1, 2, \ldots$ let $h(k)$ be a natural number such that

$$\frac{n_k}{r^{h(k)-1}} \leq n_{k+1} < \frac{n_k}{r^{h(k)}}. \tag{12}$$

Hence, it follows from (6*) that for all natural k,

$$1 \le h(k) \le N+1. \tag{13}$$

We now try to show that

$$LIM\Big(\frac{n_k^{(h(k))}}{n_{k+1}}\Big) = \{r, 1\}. \tag{14}$$

Namely, (11) implies

$$(A_k^{(m)} p^{n_k^{(m)}})\delta(b_k^{(m)} - b^{(m)}) \Rightarrow q$$

as $k \to \infty$, for $m = 1, 2, ..., N+1$. Therefore, by setting

$$\begin{array}{ll} p_{2k-1} & := (A_k^{(h(k))} p^{n_k^{(h(k))}})\delta(b_k^{(h(k))} - b^{(h(k))}), \\ p_{2k} & := (A_{k+1} p^{n_{k+1}})\delta(b_{k+1}) \end{array}$$

for $k = 1, 2, \ldots$, from (1) we have $p_k \Rightarrow q$ as $k \to \infty$. If $s \in LIM(n_k^{(h(k))}/n_{k+1})$ and $s \ne 1$, then (12) implies $r \le s < 1$. On the other hand, from (6*) and the definition of p_k, in the same way as in the proof of Theorem 2 we can see that q is *s-semi-stable*. Meanwhile, q is *(r)-semi-stable*. Consequently, $s = r$ and looking at Note in the first section, we can see that (14) holds.

The sequences (A_k^1), (b_k^1) and (n_k^1) are built as follows

$$\begin{array}{ll} A_k^1 & := A_j^{(m)}, \\ b_k^1 & := b_j^{(m)} - b^{(m)}, \\ n_k^1 & := n_j^{(m)} \end{array}$$

if $k = h(1) + h(2) + \ldots + h(j-1) + m$, $1 \le m \le h(j)$, $j = 2, 3, \ldots$. Then by using (11) and (13) we can easily verify that (1') is satisfied. Besides, for $k = h(1) + h(2) + ... + h(j-1) + m$,

a) If $k = h(1) + h(2) + \ldots + h(j-1) + m$ and $1 \le m < h(j)$ then

$$\frac{n_k^1}{n_{k+1}^1} = \frac{n_j^{(m)}}{n_j^{(m+1)}} = \frac{[n_j/r^{(m-1)}]}{[n_j/r^m]} \to r \quad \text{as } j \to \infty. \tag{15}$$

b) If $k = h(1) + h(2) + ... + h(j)$ then

$$\frac{n_k^1}{n_{k+1}^1} = \frac{n_j^{(h(j))}}{n_{j+1}}.$$

This together with (14) and (15) results (2'). The proof is complete. □

Lemma 3. *Let p and q be p.m.'s on V, q be full and $0 < r < 1$. Then*

(i) *If there exist sequences (A_k), (b_k) and (n_k) satisfying (1) and (2) then we can find sequences (A_k^1), (b_k^1) and (n_k^1) such that $(1')$ and $(2')$ hold,*

(ii) *Conversely, if there exist sequences (A_k^1), (b_k^1) and (n_k^1) such that $(1')$ and $(2')$ hold and q is not operator stable then we can build new sequences (A_k), (b_k) and (n_k) satisfying (1) and (2).*

Proof. (i) For $m = 1, 2, \ldots$ let us put

$$A^1_{2m-1} = A^1_{2m} := A_m,$$
$$b^1_{2m-1} = b^1_{2m} := b_m,$$
$$n^1_{2m-1} := n_m,$$
$$n^1_{2m} := n_m + 1.$$

Then, by the assumption, it is evident that

$$(n^1_{2m})/(n^1_{2m-1}) \to 1; \quad (n^1_{2m-1})/(n^1_{2m+1}) \to r; \quad (n^1_{2m})/(n^1_{2m+1}) \to r.$$

Consequently, the condition (2') is true. On the other hand, by virtue of Lemma 2, we have

$$A^1_k \to \Theta; \qquad A^1_k p \Rightarrow \delta(0).$$

and (1') follows straightly from (1).

(ii) Now let us suppose that conditions (1') and (2') are satisfied. By an argument analogous to that used for the proof of Theorem 2 we conclude that q is operator r-semi-stable and after Theorem 3.2 [6] there exists a positive number $r_0 < 1$ such that q is operator (r_0)-semi-stable and $r = r_0^m$ with some natural m. Then

a) If $m = 1$ then q is operator *(r)*-semi-stable. Let $q = (Aq^r)\delta(b)$ with $b \in V$ and $A \in \mathrm{Aut}V$ (see Theorem [4]). For every $k = 1, 2, \ldots$ let $h(k)$ be a natural number such that

$$n^1_{h(k)-1} \leq \frac{n^1_k}{r} < n^1_{h(k)}. \tag{16}$$

Then from (1') we see that

$$\lim_{k\to\infty} (A^1_k p^{[n^1_k/r]})\delta(b^1_k/r) = \lim_{k\to\infty} ((A^1_k p^{n^1_k})\delta(b^1_k))^{1/r} = q^{1/r} = (A^{-1}q)\delta(b^0_r)$$

with some $b^0_r \in V$. Hence

$$(AA^1_k p^{[n^1_k/r]})\delta(Ab^1_k/r - b^0_r) \Rightarrow q. \tag{17}$$

We infer that

$$LIM([n^1_k/r]/n^1_{h(k)}) = \{r, 1\}. \tag{18}$$

Indeed, it follows from (16) that $n^1_{h(k)-1}/n^1_{h(k)} \leq [n^1_k/r]/n^1_{h(k)} < 1$ and by virtue of (2') we have

$$LIM(n^1_{h(k)-1}/n^1_{h(k)}) \subset \{r, 1\}.$$

Consequently, it is clear that

$$LIM([n^1_k/r]/n^1_{h(k)}) \subset [r, 1].$$

Thus, if $s \in LIM([n^1_k/r]/n^1_{h(k)})$ and $s \neq 1$ then under the power of (1') and (17), with the same reason of Lemma 2, we can confirm $s = r$ and (18) is just proved.

Now let us define

$$\begin{aligned} K_1 &:= \{k : [n_k^1/r]/n_{h(k)}^1 \geq \tfrac{1+r}{2}\}, \\ K_2 &:= \{k : [n_k^1/r]/n_{h(k)}^1 < \tfrac{1+r}{2}\}. \end{aligned}$$

Then (18) implies

$$\left\{ \begin{array}{l} \lim\limits_{k\to\infty, k\in K_1} ([n_k^1/r]/n_{h(k)}^1) = 1, \\ \lim\limits_{k\to\infty, k\in K_2} ([n_k^1/r]/n_{h(k)}^1) = r. \end{array} \right. \tag{19}$$

Moreover, it is obvious that

$$n_k^1/[n_k^1/r] \to r \tag{20}$$

as $k \to \infty$.

The desired sequences (A_k), (b_k) and (n_k) will be constructed by the following induction: Let $A_1 := A_1^1$, $b_1 := b_1^1$ and $n_1 := n_1^1$. Further we set

$$A_2 := A_{h(1)}^1, \qquad b_2 := b_{h(1)}^1, \qquad n_2 := n_{h(1)}^1$$

if $1 \in K_1$, or

$$A_2 := AA_1^1, \qquad b_2 := (1/r)Ab_1^1 - b_r^0, \qquad n_2 := [n_1^1/r]$$

$$A_3 := A_{h(1)}^1, \qquad b_3 := b_{h(1)}^1, \qquad n_3 := n_{h(1)}^1$$

if $1 \in K_2$.

Suppose that A_i, b_i and n_i have been built for $i = 1, 2, \ldots, k$ and

$$A_k = A_{h(j)}^1, \qquad b_k = b_{h(j)}^1, \qquad n_k = n_{h(j)}^1$$

with some natural j. Then we set

$$A_{k+1} := A_{h(h(j))}^1, \qquad b_{k+1} := b_{h(h(j))}^1, \qquad n_{k+1} := n_{h(h(j))}^1$$

if $h(j) \in K_1$, or

$$A_{k+1} := AA_{h(j)}^1, \qquad b_{k+1} := (1/r)Ab_{h(j)}^1 - b_r^0, \qquad n_{k+1} := [n_{h(j)}^1/r]$$

$$A_{k+2} := A_{h(h(j))}^1, \qquad b_{k+2} := b_{h(h(j))}^1, \qquad n_{k+2} := n_{h(h(j))}^1$$

if $h(j) \in K_2$, etc.

It follows from (1') and (17) that (1) is true for the new sequences (A_k), (b_k) and (n_k). Moreover, (2) fulfils immediately from (19) and (20).

b) In the case that $m > 1$, the condition (2') allows us to suppose that

$$\frac{n_k^1}{n_{k+1}^1} \geq r_0^{m+1} > 0$$

for all k. Then the conditions of Lemma 2 are satisfied for r_0 in place of r whilst (A_k^1), (b_k^1) and (n_k^1) playing respectively the role of (A_k), (b_k) and (n_k). In that circumstance, with the new constructed sequences, by using Lemma 2 we can turn to the case when $m = 1$, and apply the above part to complete the proof. □

Proof of Theorem 3. (a) Let q be operator *(r)*-semi-stable with $0 < r < 1$. Then we can turn to the case when $LIM(n_k/n_{k+1}) = \{r, 1\}$ applying Lemma 2 and supposing in addition that $0 < c \leq r$. Therefore, the support of Lemma 3 guarantees the confirmation of $p \in DOSA((r), q)$.

(b) Let q be operator stable and for every $s \in (0, 1]$ let $B_s \in \mathrm{Aut}V$, $\beta_s \in V$ be defined by

$$(B_s q^{1/s})\delta(\beta_s) = q \tag{21}$$

(see Theorem 2 [8]). We construct the sequences (A°_m) and (b°_m) as follows:

$$A^{\circ}_m := B_{n_j/m} A_j,$$
$$b^{\circ}_m := (m/n_j) B_{n_j/m} b_j + \beta(n_j/m)$$

for $n_j \leq m < n_{j+1}$, $j = 1, 2, \ldots$. We tend to prove that

$$(A^{\circ}_m p^m)\delta(b^{\circ}_m) \Rightarrow q \tag{22}$$

as $m \to \infty$.

In the matter, let (m') be any subsequence of natural numbers. Then for all $m' \in (m')$ one can find a natural number $j(m')$ such that $n_{j(m')} \leq m' < n_{j(m')+1}$. Hence from (6*) we have

$$c \leq \frac{n_{j(m')}}{n_{j(m')+1}} < \frac{n_{j(m')}}{m'} \leq 1.$$

Therefore one can pick from (m') another subsequence (m'') such that

$$\frac{n_{j(m'')}}{m''} \to s$$

for some $s \in (0, 1]$. Then by virtue of the compactness lemma ([8, Proposition 1]), using (21) and taking a subsequence once more if necessary, we can assume that

$$B_{(n_{j(m'')}/m'')} \to B_s, \qquad \beta(n_{j(m'')}/m'') \to \beta(s).$$

Now, (1) together with (21) implies

$$\begin{aligned}
&A^{\circ}_{m''} p^{m''})\delta(b^{\circ}_{m''}) \\
&= B_{(n_{j(m'')}/m'')} A_{j(m'')} p^{m''} \delta\Big(\frac{m''}{n_{j(m'')}} B_{(n_{j(m'')}/m'')} b_{j(m'')} + \beta(n_{j(m'')}/m'')\Big) \\
&= [B_{(n_{j(m'')}/m'')}\{(A_{j(m'')} p^{(n_{j(m'')})})\delta(b_{j(m'')})\}]^{(m''/n_{j(m'')})} \delta(\beta(n_{j(m'')}/m'')) \\
&\Rightarrow (B_s q^{1/s})\delta(\beta(s)) = q.
\end{aligned}$$

Thus, Theorem 2.3 [1] yields (22), i.e. $p \in DOA(q)$, the proof is finished. □

As an immediate consequence of Theorem 3 we have

Corollary 3. *Suppose that q is a full operator stable p.m. on V. Then we have $DOA(q) = DOSA(r, q)$ for every $r \in (0, 1)$.*

After Theorem 3 we have new characterizations of DOA's and DOSA's by putting (6*) in place of (2) in the definitions. It is worthy to notice that the condition (9) used in Theorem 2 to determine semi-stability of a given p.m. is weaker than (6*) that is equivalent to

$$LIM(\frac{n_k}{n_{k+1}}) \subset (0,1].$$

Moreover, in the same way as in Example [2], we can build an example to show that in Theorem 3 the condition (6*) cannot be replaced by (9).

P.S. The earlier version of the article was published only in a preprint form. However, a part of the results has been used in a monograph written by Meerschaert and Scheffler ([7, Theorem 7.5.6, p. 281]).

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